A TEXT BOOK OF ALGEBRA

FOR HIGH SCHOOLS

BY

V. R. VENKATRAMA . . AR, B.A., L.T.,

Inspector of Schools (Retired).

Thoroughly revised and brought up-to-date

M/A.L. 26

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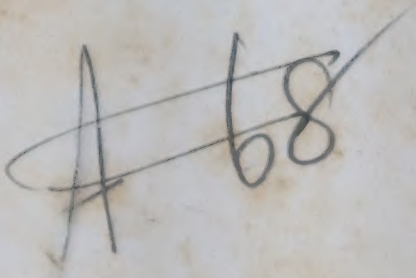

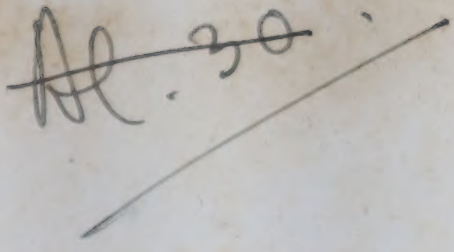

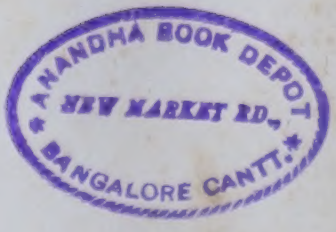
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Thoroughly revised and brought up-to-date
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PREFACE TO THE FIRST EDITION.

This work is intended as a text-book on Algebra for all classes in our High Schools which prepare their pupils either for the University Matriculation (now S.S.L.C.) Examination. Great pains have been taken to render the work intelligible to young students by copious explanations and illustrations.

The plan adopted in this work is one which I have, for many years, followed in teaching large classes in several Schools and Colleges and which I have found to succeed. A proposition is first clearly demonstrated, then copiously illustrated by select examples, and then a large number of other examples arranged progressively is given for exercise. In addition, at the end of every eight chapters, a series of examination papers, each paper containing ten questions, is inserted.

The book contains thousands of examples, of which over 400 are fully worked out. They have been, for the most part, selected from the Examination papers of English and Indian Colleges and Universities.

In the preparation of this work, most of the recent treatises on Algebra have been consulted.

SAIDAPET. V. R. V.

PREFACE TO THE FIFTH EDITION.

In this edition, the book has been thoroughly revised and considerably enlarged. Many parts have been entirely re-written while there is no part which has not been modified more or less. A good deal of elementary matter is added in the beginning so as to suit the requirements of students who now begin their study of Algebra only in the Fourth Form. Almost every exercise has been increased by adding a number of examples, interesting in themselves or illustrative of standard methods.

To specify the exact portions which a beginner ought to get up before proceeding to advanced portions is, if not quite impossible, at least no easy task.

One full chapter has, however, been devoted exclusively for harder examples in Multiplication, Division and Factorization. On the whole it would be better to reserve the later examples of every exercise for a second reading.

It must be admitted that clearness in working out problems is of the very first importance especially in a subject like Algebra. Hence the model solutions illustrate not simply the hard or tricky points in algebraical sums but even the very manner of working out the same.

By the method of verification now introduced into several chapters, the student is enabled to check his results without taking the trouble of going through the process over again.

It is hoped that this little work in its revised edition will maintain its former popularity and continue to be a help to the teacher and a guide to the student.

Any emendations or suggestions for the improvement of the work will be thankfully received by the Publishers.

MADRAS, K. R. Shenai.

PREFACE TO THE SIXTH EDITION.

In this edition the book has been carefully revised and suitably enlarged. Many changes, additions and omissions have been made so that it may be adopted as an up-to-date text book in Algebra meeting modern requirements in the subject. It will be found useful not only to beginners who learn formal Algebra in the IV f. rm but also to pupils who specialise in Algebra and Geometry and pursue an advanced course in it under the C group of the S.S.L.C. scheme in this presidency while it retains its chief and former merit of remaining a good text book for students preparing for the Matriculation Examination of the various Indian Universities. The following are some of the noteworthy features of this edition :—

(i) Simple equations, symbolical representation as preparatory and preliminary to the solution of problems leading to simple equations have been introduced early enough in the VI and VII Chapters after the first four rules.

(ii) New Chapters deal with (*a*) problems on simple equations (*b*) co-ordinates and plotting points (*c*) the straight line graph and (*d*) some applications of graph.

(iii) Some more chapters have been added at the end of the book fully illustrating the graphs of equations of the second degree in x and y and also the maximum and minimum values of quadratic functions of x.

(iv) The remainder theorem, its application in factorisation, functional notation and the transformation of $\phi(x)$ into the form $F(x+b)$ have been amply treated in the advanced chapter on multiplication and division.

(v) Some of the old fashioned sums have been omitted, for instance, in radical equations ; such omissions have secured greater attention to principles than to mere artifices and mechanical methods. In conclusion, the book, it is hoped, will commend itself to all teachers and pupils in view of the numerous graduated examples given under each exercise illustrating an algebraical principle taught in the class. With the use of this book, the difficulty now experienced by many teachers in selecting and sometimes framing appropriate sums to accompany the theoretical instruction will vanish and the anxiety felt by pupils in not having some more sums for home work on the same principle as those done in the class will equally disappear. Any suggestions will be thankfully received by the publishers.

MADRAS,
April 1925.

S. V. RAMADAS PANTULU,
M.S
V. VENKATASUBBAYYA.

PREFACE TO THE SEVENTH EDITION.

In this Seventh edition the book has undergone a careful and thorough revision to suit the modern requirements of beginners as well as advanced students of Algebra in the High School Course. Many useful changes have been introduced, but care has been taken to preserve the special characteristics of the book which have earned for it a great reputation as a very good Text-Book for students preparing for the S. S. L. C. Examination and the Matriculation Examination of the various Indian Universities.

The book has been divided into two parts with all the chapters carefully re-arranged.

Part I is complete in itself forming an easy course for the beginner. The subject of graphs has been introduced early enough in Chapters IX and X.

Part II treats of the harder portions of Algebra required for a specialist. The Remainder Theorem, its application to factorisation, functional notation and the application of the Principle of Indeterminate Co-efficients have been copiously dealt with in this part.

The most attractive feature of this edition is the omission of old fashioned sums and the introduction of a large variety of examples of quite a modern type in the several sets of Revision Papers and Miscellaneous Examples

In conclusion, as every effort has been made to combine modern methods of treatment with a good number of modern problems of varying grades of difficulty, it is hoped that the book will be found quite useful both to teachers and pupils. Any suggestions will be thankfully received by the publishers.

MADRAS,
15th July 1927.

G. V. NARAYANASWAMI IYER,
B.A., L.T.

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A
TEXT BOOK OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND EXPLANATIONS OF SIGNS.

N.B.—The definitions and explanations of signs given in this Chapter may be taught to boys whenever there are found necessary They are given in one place only for the sake of convenience.

1. Algebra, like Arithmetic, is a science of *numbers* ; but the numbers in Algebra are *generally* denoted by *letters* instead of by *figures*.

2. It is usual to represent *known* numbers by the first letters of the alphabet, as a, b, c and *unknown* numbers by the last letters, as x, y, z ; but this need not be strictly followed.

The word **quantity** is often used in the sense of *number*, integral or fractional. Any *whole* number is called an **integer** or an *integral* number Integers divisible by 2 are *even* ; those that are not divisible by 2 are *odd*. If n is taken to represent any integer, all even integers come under the general term $2n$, and all odd integers under $2n+1$.

3. The sign + (read *plus*) signifies that the quantity to which it is prefixed is to be *added*. Thus $a+b$ (read *a plus b*) means that the number denoted by b is to be added to that denoted by a. If a represents 4 and b represents 5, $a+b$ represents 9. Similarly $a+b+c$ means that b is to be added to a and that to the result c is to be added; if a, b, c, denote 4, 5, 6, respectively, $a+b+c$ denotes 15.

When no sign is prefixed to the letter at the beginning of an expression, + is understood. Thus $a+b$ means $+a+b$.

4. The sign $-$ (read *minus*) signifies that the quantity to which it is prefixed is to be subtracted. Thus $a-b$ (read *a minus b*) means that the number denoted by b is to be subtracted from that denoted by a; if a denotes 7 and b denotes 4, $a-b$ denotes 3. Similarly $a-b-c$ means that b is to be subtracted from a and that from the result c is to be subtracted; if a, b, c represent 7, 4, 2 respectively, $a-b-c$ represents 1.

Quantities to which the sign $+$ is prefixed, or before which it is understood, are called **positive quantities.**

Quantities having the sign $-$ prefixed to them are called **negative quantities.**

5. The *difference* of two numbers is denoted by the sign $\backsim$; thus $a \backsim b$ denotes the difference of the numbers denoted by a and b, and is equal to $a-b$ or to $b-a$, according as a is greater or less than b.

Note.—The sign $\mp$ denotes *plus* or *minus*; and the sign $\pm$ denotes *minus* or *plus*.

6. The sign $\times$ (read *into*) signifies that the numbers between which it stands are to be *multiplied* together.

Thus $a \times b$ (*a into b*) denotes that the number represented by a is to be multiplied by the number represented by b. If a represents 4 and b represents 3, then $a \times b$ represents 12.

The sign of multiplication is often omitted for the sake of brevity in Algebra, but never in Arithmetic; thus ab is used instead of $a \times b$ and has the same meaning, so abc is used for $a \times b \times c$. Also $3b$ denotes $3 \times b$; but 38 is used for $3 \times 10 + 8$ and not for 3×8. Sometimes a point is used instead of the sign $\times$; thus $a.b$ is used for $a \times b$ or ab. But this point must not be confounded with the decimal point. Thus 2·4 (2 *into* 4) is different from 2·4 (2 *decimal* 4).

7. The sign $\div$ (read *by*) signifies that number which precedes it is to be *divided* by the number which follows it. Thus $a \div b$ (*a by b*) signifies that the number denoted by a is to be divided by the number denoted by b. If a represents 10, and b represents 2, then $a \div b$ represents 5.

$\frac{a}{b}$ or a/b is used instead of $a \div b$ and has the same meaning.

Note.—The word *sign* is often used to denote exclusively the signs + and −. Two signs are called *like* when they are both + or both −; otherwise, they are *unlike*.

8. The sign = (read *equal to*) signifies that the numbers between which it is placed are *equal*, and is called the *sign of equality*: $a=b$, is read thus, "*a equals b*," or "*a is equal to b*." The sign $\neq$ denotes *not equal to*.

9. The sign $>$ denotes *greater than*, and the sign $<$ denotes *less than*; thus $a>b$ denotes *a is greater than b*; and $a<b$ denotes *a is less than b*. The sign $\not>$ denotes *not greater than* and the sign $\not<$ denotes *not less than*.

10. The sign $\therefore$ denotes *then* or *therefore*; the sign $\because$ denotes *since* or *because*.

11. **A vinculum** ———, or a pair of **brackets** ( ), { }, or [ ], signifies that all quantities under or within it are equally affected by any quantity without. Thus $a(b+c)$ or $a.b+c$ denotes that a is to be multiplied by the sum of $b+c$; whereas $a\,b+c$ or $ab+c$ would signify that a was to be multiplied by b, and their product increased by c. If $a=2$; $b=3$; $c=4$, the first would $=14$; the second would $=10$.

12. The **reciprocal** of any quantity is that quantity inverted, or unity divided by it. The reciprocal of a is $\frac{1}{a}$; that of $a+b$ is $\frac{1}{a+b}$. Conversely the reciprocal of $\frac{1}{a}$ is a; that of $\frac{1}{a+b}$ is $a+b$.

13. Any collection of algebraical symbols is called an **algebraical expression**.

Those parts of an expression which are connected by the signs + or − are called **its terms**.

When an expression consists of *one* term, as a, bx, $-abx$, it is called a **monomial expression**; when it consists of *two* terms, as $a+b$, $ax-by$, it is called a **binomial expression**; when it consists of *three* terms, it is called a **trinomial expression**; any expression consisting of several terms is called **multinomial** or **polynomial expression**.

14. When an expression consists of *one* term, it is called a **simple** expression; when it consists of *more than one term*, it is called a **compound** expression.

15. When two or more quantities are multiplied together the result obtained is called their **product** and each of the quantities so multiplied is called a **factor** of that product. Thus a, b and c are factors of the product abc.

16. When a number consists of the product of two factors, each factor is called the **co-efficient** or **co-factor** of the other factor. In the product $3a$, 3 is the co-efficient of a; where there is no other arithmetical factor, we may supply unity; in the product ab, the co-efficient of ab is unity.

In the product abc, we may call a the co-efficient of bc, or b the co-efficient of ac, or c the co-efficient of ab. These co-efficients are called **literal** co-efficients as distinguished from the former which are called **numerical** co-efficients.

In $5a^2b$, 5 is the numerical co-efficient of a^2b; $5a^2$ and $5b$, the literal co-efficients of b and a^2 respectively.

The co-efficient must always be given with the sign + or − preceding it as the case may be. In $-4a^2b^2$, the co-efficient of b^2 is $-4a^2$, and of a^2, $-4b^2$.

Exercise—State the co-efficient of—

1. a in $7abc$.—$4ab^2$.
2. b in—$5a^3bc$, $10a^4c^2b$, a^2bc^3d.
3. x^2. in—$10a^2x^2y^2$, x^2yz,—$b^2x^2c^2$.
4. ab in a^2b^2—$3abc$.
5. xy in $4x^3$—$2xy$+1.
6. a^2 in a^2+3a^3b.

17. If a number be multiplied by itself any number of times, the product is called a **power** of that number. Thus $a \times a$ is called the *second power* or **square** of a; $a \times a \times a$ is called the *third power* or **cube** of a; $a \times a \times a \times a$ is called the *fourth power* of a, and so on. The number a itself is called the *first power* of a.

Any power of a quantity is usually expressed by placing above the quantity the number which represents how often it is repeated in the product. Thus a^2 (*a squared*) is used to express $a \times a$; a^3 (*a cubed*) is used to express $a \times a \times a$; a^4 is used to express $a \times a \times a \times a$, and so on a^1 has the same meaning as a.

The number placed above a quantity to express the power of that quantity is called an **index** or **exponent** of the power.

(a^n is the product of n factors each equal to a, and n, is called the *index* or *exponent* of a^n where n is any whole number).

The symbol a^4 is read thus, "*a to the fourth power*," or "*a to the fourth*" and a^n is read thus, "*a to the n^{th}*."

When the different powers of the *same* quantity occur in a multinomial expression, it is usual to arrange the terms according to *descending* or *ascending* powers of that quantity. For example, in the expression $4a^4+3a+5a^2-6a^3$, different powers of a, namely, a^4, a, a^2, a^3 occur. We can arrange the terms in two ways; *first*, beginning with the *highest* power of a, we may write the other terms so as to contain the lower powers in *descending* order, thus: $4a^4-6a^3+5a^2+3a$; *secondly*, beginning with the *lowest* power of a, we may write the other terms so as to contain the highest powers in *ascending* order, thus: $3a+5a^2-6a^3+4a^4$.

Note.—If there be one or more terms not involving a, such terms should come *last* when the terms are arranged according to *descending* powers of a, and *first* when the terms are arranged according to *ascending* powers of a.

Exercise.—Arrange according to *ascending* and *descending* powers of x each of the following expressions, and of y, expression (2), and of b, expression (3).

1. $x-3x^4+4x^3-x^5+x^6+3x^2+5$.
2. $x^3y^2=xy+x^2y^2+x^5y-x^4y^3+y^4$.
3. $b^5x^4-bx^3+b^4x+b^4x^3+b^3x^2+b$.

18. The square root of any proposed quantity is that quantity whose second power or square gives the proposed quantity. Thus the **square root of 64 is 8**, for $8^2=64$.

The **cube root** of any proposed quantity is that quantity whose third power or cube gives the proposed quantity. Thus the cube root of 64 is 4, for $4^3 = 64$.

The m^{th} root of any quantity is that quantity whose m^{th} power gives the proposed quantity.

The square root of a is denoted by $\sqrt{a}$ or simply by $\surd a$ usually

The cube root of a is denoted by $\sqrt[3]{a}$ and the m^{th} root of a by $\sqrt[m]{a}$.

The sign $\surd$ is said to be a corruption of the letter r, the first letter of the word *radix*. The sign is called the **radical sign**.

$\surd a$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, . . . $\sqrt[m]{a}$ can also be represented by $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, . . . $\frac{1}{a^m}$ respectively.

19. Algebraical quantities are said to be *like* or *unlike*, according as they contain the *same* or *different* combinations of letters.

Thus a and $5a$,—$5a^2b$ and $7a^2b$, $3a^3bc$ and $-a^3bc$, are pairs of *like* quantities; a^3 and a^2, $3ab$, and $-7a$. $3a^2b$ and $4a^3b$, of *unlike* quantities.

Exercise.—Collect all the *like* terms from each of the following :—

1. $2a - 3b + 4ab - 5a^2b - 4a^3b^2 + 5a + 4b - a^3b^2 + 2ba^2 + 14ba$.
2. $2x + 3y + 4z - 7z^2 - 4z^3 - 9z - 4y^2 - 3y + 10z^3 - 4x^2 - 10x + 9x - 3y^2$
3. $4ab - 3xy - 10ba - 4yz + 9xy - 6ab + 7yz + 2xy - 3yz + 17ab$

20. Each of the letters which occur in an algebraical product is called a **dimension** of the product, and the number of the separate letters contained in the product (as if they were all written out in full) is the **degree** of the product. Thus a^2b^2c or $a \times a \times b \times b \times c$ is of five dimensions or of the fifth degree. Numerical co-efficients should not be counted; thus $9a^2b$ and ab^2 are of the same dimensions, namely, of 3 dimensions.

The *number of dimensions* of a term is the *sum* of the exponents or indices of the several factors of the term.

When all the terms of an algebraical expression are of the same dimensions, the expression is said to be **homogeneous**. Thus $4a^3b^2$

$+8a^{3}b^{2}+2a^{2}b^{3}+a^{2}bcd$ is a homogeneous expression, for each term is of 5 dimensions.

21 The square root of any even root of a negative quantity is called an *impossible* or **imaginary quantity**. Thus $\sqrt{-3}$ is an impossible or imaginary quantity.

22. When any root of a quantity cannot be exactly extracted, the quantity which represents the root is called a **surd** or an **irrational quantity**. Thus $\sqrt{3}$ and $\sqrt{ab}$ are surds.

NUMERICAL VALUES.

Ex. 1. If $a=5$.

$4a=4\times a=4\times 5=20.$

$a^{4}=a\times a\times a\times a=5\times 5\times 5\times 5=625.$

$6a^{3}=6\times 5\times 5\times 5=750.$

$\frac{3}{7}a^{2}=\frac{3}{7}\times 5\times 5=\frac{75}{7}=10\frac{5}{7}.$

$1^{a}=1\times 1\times 1\times 1\times 1=1.$

Note.—Every power of 1 is 1.

Ex. 2. If $a=2$, $b=3$, $x=1$, $y=0$,

find the values of $4abx$ and $6abxy$.

$4abx=4\times 2\times 3\times 1=24.$

$6abxy=6\times 2\times 3\times 1\times 0=36\times 0=0.$

Note.—If one factor of a product is equal to 0, the whole product is equal to 0, whatever values the other factors may have.

Ex. 3. If $a=3$, $b=2$, $x=4$,

find the values of a^{b}, x^{a} and $3b^{x}$,

$a^{b}=a^{2}=a\times a=3\times 3=9.$

$x^{a}=x^{3}=x\times x\times x=4\times 4\times 4=64.$

$3b^{x}=3\times b^{4}=3\times b\times b\times b\times b=3\times 2\times 2\times 2\times 2=48.$

$0=0\times 0\times 0=0.$

Note.—Every power of 0 is 0.

If $a=1$, $b=2$, $c=3$, $d=4$, $e=0$, find the values of :—

Ex. 4. $a-2b+3c+4d-6e$
$=1-2\times2+3\times3+4\times4-6\times0$
$=1-4+9+16-0$
$=1+9+16-4$
$=26-4$
$=22.$

Ex. 5. $7abc-5bcd+4cde+3acd$
$=7\times1\times2\times3-5\times2\times3\times4+4\times3\times4\times0+3\times1\times3\times4.$
$=42-120+48\times0+36$
$42+0+36-120$
$=78-120$
$=-42.$

Ex. 6. If $a=4$, $b=3$, $c=5$, find the values of $3a^2b^2$, $4a^3c^2$ and $5ab^2c$

$3a^2b^2=3\times4^2\times3^2=3\times16\times9=432.$
$4a^3c^2=4\times4^3\times5^2=4\times64\times25=6400.$
$5ab^3c=5\times4\times3^3\times5=5\times4\times27\times5=2700.$

Ex. 7. If $x=2$, $y=3$, $z=4$, find the values of

$\frac{4x^2y}{3z^2}$ and $\frac{5y^2z^3}{27x^4}$

$\frac{4x^2y}{3z^2}=\frac{4\times2^2\times3}{3\times4^2}=\frac{4\times4\times3}{3\times4\times4}=1.$

$\frac{5y^2z^3}{27x^4}=\frac{5\times3^2\times4^3}{27\times2^4}=\frac{5\times9\times64}{27\times16}=6\frac{2}{3}.$

EXERCISE 1.

If $a=1$, $b=2$, $c=3$, $x=4$, $y=0$, find the numerical values of :—

1. $14a^3$; $7c$; $8x^2$; $4bcx$; $3abc$; $5bcxy$.
2. xa ; ax ; bc ; $4b$; $5x$; $6a$.
3. $\frac{1}{6}x^2$; $\frac{7}{12}acx$; $\frac{3}{4}bxy$; $\frac{a^3}{4}$; $\frac{c^3}{3}$; $\frac{x^4}{10}$.
4. ya ; $10y^2$; $\frac{yz}{81}$; $\frac{10abcy}{32}$.

If $a=4$, $b=3$, $c=5$, $d=6$, $e=1$, $f=0$, find the numerical values of :—

5. $3a+2b-c$.
6. $2c-d+3e$.
7. $12c+3a-5b-2f+4e$.
8. $ab+bc+ca$.
9. $bc-de-af$.
10. $-6af+abc-2cd+e$.
11. $4ab+3cf-6d+7be$.
12. $abcd-bce+fac$.
13. $\frac{1}{2}ab-3bcd+5ef-6cde$.
14. $4abcd-7ade-2afe+2bcde$.
15. $4ac-9ae+11bcd-10afc$.
16. $42a-21bc-14ae+36ace$.
17. $20c-\frac{1}{5}cde+\frac{1}{4}abd-2cd$.
18. $12b-6cf+7cde-9abde$.

Ex. 8. Find the value of :—

$\frac{2d}{a}-\frac{6e}{b}-\frac{15a}{cd}+\frac{4b}{d}$ (for values of a, b, &c., as in **Ex. 4.**)

$$\equiv\frac{2\times4}{1}-\frac{6\times0}{2}-\frac{15\times1}{3\times4}+\frac{4\times2}{4}$$

$$=\ 8\ -\ 0\ -\ \frac{5}{4}\ +\ 2$$

$$\equiv 10-\frac{5}{4}$$

$$\equiv\ 8\frac{3}{4}$$

Ex. 9. If $a=4$, $b=3$, $c=2$, find the value of :—

$$ab(a-b)-ac(a-c)+bc(b-c)$$
$$=4\times3(4-3)-4\times2(4-2)+3\times2(3-2)$$
$$=\ 12\times\ 1\ -\ 8\times\ 2\ +\ 6\times\ 1$$
$$=\ 12\ -\ 16\ +\ 6$$
$$=18-16$$
$$=2.$$

EXERCISE. 2.

If $a=6$, $b=5$, $c=0$, $x=2$, $y=3$. $z=4$,
find the numerical values of—

1. $5a^3b^2x$; $7abx^2y$; $4a^2b^2c$.
2. $5a^4x^2y^3c^4$; $4x^2y$; $axbycb$; zyx^2ab.
3. $\frac{4a^2b^2x}{3y^3z}$; $\frac{7z^3abx}{3y^3}$; $\frac{3a^3z^3}{4b^4y^3}$.
4. $\frac{45}{y^2b^2}$; $\frac{4a^2}{xyz}$; $\frac{x^2ab}{y^2z}$; $\frac{a^2bc}{xy^2}$; $\frac{b^2c}{3x^2y^2z^3}$.

If $a=3$, $b=4$, $c=6$, $d=5$, $e=1$, $f=0$.
find the values of :—

5. $\frac{a}{b}-\frac{b}{c}+\frac{c}{a}$.
6. $\frac{ab}{c}+\frac{bc}{a}-\frac{ca}{b}$.
7. $\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}$.
8. $\frac{a}{b-e}+\frac{b}{a-b}+\frac{c}{d-c}$.
9. $ab+cd-bd+ac$.
10. $a-b(d-e)+c(a+b)$.
11. $(a+b)(b+c)(c+a)$.
12. $\{a+(b+c)-d\}\ \{(a-b)-(c-d)\}\ (a-\overline{b+c+d})$.
13. $\{12a-(3b-3e)\}-4f-\overline{4b-3d+5c}\}$.
14. $\frac{ab}{cd}-\frac{bc}{a}+\frac{ac}{e}+\frac{abc}{d}-\frac{cde}{b}$.
15. $a(b-c)+b(c-a)+c(a-b)+d)(a-e)$.
16. $4a-e(2b-3c)+d(b-2c)+a(2d-3e)$.
17. $\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}$.
18. $\frac{a-b+c}{a+b-c}+\frac{b-c+a}{b+c-a}+\frac{c-a+b}{c+a-b}$.

Ex. 10. Find the values of :—

$\sqrt{a^2b}$ and $\sqrt[3]{bca}$ when $a=2$, $b=4$, $c=1$.

$\sqrt{a^2b}=\sqrt{2^2\times4}=\sqrt{4\times4}=\sqrt{16}=4$.

$\sqrt[3]{bca}=\sqrt[3]{4\times1\times2}=\sqrt[3]{8}=2$.

Ex. 11. Find the values of :—

$3\sqrt[3]{\frac{y^2bc}{6}}$ and $\frac{3}{4}ac-\sqrt{\frac{2b^2x}{9y}}-\sqrt[3]{\frac{by}{9x^2}}$

when $a=1$, $b=4$, $c=9$, $x=3$, $y=6$.

$$3\sqrt[3]{\frac{y^2bc}{6}}=3\times\sqrt[3]{\frac{6^2\times4\times9}{6}}=3\sqrt[3]{216}$$

$$=3\times6=18$$

$$\tfrac{3}{4}ac-\sqrt{\frac{2b^2x}{9y}}-\sqrt[3]{\frac{by}{9x^2}}$$

$$=\tfrac{3}{4}\times1\times9-\sqrt{\frac{2\times4^2\times3}{9\times6}}-\sqrt[3]{\frac{4\times6}{9\times3^2}}$$

$$=6\tfrac{3}{4}-\sqrt{\tfrac{16}{9}}-\sqrt[3]{\tfrac{8}{27}}$$

$$=6\tfrac{3}{4}-\tfrac{4}{3}-\tfrac{2}{3}$$

$$=4\tfrac{3}{4}.$$

Ex. 12. If $a=1$, $b=2$, $c=3$, find the value of :—

$$\begin{aligned}
&(a+b)^2(a^2-ab+b^2)+(a+c)^2(a^2-ac+c^2)\\
=&(1+2)^2(1^2-1\times2+2^2)+(1+3)^2(1^2-1\times3+3^2)\\
=&\quad 3^2(1-2+4)\quad+\quad 4^2(1-3+9)\\
=&\quad 9\times3\quad+\quad 16\times7\\
=&\quad 27\quad+\quad 112\\
=&139.
\end{aligned}$$

Ex. 13. If $a=2$, $b=3$, $x=6$, $y=5$, find the value of :—

$$\begin{aligned}
&\sqrt[y]{(a+b)^2y}+\sqrt[y]{a+x)(y-2a)}\quad+\sqrt[y]{(y-b)^2a}.\\
\equiv&\sqrt[y]{(2+3)^25}+\sqrt[y]{(2+6)(5-2\times2)}\quad+\sqrt[y]{(5-3)^22}\\
=&\sqrt[y]{5^2\times5}\quad+\sqrt[y]{8\times1}\quad+\sqrt[y]{2^3\times2}\\
=&\sqrt[y]{5^3}\quad+\sqrt[y]{8}\quad+\sqrt[3]{2^3}\\
=&5\quad+2\quad+2\\
\equiv&9.
\end{aligned}$$

EXERCISE 3.

If $x=4$, $y=1$, $z=2$, $a=9$, $b=5$, $c=8$, find the values of :—

1. $\sqrt{b^2cz}$; $\sqrt[3]{6az^2y}$; $\sqrt{ab^2yx}$; $\sqrt[3]{3ac}$

2. $\sqrt{\left(\frac{5z^3y^2}{bzc^2}\right)}$; $\sqrt[3]{\left(\frac{-5y^3}{b^3c}\right)}$; $\sqrt[3]{\left(\frac{c^2a^2}{3x^3}\right)}$; $2xy\sqrt{4ab^2}$

3. $\frac{\sqrt{xzc}}{\sqrt[y]{3axz}}$; $\frac{\sqrt{b^3c}}{\sqrt{4a^2x}}$; $\frac{\sqrt{cz}}{\sqrt{yx}}$; $\frac{\sqrt[y]{cby+z}}{\sqrt{ayzx}}$ $\Rightarrow$ $\frac{\sqrt[y+z]{c^{y+z}}}{\sqrt{a^y z^x}}$.

If $a=5$, $b=2$, $c=1$, $d=4$, $e=8$, $f=0$, $m=6$, $n=4$, $p=9$, find the values of :—

4. $\sqrt{2a-1}+\sqrt[3]{an+p-b}-\sqrt{a^2-d^2}$.

5. $(a+b)^2-(d-c)^2+(f+e)^2-(p-m+n)^2$.

6. $\sqrt{p}-\sqrt{n+e}-\sqrt{c}+\sqrt{cd}-\sqrt[3]{ef}$.

7. $\frac{a+b+c}{m+n}-\frac{b+c+d}{n+p}+\frac{c+d+e}{m+p}-\frac{d+e+f}{m+n+p}$.

8. $a^c-bd+db-mb+nc-pb$.

9. $\sqrt{pn-df-p-b}+\sqrt[a]{m^2-pd}$.

Find the value of $(9-a)(b+1)+(b+5)(a+7)-112$, when $a=5$ and $b=3$.

11. If $V=5a+4b-6c$, $X=-3a-9b+7c$, $Y=20a+7b-5c$, $Z=13a-5b+9c$, find the value of $V-(X+Y)+Z$ if $a=b=c=1$.

12. If $x=3$, $y=-4$, find the value of :—

$$21\left(\frac{2x+4y}{3}-\frac{4x-y}{7}\right)$$
$$-24\left[\frac{2x-y}{12}-\left\{\frac{5x-7y}{8}-\frac{1}{6}\left(\frac{x}{2}-\frac{5x-y}{4}\right)\right\}\right].$$

13. If $a=1$, $b=3$, $c=5$, find the value of:—
$\{a-(b-c)\}^2+\{b-(c-a)\}^2+\{c-(a-b)\}^2$.

14. If $x=11$, find the value of:—
$\sqrt[3]{[(x+2)\sqrt{x-2}-2\{\sqrt[3]{11x^2-x+2}\sqrt{x-2}\}]}$.

15. If $a=16$, $b=10$, $x=5$ and $y=1$, find the values of:—
(i) $(b-x)\sqrt{(a+b)}+\sqrt{\{(a-b)(x+y)\}}$.
(ii) $(a-y)\{\sqrt{(2bx)}+x^2\}+\sqrt{\{(a-x)(b+y)\}}$.

16. If $a=3$, $b=-2$, $c=1$, find the value of:—
$a(a-b)(a-c)\{\sqrt{2a^2+3b+4c}-\sqrt{(a+b)(a+c)}\}$.

17. Find the value of $a+b\sqrt{(x+y)}-(a-b)\sqrt{(x-y)}$, when $a=10$, $b=8$, $x=12$ and $y=4$,

18. If $x=5$, and $a=8$, find the value of:—
$$a\sqrt{(x^2-3a)}+x\sqrt{(x^2+3a)}+\frac{x+2a}{a-x}.$$

19. If $x=\frac{1}{4}$, $y=\frac{1}{9}$, $z=4$, find the value of:—
$$\frac{x+\sqrt{y}}{x+\sqrt{y}+\sqrt{z}}+\frac{y+\sqrt{z}}{z-\sqrt{x}-\sqrt{y}}+\frac{z+\sqrt{x}}{y-\frac{1}{8}\sqrt{z}+\sqrt{x}}.$$

20. If $a=1$, $b=2$, $c=3$, $d=0$, find the value of:—
$\sqrt[3]{b-a}+\sqrt[3]{4(c-a)}-\sqrt[3]{(8a+5b+3c-2d)}$.

21. If $x=2$, $y=3$, $a=6$, $b=5$, find the value of:—
$\sqrt{b(x+y)^2}+\sqrt[3]{(x+a)(b-2x)}+\sqrt[3]{x(b-y)^2}$.

Ex. 14. Make a table of values of y. When $y=2x^2-x+3$ corresponding to the following values of x 0, 1, 2, 3, −1, −2.

When	$x=$	0	1	2	3	−1	−2
	$2x^2=$	0	2	8	18	2	+8
	$-x=$	0	−1	−2	−3	+1	+2
	$+3=$	+3	+3	+3	+3	+3	+3
$y=$	$2x^2-x+3=$	3	4	9	18	6	13

22. Find the values of $\frac{2x-3}{4}$ for the following values of x:—

2, 4, 0, $1\frac{1}{2}$, $\frac{1}{2}$.

23. Find the values of y corresponding to the values of 0, 1, 2, 3, 4 of x if $y=\frac{4x+5}{2}$.

24. Find what values will $\frac{6x+7}{3}$ take when $x=-1, 2, 3, 0, -2$.

25. Find the values of $2x^2-3x+4$ corresponding to the following values of x:—0, 1, 2, −1, −2.

26. $y=x^2-3x+2$. Make a table of values of x and y when $x=-2, 3, 1, 4$.

27. $-3x^2+5x-6=y$. Make a table of values of x and y for values 0, 1, 2, 3 −2, −3 of x.

28. $y=-4x^2+2x-6$. Make a table of values of x and y corresponding to values $-1, -2, 0, +1, +2$ of x.

29. Show that $x^2-5x+6=0$ when $x=2$ or 3. Find the value of the expression when $x=5$.

30. Show that $2x^2+5x-12=0$ when $x=-4$. Find the value of the expression when $x=\frac{1}{2}$, 3.

31. Find the value of the expression $x^3-6x^2+11x-6$ when $x=1$, 2, 3.

32. Show by substituting 16 for x and 10 for y that the two expressions $3(x-y)+4(x+y)$ and $6(x+2y)-5(2x-y)$ are equal.

33. Show that p^2+23p is equal to $3(3p^2+5)$ when $p=1$, 3, or 5. Which expression is greater when $p=2$.

CHAPTER II.

ADDITION.

POSITIVE AND NEGATIVE QUANTITIES.

23. The signs + and − have been defined in Arts. 3 and 4 of the previous chapter. They denote the operations of addition and subtraction when placed between two quantities. Thus the expression $4a+5b-10c-7d$ means that the symbols $4a$ and $5b$ are to be *added* while $10c$ and $7d$ are to be *subtracted*. So long as the sum of the *additive* terms is greater than the sum of the *subtractive* terms, the meaning is quite intelligible; but when the sum of the subtractive terms exceeds that of the additive, it might at first sight seem to the student unlikely that such an expression should occur in practice.

24. Let us consider one or two simple illustrations. Suppose a merchant was to gain in one year a certain number of rupees and to lose a certain number of rupees in the following year, what change has taken place in his capital? Let a denote the number of rupees gained in the first year and b the number of rupees lost in the second. Then if a is greater than b, the capital of the merchant has been increased by $a-b$ Rupees. For example if $a=$ Rs. 200, and $b=$ Rs. 150, the merchant's capital has *increased* by Rs. 50. If, however, b is greater than a, the capital has been diminished by $b-a$ Rupees. For example if $a=150$, and $b=200$, the merchant's capital has suffered a diminution of Rs. 50.

The above statements are indicated thus:

Rs. 200 − Rs. 150 = + Rs. 50.
Rs. 150 − Rs. 200 = − Rs. 50.

In the first case the merchant has *gained* Rs. 50, while in the second case he has *lost* Rs. 50. If we denote by 50 or by +50 a *gain* we may denote by − 50 a *loss* of the same extent.

25. Again a distance measured in a particular direction say north, from a starting point bears to a distance measured in the *opposite* direction (south) the same relation as gain bears to loss. A date *before* a given point of time bears the same relation to a date *after* that point of time. These pairs of related magnitudes are distinguished by means of the signs + and −.

There are two points to be noticed; *first* that when no sign is used, + is to be understood; *secondly* the sign + may be ascribed to either of the two related magnitudes and then the sign − will belong to the other magnitude.

26. In Arithmetic we are concerned only with the numbers represented by the symbols 1, 2, 3, &c., and intermediate fractions. In Algebra, besides these, we consider another set of symbols $-1, -2, -3$, &c., and intermediate fractions. So it is usual to divide all algebraical quantities into **positive quantities** and **negative quantities** according as they are expressed with the sign + or the sign −.

The *absolute value* of any quantity is the number represented by this quantity taken independently of the sign which precedes the quantity.

27. When two or more algebraical quantities are united together, the result is called their **sum** and the process of finding the result is called **Addition**. An **algebraic sum** is not necessarily the sum of positive quantities.

28. When the terms of an expression are connected by the sign +, we may write the terms *in any order*. Thus $a+b$ and $b+a$ will give the same result.

When an expression consists of some terms preceded by the sign + and some terms preceded by the sign −, we may write the former terms first in any order we please, and the latter terms after them in any order we please.

Thus $a+d-b-c=a+d-c-b=d+a-c-b=d+a-b-c$.

29. *There are three cases of Addition:—*

First case. Where the quantities are *like*, and have *like* signs.

Rule.—"*Add the co-efficients together, and to their sum adjoin the letters common to each term, prefixing the common sign.*"

Ex. 1. Add together $3a$, $5a$, $9a$ and $7a$.

$3+5+9+7=24$ ∴ The required sum $=24a$.

Ex. 2. Add together $-2b$, $-4b$, $-5b$ and $-6b$.

$2+4+5+6=17$ ∴ The required sum $=-17b$.

Ex. 3. Add together $4a-3b$, $5a-6b$ and $2a-7b$.

$4a-3b+5a-6b+2a-7b$

$=4a+5a+2a-3b-6b-7b$

$=11a-16b$ or thus :—

$$\begin{array}{ll} 4a-3b & 4+5+2=11 \\ 5a-6b & \\ 2a-7b & \\ \hline 11a-16b & -3-6-7=-16. \end{array}$$

Ex. 4. Add together $\frac{2}{3}ax-b$, $\frac{1}{2}ax-\frac{3}{4}b$ and $\frac{1}{4}ax-\frac{1}{2}b$.

$\frac{2}{3}ax-b+\frac{1}{2}ax-\frac{3}{4}b+\frac{1}{4}ax-\frac{1}{2}b$

$=\frac{2}{3}ax+\frac{1}{2}ax+\frac{1}{4}ax-b-\frac{3}{4}b-\frac{1}{2}b$

$=1\frac{1}{4}ax-\frac{9}{4}b$ or thus

$$\begin{array}{ll} \frac{2}{3}ax-b & \frac{2}{3}+\frac{1}{2}+\frac{1}{4}=1\frac{1}{12} \\ \frac{1}{2}ax-\frac{3}{4}b & \\ \frac{1}{4}ax-\frac{1}{2}b & \\ \hline 1\frac{1}{12}ax-\frac{9}{4}b & -1-\frac{3}{4}-\frac{1}{2}=-\frac{9}{4}. \end{array}$$

EXERCISE 4.

Add together :—

1. $2a$, $5a$, a; $-2b$, $-3b$, $-b$; $\frac{1}{2}a$, $\frac{1}{4}a$, $\frac{5}{4}a$.
2. $10ab$, $2ab$, $9ab$, $16ab$; $6a^2b$, $4a^2b$, $3a^2b$.
3. $2a-3b$, $4a-10b$, $a-2b$, $2a-b$, $14a-13b$.
4. $x-2y-3z$, $-3y-4z+5x$, $-5z+6x-7y$.
5. a^2-ab+b^2, $2a^2-3ab+3b^2$, $3a^2-4ab+7b^2$.
6. $\frac{1}{3}ab+3bc-2ca$, $\frac{1}{4}bc-\frac{1}{2}ca+\frac{3}{4}ab$, $\frac{5}{3}ca+\frac{3}{4}ab+\frac{1}{2}bc$.
7. $x^2-2xy+y^2$, $\frac{1}{2}x^2-\frac{1}{4}xy+2y^2$, $\frac{3}{4}x^2-xy+\frac{1}{2}y^2$.
8. $1\frac{1}{2}-ax+by-ab$, $2\frac{1}{2}-3ax+\frac{1}{2}by-3ab$,
$4\frac{1}{2}-\frac{1}{3}ax+3by-\frac{3}{2}ab$, $4-10ax+by-\frac{3}{8}ab$.

9. $m^3+n^3-3m^2n+3mn^2$, $\frac{1}{2}m^3+\frac{1}{2}n^3-6m^2n+9mn^2$,
$\frac{1}{4}m^3+n^3-8m^2n+3mn^2$, $m^3+n^3-3m^2n+6mn^2$.

10. $a^3+b^3-c^3$, $2a^3+\frac{1}{2}b^3-\frac{1}{4}c^3$, $3a^3+\frac{1}{4}b^3-\frac{1}{8}c^3$.

11. $6a-4$, $7a-3$, $8a-4$, $9a-6$, $10a-8$.

12. x^2+6x-8, $5x^2+9x-3$, $4x^2+13x-10$, $8x^2+7x-6$.

13. $x^2+5xy+6xz-3yz$, $4x^2+3xy+8xz-9yz$,
$3x^2+4xy+2xz-10yz$.

30. *Second case.* Where the quantities are all *like*, but the signs not all *like*.

Rule.—"*Add all the positive co-efficients into one sum, and all the negative co-efficients into another; subtract the less of these sums from the greater, and to the difference prefix the sign of the greater sum, and then adjoin the common letters. If the aggregate of the positive terms be equal to that of the negative ones, then the difference* $=0$."

Ex. 5. Add together :—

$3a, -5a, 9a, -7a$

$+3$ -5
$+9$ -7
$+12$ -12

$\therefore$ The required sum $+12a-12a=0$ Ans.

Ex. 6. Add together :—

$-2b, +4b, -5b, +6b, -13b.$

-2	
-5	$+4$
-13	$+6$
-20	$+10$

$-20+10=-10.$

The required sum $=-10b$.

Ex. 7. Add together :—

$4x^2+3xy+y^2$, $-3x^2+xy-2y^2$ and $2x^2-5xy+3y^2$.

Proceeding by the rule,

$4x^2+3xy+y^2$	$4+2-3=3$
$-3x^2+xy-2y^2$	$3+1-5=-1$
$2x^2-5xy+3y^2$	$1+3-2=2$
$3x^2-xy+2y^2$	

Ex. 8. Add together :—

$\frac{1}{2}a+\frac{1}{3}b-\frac{1}{6}c$, $\frac{1}{3}a-\frac{1}{6}b+\frac{1}{3}c$, $\frac{1}{6}a-\frac{1}{2}b-\frac{2}{3}c$ and $a-b+c$.

Proceeding by the rule,

$\frac{1}{2}a+\frac{1}{3}b-\frac{1}{6}c$ $\quad\quad$ $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+1=2$

$\frac{1}{3}a-\frac{1}{3}b+\frac{1}{4}c$ $\quad\quad$ $\frac{1}{3}-\frac{1}{3}-\frac{1}{2}-1=-\frac{4}{3}$

$\frac{1}{4}a-\frac{1}{2}b-\frac{1}{3}c$ $\quad\quad$ $-\frac{1}{6}+\frac{1}{4}-\frac{3}{4}+1=\frac{1}{2}$

$a-b+c$

$2a-\frac{1}{3}b+\frac{1}{2}c$

EXERCISE 5.

Add together :—

1. $3a, -5a, -6a, 8a,$
2. $5b, -7b, -9b, +13b.$
3. $-4ab, 7ab, \frac{1}{3}ab-\frac{1}{4}ab.$
4. $\frac{1}{5}a^2-\frac{1}{3}a^2, -\frac{1}{6}a^2, \frac{1}{8}a^2.$
5. $a+b+c+d, -a+b+c+d, a-b+c+d, a+b-c+d$ and $a+b+c-d.$
6. $a+2b+3c-4d, b+2c+3d-4a, c+2d+3a-4b$ and $d+2a+3b-4c.$
7. $a^2-\frac{1}{2}b^2+\frac{1}{3}c^2-\frac{1}{4}d^2, b^2-\frac{1}{2}c^2+\frac{1}{3}d^2-\frac{1}{4}a^2,$ $c^2-\frac{1}{2}d^2+\frac{1}{3}a^2-\frac{1}{4}b^2$ and $d^2-\frac{1}{2}a^2+\frac{1}{3}b^2-\frac{1}{4}c^2.$
8. $a^4+a^3-a^2+1, a^3-2a^2+3a^2-3$ and $4a^2-3a^3-4a+4.$
9. $ax+by+cz, -3ax-2by-3cz$ and $4ax-3by+4cz.$
10. $a^4+4a^3+6a^2+4a+1$ and $a^4-4a^3+6a^2-4a+1.$
11. $a^3-\frac{3}{4}a^2x+\frac{1}{8}ax^2-\frac{5}{16}x^3, a^3+\frac{1}{2}a^2x+\frac{1}{4}ax^2+\frac{2}{3}x^3$ and $2a^3-\frac{1}{4}a^2x+\frac{1}{12}ax^2+\frac{1}{384}x^3.$
12. $a^2+b^2+c^2+2ab+2ac-2bc,$ $a^2+b^2c^2-2ab+2ac-2bc.$ $a^2+b^2+c^2-2ab-2ac+2bc.$

31. *Third case.* Where the quantities are *unlike*.

Rule.—"*Set all the quantities down one after another with their co-efficients and proper signs prefixed, and collect all the like quantities together (if there be any) by the foregoing rules.*"

Ex. 9. Add together :—

$3a, 5b, 6c, 7d.$

The sum $=3a+5b+6c+7d.$

Ex. 10. Add together :—

$3a, 6b, +8c, -7x.$

The sum $=3a+6b+8c-7x.$

Ex. 11. Add together :—

$ab+ac+bc$, $ax-2ab+by$, $bz-2ax+cy$ and $bc-cz+3ac$.

Proceeding by the rule,

$$\begin{array}{llll} ab+ac+bc & & & \\ -2ab & +ax+by & & \\ & -2ax & +bz+cy & \\ & +3ac+bc & & -cz. \\ \hline -ab+4ac+2bc & -ax+by+bz+cy & & -cz \end{array}$$

EXERCISE 6.

Add together :—

1. $3b, -8a^2b, 10ab^2$.
2. $6ac, -7xy, 10xy^2, 9xy^3$.
3. $5x^2+1, x^2-4x+3, 6x^2+8x-9$.
4. $6a^3+7ba^2+8b^2a+2b^3, 3a^2b+5b^3+11a^3, 7ba^2+8a^3-9ab^2, 6a^2b+12ab^2-10ab^3$.
5. $m^3+n^3-3mn+1, n^3+l^3+3nl-1, l^3+m^3-3lm+1$.
6. $a-x-y, b-c-d, x+y+c$ and $d+c-x-y$.
7. ay^2+by+c, by^2+cy+a and cy^2-by-a.
8. $x^2+xy+xz, y^2+xy+yz, z^2+zx+zy$ and $x^2+y^2+z^2-2xy-2yz-2zx$.
9. $3ab^2-4a^2b+a^3, -4ac^2+5ab^2-c^3, -7b^3+2a^2b-6ac^2$ and $5a^3-11ab^2-12ac^2$
10. $a-\frac{1}{3}b+\frac{1}{4}c-\frac{1}{5}d, -\frac{1}{2}c+\frac{1}{3}a-\frac{1}{4}b+d, \frac{1}{4}d-\frac{1}{3}b+c-a, \frac{1}{3}a-3d+b-\frac{1}{5}c$ and $8a-6b+3c-4d$.
11. $a-\frac{b}{2}+\frac{c}{2}-\frac{d}{4}, -\frac{b}{3}-\frac{a}{4}+d, -\frac{c}{4}$ and $\frac{a}{5}-\frac{b}{4}+\frac{c}{3}-\frac{d}{2}$.
12. $1+x+x^2+x^3+x^4, 1-2x+3x^2-4x^4, x+3x^2+5x^3-7x^4$ and $1-x^4$.
13. $2x^2-6xy-\frac{3}{11}y^2, 4y^2-\frac{4}{3}y^2+2z^2, 2xy-\frac{2}{3}y^2+2y^3$ and $4xy-\frac{2}{3}y^2-3x^2-3y^2-3z^2$.
14. $2a^3+4a^2+6ab, a^2+b^2+ab^2+b^3, ab^2-4ab-2a^2, 3a^2b-a^3, ab^2+a$, and $b-2a^2$.
15. $11x^4+4x^3y-20xy^3+3y^4, 3x^3y-10x^2y^2+2y^4-4x^4, 7x^2y^2+6xy^3-6x^4-3x^3y$ and $18xy^3+9x^2y^2-4y^4$.
16. a pies, b pies, $-c$ pies.
17. x rupees, $3y$ rupees, and $5z$ rupees.
18. $2p$ pounds, $3q$ pounds, $4r$ pounds.

CHAPTER III.

SUBTRACTION AND REMOVAL OF BRACKETS.

32. Any quantity B is said to be subtracted from any other quantity A when a third quantity C is found such that the *sum* of B and C is equal to A.

i.e., $C = A - B$ when C is such that $B + C = A$.

The quantity *from* which another quantity is subtracted is called the **minuend**.

The *quantity subtracted* is called the **subtrahend.**

The result is called the **difference** or the **remainder**.

A is the minuend, B the subtrahend and C the remainder.

Graphical representation of subtraction and negative numbers. Subtraction as in Arithmetic is the opposite process of addition. Subtraction of 3 from 4 gives $4 - 3$. Similarly subtraction of 8 from 4 may be taken as $4 - 8$ but $4 - 3$ has a meaning whereas $4 - 8$ is not easily intelligible. If $+$ is regarded as gain $-$ may be regarded as loss; so $4 - 3$ means an initial gain of 4 followed by a loss of 3 resulting in a net gain of 1; but $4 - 8$ may similarly mean an initial gain followed by a loss of 8 resulting in a net loss of 4 which is denoted by -4.

$\therefore 4 - 8 = -4$ or graphically thus.

Suppose a person starts from O and takes 4 steps, to the right, to A. This is denoted by $+4$. -3 means from A he takes 3 steps in the opposite direction, *i.e.*, to the left to B. (See the figure.) Now he is finally at B which is 1 step to the right of O. $\therefore +4 - 3 = +1$. Similarly $+4 - 8$, means taking 4 steps from O to A to the right and then taking 8 steps from A to the left which brings him to C which is 4 steps to the left of O. $\therefore +4 - 8 = -4$. Thus if $+$ denotes advance in one direction $-$ denotes that in the opposite direction.

33. From $2a$ take b, and the difference is denoted by $2a-b$; because the sign $-$ prefixed to b shows that it is to be subtracted from the other and $2a-b$ is the sum of $2a$ and $-b$.

Again, from $2a$ take $-b$ and the difference is $2a+b$; because $2a = 2a+b-b$.

Take away $-b$ from these equal quantities.

$\therefore$ The difference between $2a$ and $-b$ is $2a+b$, *i.e.*, the sum of $2a$ and $+b$.

$\therefore$ *To take—b is the same as to add +b.*

Hence we have the following rule :—

Rule.—"*Change the signs of all the quantities to be subtracted or conceive them to be changed, and then collect the different terms together as in addition.*"

Ex. 1. Subtract $3x-5b$ from $4x-8b$. Changing the sign of $3x-5b$, we have, $-3x+5b$. Add $-3x+5b$ to $4x-8b$ thus:

$$\begin{array}{r} 4x-8b \\ -3x+5b \\ \hline x-3b. \end{array}$$

Ex. 2. Subtract $x^2-2xy+y^2$ from $x^2+2xy+y^2$.

Changing the sign of every term of the subtrahend we have $-x^2+2xy-y^2$.

$$\begin{array}{r} x^2+2xy+y^2. \\ -x^2+2xy-y^2 \\ \hline 4xy \end{array}$$

Note.—The signs need not be actually changed, the operation may be performed mentally.

Ex. 3. Subtract $2a^2+3ab-5b^2$ from $-3a^2+2ab-4b^2$.

$$\begin{array}{r} -3a^2+2ab-4b^2 \\ 2a^2+3ab-5b^2 \\ \hline -5a^2 \quad -ab+\ b^2 \end{array}$$

Ex. 4. Subtract $a^2-4ab+5x^2-y^2$ from $6x^2+3y^2-6a^2$.

$$\begin{array}{r} 6x^2+3y^2-6a^2 \\ 5x^2-\ y^2+a^2-4ab \\ \hline x^2+4y^2-7a^2+4ab \end{array}$$

EXERCISE 7.

Subtract :—

1. -3 from 5, -3 from -6, -8 from 9.
2. $2d$ from $3c$, $-4x$ from $6y$, $-8z$ from $-9x$.
3. $a+b$ from $c+d$, $3x-2y$ from $6x-8y$, $10z+6c$ from $3x-4z$.
4. $a-b+c$ from $3a-2b+c$.
5. $-x-y+z$ from $3x+4y-5z$.
6. $-4a^2-2ab+6$ from $-5a^2-ab+8$.
7. $2x^3+3x^2+4x+5$ from x^3-4x^2+6x+7.
8. $a^3+3a^2b+3ab^2+b^3$ from $a^3-3a^2b+3ab^2-b^3$.
9. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ from $a^4+4a^3b+6a^2b^2+4ab^3+b^4$ and find the value of the difference, when $a=2$, and $b=3$.
10. From $\frac{1}{2}x-\frac{1}{3}a+\frac{1}{4}b-\frac{1}{5}y$ take $\frac{1}{3}x+\frac{1}{3}a-\frac{1}{4}b-\frac{1}{2}y$.
11. From $\frac{2}{3}a+3b+\frac{1}{4}c+\frac{1}{2}bc$ take $\frac{1}{2}a-7b-3bc+\frac{1}{2}c$.
12. From $\frac{1}{2}y-\frac{4}{3}a-\frac{3}{4}x+\frac{1}{3}a$ take $3y+\frac{1}{4}a-\frac{2}{3}x$.
13. From $10am-14bn-cp-5d$ take $-9am+2bn+cp-5d$.
14. Subtract $\frac{1}{3}a-b-\frac{1}{4}$ from the sum of $b+\frac{1}{2}-\frac{1}{3}a$, $1+\frac{1}{2}a-\frac{1}{3}b$ and $a+\frac{1}{2}b-\frac{1}{3}$.
15. Subtract $1+a-a^2$, from the remainder after subtracting $3a^2-5a-\frac{1}{4}$ from $a^3+a^2-2a-\frac{1}{3}$.
16. Subtract the sum of $3a-b+2c-5d$ and $a+b-2c+3d$ from the excess of $6a-c-d$ over $a-b-c$.
17. From the sum of the first four of the following expressions $x^2+y^2+z^2+a^2$, $a^2+y^2+z^2$, $y^2-z^2+y^2-a^2$, $x^2-y^2+z^2+a^2$, $y^2+z^2+a^2-y^2$, subtract the sum of the last four.
18. What must be added to $3a^2-5ab+6b^2+7bc$ in order that the sum may be $-a^2-b^2-bc$?
19. What must be added to $-5a^3+13a^2b^2-x^2ya+5ab^2+7abxy$ in order that the sum may be $a^3+a^2b^2+x^2ya-2yab^2-2xyab$?
20. What must be subtracted from $4a^2-3ab-6b^2$ in order that the remainder may be a^2+3b^2?
21. What must be subtracted from $10x^2-7xy+5y^2-9x-11y+14$ in order that the remainder may be $x^2+xy+y^2+x+y+1$?

22. From what expression must $3x^2-7xy-8yz+9y^2$ be subtracted in order that the remainder may be $2x^2+3xy+3yz+2y^2$?

23. The sum of two expressions is $2x^2+2y^2-4zx$, and one of them is $x^2+y^2-z^2+2xy+2yz-2zx$. Find the other.

24. By how much does x^2+xy+y^2 exceed x^2-xy+y^2 ?

25. A ship sails 30 miles due north and then 40 miles due south. How far north is she from her starting point? How far south?

26. In a certain year the revenue of a country is increased by 10 lakhs. The next year it is decreased by 15 lakhs. By how much has the revenue decreased? Express it as an increase

34. When any expression is bracketed the whole expression is treated as a single term; and the operations of addition, subtraction, &c., are performed upon it as if it were only a symbol like a, b and c. For instance, $a+(b-c)$ means, that performing possible simplifications upon $b-c$ we are to add the *result* to a.

Again $a+(b-c)$ itself may be treated as a single term, and added to, or subtracted from, any other term. In that case the whole expression $a+(b-c)$ should be bracketed.

Thus $d-\{a+(b-c)\}$ means that, simplifying $a+(b-c)$ we are to take the result from d. This again may be treated as a single term and put within brackets thus $[d-\{a+(b-c)\}]$.

A straight line placed over an expression also serves the purpose of brackets. Thus $a+b-c$ has the same meaning as $a+(b-c)$. This straight line is called a **vinculum.**

35. To prove that (1) $a+(b+c)=a+b+c$.

(2) $a+(b-c)=a+b-c$.

$a+(b+c)$ means that the sum of b and c is to be added to a. Therefore $a+(b+c)=a+b+c$.

Again $a+(b-c)$ means that we are to add $b-c$ to a. Now $b-c$ means that b is affected by a loss c; therefore $b-c$ is less than b. If therefore to a we add b instead of $b-c$, we add too much by c. Hence $a+b$ is too large by c; the correct result must be less than this by c. Therefore $a+(b-c)=a+b-c$.

Examining the results in the two cases, it appears that we simply write down the terms in a line with their proper signs, taking no notice of the brackets. Hence the following rule :—

Rule —*Plus sign before a pair of brackets does not affect the signs of terms within it.*

Graphical proof of :—$+(b+c)=+b+c$.
$+(b-c)=+b-c$.

Let O be the starting point. All distances measured to the right are positive, i.e., +and those measured to the left are negative, *ie.*—

Let $OA=b$ units, $AB=c$ units, $AC=AB$ in length but measured in an opposite direction to $Ac=-c$ units.

$\therefore OB=b+c$ units, $CC=b-c$ units.

I. Now walking from O to B is equivalent to walking from O to A and then from A to B (all in the same direction) $\therefore +(b+c)=+b+c$.

II. Now walking from O to C is equivalent to walking from O to A and then walking from A to C in the opposite direction.

$\therefore +(b-c)=+b-c$.

∴ +sign before a bracket does not affect the sign of the terms when it is removed.

36. To prove that (1) $a-(b+c)=a-b-c$.
(2) $a-(b-c)=a-b+c$.
(3) $a-(b-c-d+e)=a-b+c+d-e$.

(1) As each of the numbers b and c is to be taken from a the result is denoted by $a-b-c$. We enclose the term $b+c$ in brackets, because, *both* the numbers b and c are to be taken from a. Therefore $a-(b+c)=a-b-c$

(2) If we take b from a, we obtain $a-b$; but we have thus taken too much from a, for we are required to take, not b but, b diminished by c. Hence we must increase the result by c ; thus $a-(b-c)=a-b+c$.

(3) We have to take $b-c-d+e$ from a. This is the same thing as taking $b+e-c-d$ from a. Take away $b+e$ from a, and the result is $a-b-e$, then *add* $c+d$, because we were to take away not $b+e$ but $b+e$ diminished by $c+d$; thus $a-(b-c-d+e)=a-b-e+c+d=a-b+c+d-e$.

Examining the results in these cases, we obtain the following rule :—

Rule.—*Minus sign before a pair of brackets changes the sign of every term within it, plus to minus and minus to plus.*

Graphical proof of :—

$$-(b+c)=-b-c.$$
$$-(b-c)=-b+c.$$

Now in this figure O A' = O A but is measured in the opposite direction.

Similarly A'B' = A B but is measured in the opposite direction

Do. A'C' = A C do.

$$\left.\begin{array}{l} \therefore \text{O A}' = -b \\ \text{A}'\text{B}' = -c \\ \text{A}'\text{C}' = +c \end{array}\right\} \quad \begin{array}{l} \therefore \text{O B}' = -(b+c). \\ \therefore \text{O C}' = -(b-c). \end{array}$$

III. A man walking O to B' is equivalent to going first to A' and then to B' $\therefore -(b+c)=-b-c$ all in the same direction.

IV. Again walking O to C' is equivalent to walking from O to A' and thence to C' in the opposite direction $\therefore -(b-c)=-b+c$.

$\therefore$ − sign before brackets changes the sign of all the terms within them if they are removed.

EXERCISE 8.

Simplify :—

1. $2x-(3x-x)$.
2. $-2y-(3y-5y)$.
3. $3b-(4b-7b)$.
4. $-4c+(4c-7c)$.
5. $5x-(3x+x)$.
6. $10a-(6a+5a)$.
7. $2a+3b+(3b-2a)$.
8. $x+y-(y-x)$.
9. $\frac{1}{2}b-(\frac{1}{3}x-\frac{1}{4}x)$.
10. $\frac{1}{2}a+\frac{1}{3}b-(\frac{1}{3}a-\frac{1}{4}b)$.

37. The following **rules** must be observed in the *removal of brackets* :—

(1) If any number of terms be enclosed within a pair of brackets preceded by the sign +, the brackets may be removed *without* altering the signs of the terms.

Thus $a+(b+c-d-e)=a+b+c-d-e$.

(2) If any number of terms be enclosed within a pair of brackets preceded by the sign—, the brackets can be removed only if the signs of all the terms within be changed, + to − and − to +. Thus $a-(b+c-d-e)=a-b-c+d+e$.

The following rules must be observed in the *insertion of brackets* :—

(1) Any number of terms may be put within a pair of brackets with the sign + prefixed, without any other changes.

Thus $a+b-2c=a+(b-2c)$.

(2) If the sign − be prefixed, the signs of all the terms within the pair of brackets must be reversed.

Thus $a-b+2c=a-(b-2c)$.

Note.—If there be brackets within brackets in an expression, it is the safest method to remove the innermost brackets first, then the next, and so on.

The following brackets are generally used :—

( ) These are called **simple brackets**
{ } .....**double**.....
[ ] .....**square**.....
——— This is called a **vinculum**.

Ex. 1. Simplify $\{a-(b-c)\}-\{b-(c-a)\}$.

The given expression $=\{a-b+c\}-\{b-c+a\}$
$=a-b+c-b+c-a$
$=2c-2b$.

Ex. 2. Simplify $a-[b-\{c-(d-e)\}-f]$.

The given expression $=a-[b-\{c-d+e\}-f]$
$=a-[b-c+d-e-f]$
$=a-b+c-d+e+f$.

Ex. 3. Simplify $7a-\{3a-[4a-(5a-2a+a)]\}$

The given expression $=7a-\{3a-[4a-(5a-2a-a)]\}$

$=7a-\{3a-[4a-5a+2a+a]\}$

$=7a-\{3a-4a+5a-2a-a\}$

$=7a-3a+4a-5a+2a+a$

$=14a-8a$

$=6a.$

Ex. 4. Remove the brackets from

$x-2y-[4x-6y-\{3x-z+(5x-2y-3x-z+2y)\}].$

The given expression

$=x-2y-[4x-6y-\{3x-z+(5x-2y-3x+z-2y)\}]$

$=x-2y-[4x-6y-\{3x-z+5x-2y-3x+z-2y\}]$

$=x-2y-[4x-6y-3x+z-5x+2y+3x-z+2y]$

$=x-2y-4x+6y+3x-z+5x-2y-3x+z-2y$

$=x-4x+3x+5x-3x-2y+6y-2y-2y-z+z$

$=9x-7x+6y-6y+z-z$

$=2x$

Ex. 5. Enclose $2a+b-c+2d-3e-4f$ in brackets.

(1) Taking the terms two together each pair of brackets being preceded by the sign :—

$2a+b-c+2d-3e-4f=-c+2a-3e+b-4f+2d$

$=-(c-2a)-(3e-b)-(4f-2d).$

(2) Taking the terms three together each pair of brackets being preceded by the sign :—

$2a+b-c+2d-3e+4f=-c+2a-3e-4f+b+2d$

$=-(c-2a+3e)-(4f-b-2d).$

Introducing a pair of inner brackets in each, the expression

$=-\{c-\overline{2a-3e}\}-\{4f-b+2d\}.$

EXERCISE 9.

Remove the brackets from :—

1. $2a-\{2a-(b+2x)\}+\{b-(2x-2b)\}.$
2. $\{6a+2b-(3a+2b)\}-\{2a+4b-(4a-b)\}.$
3. $a-[2b-\{3c-(a-\overline{2b-3c})\}].$
4. $a^2-(b^2-c^2)-b^2+(c^2-a^2)+c^2-(b^2-a^2).$

5. $-a-[-b-\{-c-(-a-b-c)\}]$.

6. $11v-[7x-\{8x-(9v+2x-6x)\}]$.

7. $2-\{4-(6-\overline{7-9})\}$.

8. $-2-[-3-\{-4-(-5-\overline{6+2})\}]$.

9. $a-[5b-\{a-(3c-3b)+(2c-\overline{a-2b-c})\}]$.

10. $7a-[2a-\{b-(3a-5\overline{a-2b})-4a\}-2b]$.

11. $-[+\,+(-x)\,]-)+[-(-x)]$.

12. $-3x-[-5y-\{-7z-(-9x-\overline{-11y-13z})\}]$.

13. $-2x-[-4y-\{-5z-(-2x+\overline{-5y-8z})\}]$.

14. $-x-[-2y+\{3z-(4x-\overline{-2y-9z})\}]$.

15. $-a-[-b-\{c-(a-\overline{-b-c})\}]$.

16. $-4a-[-3b-\{-4c-(3a+\overline{-4b-5c})\}]$.

Add together :—

17. $a-(d-2c)$, $3a-(b-4c)+2d$ and $3a-b-(2c-d)$.

18. $1-\{1-(1-x)\}$, $2x-(3-5x)$ and $2-(-4+5x)$.

19. $a-(2d-3c)$, $4a-(b+4c)+3d$ and $3a-\{(2b+2c)-d\}$.

20. $(ax-bx)-(by-cy)$, $ax+bx-(by+cy)$ and $(ax-by)-(bx-cy)$.

21. $\{6a-(b-\frac{1}{2}c)\}$, $\{\frac{1}{3}b-(\frac{1}{3}c-\frac{1}{12}a\}$ and $\{(\frac{2}{3}a-b)-(2c-\frac{1}{3}a)\}$.

Find the differences between

22. $5x-[4y-\{3z+2x-(y-z)\}]$ and $6x-\{6y-3z-3x\}$.

23. $x-\{-(1-3x)+(2x^2-3x^3)\}$ and $[1-\{x-(x^2-3x^3)\}]$.

24. $6a+[4a-\{8b-(2a+4b-22b-7b]$ and $7b-\{4a-(14a-5b)\}$.

Find the values of

25. $130-[13x-\{4x-11-(x-8)\}+\{5x-13-(x-5)\}]$ when $x=\frac{1}{6}$.

26. $[8a-(2a-4)+\{13-a)-2a-9\}]-11a$ when $a=7$.

27. $[\{2a-(b-a)\}-\{(2a-b)-(3a-2b)\}]+(a+21b)$ when $a=\frac{1}{10}$ and $b=\frac{1}{21}$.

28. $2x-[a-(3b-5c)+\{x+(4a-\overline{b+x-c})+3c\}-(2b-\overline{3a+3c})]$ when $b=\frac{3}{4}$, and $c=1$.

29. $3x-[16y-(3z-4y)-15z+\{2x-6y-5z-4x\}]$
when $x=2$, $y=\frac{3}{10}$ and $z=\frac{1}{15}$.

30. $x-[3x-5y-\overline{7x-(9y+11a)-(13x-15y-17a+19b)}]$
when $x=1$, $y=2$, $a=6$ and $b=\frac{2}{17}$.

31. $[2a-(b-c)-\{3b-(2a-c)\}-\{-2a+(c-4b)\}$
$-[-3b-(2a-4c)+\{6c-(2b-3a)\}-\{-5c+(6a-7b)\}]$
when $a=6$, $b=\frac{1}{4}$ and $c=-\frac{1}{3}$.

32. In the expression $a-2b+3c-4d+5e-6f+7g-8h$.

Enclose the 1st, 3rd, 7th and 8th terms within brackets preceded by the sign − and the other terms within brackets preceded by the sign +.

33. Include the 1st three in brackets preceded by the sign −, the next two in brackets preceded by the sign +, and the last three in brackets preceded by the sign −.

34. In the last question, introduce inner brackets in the first and the third preceded by the sign −.

35. Enclose in pairs the terms of $-a-b+c-d-e+f$ in brackets, so that each pair of brackets shall be preceded by the sign −.

CHAPTER IV.

MULTIPLICATION.

38 The quantity multiplied is called the **multiplicand**. The quantity by which it is multiplied is called the **multiplier**. The result is called the **product.**

39. In Art. 6 it has been stated that when there is no sign between two quantities, the sign of multiplication is to be understood. Thus $ab = a \times b$; $4xy = 4 \times x \times y$;

$a(b+c) = a \times (b+c)$; and $(a+b)(b+c) = (a+b) \times (b+c)$.

40. *The order of multiplication is indifferent.* This means that the product is the same in whatever order the factors are written.

Thus $4ab = 4 \times a \times b = b \times a \times 4 = a \times b \times 4$

$abc = a \times b \times c = b \times a \times c = c \times a \times b =$ &c.

41. *To multiply a numerical quantity and an algebraical quantity.* We will first consider cases like

$3a \times 4 = 3a + 3a + 3a + 3a = 12a.$

To multiply $3a$ by 4, we must multiply the co-efficient 3 by 4 and write $12a$ for the product ; for $3a = 3 \times a$; and $3a \times 4 = 3 \times 4 \times a = 12a$.

Ex. 1. Multiply $\frac{2}{3}a$ by $\frac{1}{2}$.

$\frac{2}{3}a = \frac{2}{3} \times a$ $\therefore$ $\frac{2}{3}a \times \frac{1}{2} = \frac{2}{3} \times a \times \frac{1}{2} = \frac{2}{3} \times \frac{1}{2} \times a = \frac{1}{3}a$.

(The order of multiplication is indifferent.

Ex. 2. Multiply $4ac$ by $3b$.

$4ac = 4 \times a \times c$ and $3b = 3 \times b$

$\therefore 4ac \times 3b = 4 \times a \times c \times 3 \times b = 4 \times 3 \times a \times c \times b = 12acb$.

Hence the following rule for multiplying two *positive simple* algebraical quantities.

Rule.—"*Multiply together the numerical co-efficients and write the letters after the product of the numbers.*"

EXERCISE 10.

Multiply—

1. $6ab$ by $5c$.
2. $4b^2$ by $3x^2$.
3. $\frac{3}{4}bc$ by $\frac{1}{3}a$.
4. $2p^2q$ by $3r$.
5. $\frac{1}{3}a^2b^2$ by $\frac{1}{5}c^2x^2$.
6. abc by $\frac{1}{2}xy$.
7. a^2xy by $\frac{3}{4}b^2z$.
8. $10b^2y$ by $\frac{5}{8}c^2z$.
9. $\frac{5}{3}px$ by $\frac{1}{3}qy$.
10. $\frac{10}{9}dc^2$ by $\frac{9}{7}a^2b$.

42. *To multiply two powers of the same quantity.*

Ex. 1. Multiply a^2 by a^4.

$a^2 = a \times a$ and $a^4 = a \times a \times a \times a$.

$\therefore a^2 \times a^4 = a \times a \times a \times a \times a \times a = a^6$, i.e., a^{2+4}.

Ex. 2. Multiply $5b^2$ by $3b^3$.

$5b^2 = 5 \times b \times b$ and $3b^3 = 3 \times b \times b \times b$

$\therefore 5b^2 \times 3b^3 = 5 \times b \times b \times 3 \times b \times b \times b$

$= 5 \times 3 \times b \times b \times b \times b \times b \equiv 5 \times 3b^5$.

$= 15b^5$ *i.e.*, $5 \times 3b^{2+3}$.

To prove that $a^m \times a^n = a^{m+n}$ when m and n are positive integers.

Since $a^m = a \times a \times a$ . . . . . to m factors.

And $a^n = a \times a \times a \times a$ . . . . . to n factors.

$\therefore a^m \times a^n = a \times a \times a \times a$ . . . . . to $m+n$ factors.

$= a^{m+n}$.

Similarly $a^m \times a^n \times a^p = a^{m+n} \times a^p$

$= a^{m+n+p}$.

To prove that $(a^m)^n = a^{mn}$ when m and n are positive integers.

$(a^m)^n = a^m \times a^m \times a^m$. . . . to n factors.

$= a^{m+m+m}$ . . . . to n terms

$= a^{mn}$.

Hence the following rule :—

Rule.—"*Multiply the co-efficients and add the indices of the powers of the proposed quantity and write the quantity with an index equal to the sum after the product of the co-efficients.*"

This rule may be extended to cases where *more than two powers* of the same quantity are to be multiplied together.

Ex. 1. Multiply together y^2, y^3, y.

Now $y^2 \times y^3 = y^{2+3} = y^5$

and $y^5 \times y^1 = y^{5+1} = y^6$.

Ex. 2. Multiply together $2x^2$, $3x^1$, $4x^5$.

Now $2x^2 \times 3x^1 = 2 \times 3 \times x^{2+1} = 6x^6$.

and $6x^6 \times 4x^5 = 6 \times 4x^{6+5} = 24x^{11}$.

The rule holds good also when m and n are *negative*, *integral* *fractional*.

Thus: $a^{-m} \times a^{-n} = a^{-m-n}$; $a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}$

$a^{\frac{p}{q}} \times a^{-m} = a^{\frac{p}{q}-m}$; $a^{-m} \times a^n = a^{n-m}$.

EXERCISE 11.

Multiply—

1. $2a^2$ by $3a^7$.
2. $4b^2$ by $5b^3$.
3. $\frac{1}{2}x^2$ by $\frac{1}{3}x$.
4. $3p^2$ by $7p^6$.
5. $10y^1$ by $\frac{5}{2}y^9$.
6. $\frac{3}{2}d^5$ by $\frac{4}{3}d$.
7. $27z^{10}$ by $3z^{11}$.
8. $\frac{10}{3}q^8$ by $21q$.
9. $15r^{12}$ by $\frac{8}{3}r^6$.
10. $\frac{12}{5}e^{21}$ by $10e^5$.

Multiply together :—

11. $2a^2$, $3a^1$, $\frac{1}{2}a^5$.
12. $4p^2$, $10p^7$, $\frac{8}{21}p^9$.
13. $5x^3$, $9x$, $10x^{11}$.
14. y^4, $2y^1$, $3y^5$.
15. $\frac{1}{3}b^3$, $2b^{11}$, $3b^{12}$.

43. The rule of the previous Article is applicable to cases where the expressions to be multiplied together contain *powers of different letters*.

Ex. 1 Multiply $2a^2b^2$ by $3ab^3$.

Now $2a^2b^2 = 2 \times a^2 \times b^2$ and $3ab^3 = 3 \times a \times b^3$

$\therefore 2a^2b^2 \times 3a \times b^3 = 2 \times a^2 \times b^2 \times 3 \times a \times b^3$

$= 2 \times 3 \times a^2 \times a \times b^2 \times b^3$

$= 6 \times a^{2+1} \times b^{2+3}$

$= 6a^3b^5$.

Ex. 2. Multiply $5x^3y^2$ by $6x^4y^3z^2$.

Now $5x^3y^2 = 5 \times x^3 \times y^2$

And $6x^4y^3z^2 = 6 \times x^4 \times y^3 \times z^2$

$\therefore 5x^3y^2 \times 6x^4y^3z^2 = 5 \times x^3 \times y^2 \times 6 \times x^4 \times y^3 \times z^2$

$= 5 \times 6 \times x^3 \times x^4 \times y^2 \times y^3 \times z^2$

$= 30 \times x^{3+4} \times y^{2+3} \times z^2$

$= 30x^7y^5z^2$.

Note.—In the process of multiplication, the indices of one letter should not be combined with those of another. Thus the product of a^3b^2 and x^2y^3 is simply $a^3b^2x^2y^3$.

EXERCISE 12.

Multiply :—

1. $3a^2b$ by $4a^3b^2$.
2. $10x^4y^5$ by $4x^7y^2$.
3. $\frac{1}{2}x^2y$ by $\frac{1}{3}z^2y^3$.
4. $10a^3x$ by $15a^4x^2$.
5. $\frac{10}{3}a^2b^3x^2$ by $\frac{3}{4}b^5x^4y^2$.
6. $4p^3q^7$ by $13p^7qr^2$.
7. $\frac{20}{3}d^2c^3$ by $30d^3c^4a^4$.
8. $77x^2y^2i^2$ by $5xy$.
9. $a^2b^2c^3$ by $6a^3b^3x^7$.
10. p^3x^3 by $14a^6x^2p^4$.

Multiply together:—

11. a^2x, $2a^3y^2$, $3ay^4x^2$.
12. a^2b, b^2x, x^2ab.
13. $2p^2q$, $3q^2p$, $4p^2r^2$.
14. $10x^2z^2$, $2x^2y^2$, $3y^2xz$
15. $15d^2c^2$, $12d^2a$, $3a^2c$.

44. *To multiply a compound expression by a monomial.*

Let us multiply $b+c$ by a.

We know $3a$ means *three times* a, that is, $a+a+a$.

Similarly $8a = a+a+a+a+$.....to eight terms

and $ba = a+a+a+a+$.....to b terms,

that is, a is to be taken b times.

Now $(b+c)a = a+a+a+$.....taken $b+c$ times.

$= a+a+a+$.....taken b times

together with or $+(a+a+a+$.....taken c times)

$= ba+ca$.

Conversely, $ba+ca = (b+c)a$.

Again, let us multiply $b-c$ by a.

$(b-c)a = a+a+a+$.....taken $b-c$ times

$= a+a+a+$.....taken b times

diminished by or $-(a+a+a+\ldots\ldots$taken c times)

$$=ba-ca.$$

Similarly $(b-c+d)a=ba-ca+da$

and $(b-c+d-e)a=ba-ca+da-ea.$

Hence we get the following rule for multiplying a compound expression by a monomial.

Rule.—"*Multiply each term of the expression by the monomial and take the algebraic sum of all these partial products.*"

Ex. 1. Multiply $3a+4b-5c$ by $2x$.

$(3a+4b-5c)\ 2x=6ax+8bx-10cx.$

Ex. 2. Multiply $2x^3-4y-5a^2$ by $2ax^2$.

$(2x^3-4y-5a^2)\times 2ax^2=4ax^5-8ax^2y-10a^3x^2.$

Ex. 3. Multiply $\frac{1}{2}x-\frac{1}{3}y-\frac{1}{4}z$ by $\frac{3}{4}xy^2$.

$(\frac{1}{2}x-\frac{1}{3}y-\frac{1}{4}z)\times\frac{3}{4}xy^2=\frac{3}{8}x^2y^2-\frac{1}{4}xy^3-\frac{3}{16}xy^2z.$

EXERCISE 13.

Multiply :—

1. $a+2b-3c$ by $4ac$.
2. $4x-5y-a^2$ by $3x^2a$
3. $ab+ac-bc$ by ab^2.
4. $ab+ac$ by a^4b.
5. $a^2+b^2-c^2$ by $2a^4b^2$.
6. $7x^2-2x+4a$ by $3a^2$.
7. $4x^2y-2xy+4x-3y$ by $2xy$
8. $\frac{1}{2}x^2-\frac{1}{3}x+\frac{1}{4}$ by $\frac{2}{3}x^2$.
9. $7x^4-3x^3y-4x^2y^2$ by $\frac{1}{2}x^2y^2$.
10. $4a^3b-5a^2b^2+6ab^3-2b^4$ by $\frac{1}{3}ab$.
11. $\frac{1}{2}p^2q^2-\frac{1}{3}p^3q$ by $\frac{1}{4}p^2q$.
12. $10x^3y-\frac{4}{3}x^2y^2+\frac{2}{3}xy^3$ by $2xy^2$.
13. $\frac{1}{2}x^2-\frac{2}{5}y^2$ by $\frac{3}{4}x^2y$.
14. $3a^2+4b^2-5c^2+6d^2$ by $2a^2bc$.
15. $2a^2b^2-3b^2c^3-\frac{1}{2}a^2c^2$ by $4abc$.
16. $a^2x^2-\frac{1}{2}a^4y^2-\frac{1}{3}az^2$ by $\frac{1}{2}a^2y^2z$.
17. $2m^2n-\frac{1}{2}n^2p+3m^2p^2$ by $4mp$.
18. $abc+2fgh-af^2-bg^2-ch^2$ by afg.
19. $a^2x+b^2y-c^2z-a^2b^2xy$ by $\frac{1}{2}ayz$.
20. $x^4-4x^3+6x^2-4x+1$ by $2x^2y$.

Simplify :—

21. $a^2+(b+a)a+b^2$. 22. $a^2+(b-a)b-ab$

23. $a(b-c)+b(c-a)+c(a-b)$.

24. $a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)$.

25. $a(a^2-bc)+b(b^2-ac)+c(c^2-ab)$.

45. It was shown in Art. 44 that $(b-c)\,a=ba-ca$.

Putting $l-m$ for a, we get

$$(b-c)(l-m)=(l-m)b-(l-m)c$$
$$=lb-mb-(lc-mc)$$
$$=lb-mb-lc+mc\text{...(removing the brackets)}$$

$+lb$ in the product arises by multiplying $+l$ and $+b$.

$-mb$ ..... $-m$ and $+b$.

$-lc$ ..... $+l$ and $-c$.

$+mc$ ..... $-m$ and $-c$.

That is, $+l\times+b=+lb$

$$-m\times-c=+mc$$
$$-m\times+b=-mb$$
$$+l\times-c=-lc.$$

From the above results we deduce what is known as the *rule of signs.*

Rule of Signs.—"*Two like signs multiplied together give plus, and two unlike signs give minus in the product.*"

More briefly, "*Like signs produce + and unlike signs —.*"*

46. The following is taken from Kelland's Algebra with some modifications.

$+b-b$ is always zero.

$\therefore +a(+b-b)$ is always zero.

Or $(+a)(+b)+(+a)(-b)=0$

But $(+a)(+b)=+(ab)$

$\therefore (+a)(-b)=-(ab)$ (1).

Again, $+a-a$ is always zero.

$\therefore -b(+a-a)$ is always zero.

---

* The fact that —into—gives + may be explained thus: the sign — is the symbol of reversal: if a thing is *twice reversed*, it is necessarily restored to its original condition.

Or $(-b)(+a)+(-b)(-a)$ is always zero.

But $(-b)(+a)$ has been shown to be $=-(ab)$.

$\therefore (-b)(-a)$ must be such as to cancel $-(ab)$

$\therefore (-b)-a)=+(ab)$.....(2).

From (1) and (2), we have the **rule of signs** in multiplication already enunciated.

Note i.—Since $(+x)\times(+x)=x^2$ and $(-x)\times(-x)=x^2$, we have $\sqrt{x^2}=+x$ or $-x$. Every algebraical quantity has got two square roots equal in value but opposite in sign.

$\sqrt{4}=+2$ or -2; $\sqrt{9}=+3$ or -3, and so on.

Note ii.—The product of any number of factors is *positive* or *negative* according as an *even* or *odd* number of the factors is negative.

An *even* power of a negative quantity is *positive* and an *odd* power of the same is *negative*. Thus: $-a\times+b\times-c=+abc$; $-a\times-b\times-c=-abc$; $(-a)^3=(-a)\times(-a)\times(-a)=-a^3$; and $(-a)^4=(-a)\times(-a)\times(-a)\times(-a)=+a^4$.

Ex. 1. Multiply $5a^2-6b^2-7c^2$ by $-2ab$.

$(5a^2-6b^2-7c^2)\times(-2ab)$

$=5a^2\times(-2ab)+(-6b^2)(-2ab)+(-7c^2)\times(-2ab)$

$=-10a^2b+12ab^3+14abc^2$..... (by the rule of signs).

Ex. 2. Multiply $-5x^2y^3$ by $+4x^5y^4$.

$(-5x^2y^3)\times(+4x^5y^4)=-20x^{2+5}y^{3+4}=-20x^7y^6$.

EXERCISE 14.

Multiply :—

1. $4ab$ by $-3bc$
2. $-\frac{1}{2}abc$ by $-\frac{1}{3}a^2b$.
3. $4a^2b^3$ by $-10a^4bx^2$.
4. $-xyz$ by $-3xy^2za$.
5. pqr^3 by $-\frac{1}{2}p^3qr^4$.
6. $ab+ac+bc$ by $-ab$.
7. $xy-yz$ by $-2xyz$.
8. $a^2b-a^2c-b^2c$ by $-abc$
9. $a^2bc-b^2ac-c^2ab$ by $-a^2b^2$.
10. $-4ab^2+5b^2a-6b^2c$ by $2a^2c^2$.
11. $\frac{1}{2}ab-\frac{1}{3}bc-\frac{1}{4}ac$ by $-\frac{1}{2}ac$.
12. $\frac{1}{4}xy-\frac{2}{3}yz-\frac{1}{3}zx$ by $\frac{1}{4}x^2y$.
13. $-8a^2bc-4b^2xy-6c^2xz$ by $-2axy$.
14. $a^2x-b^2y-\frac{1}{2}c^2z-abc$ by $-\frac{1}{2}axyc$.
15. $\frac{1}{3}a^2b^2-\frac{1}{4}b^2c^2-\frac{1}{3}c^2a^2$ by $-12a^2b^2c^2$.

16. $x^2a - y^2b^2 - z^2c^2 - xyz$ by $-xyab$.

17. $\frac{7}{3}p^2q^2r - \frac{5}{4}r^2p^2q - \frac{7}{2}q^2r^2p$ by $-pqr$.

18. $2a^4 - 3a^6 - 7a^5 - 4a^2$ by $-5a^2$.

19. $\frac{1}{2}x^4 - \frac{1}{3}x^5 - \frac{1}{4}x^2 - \frac{5}{3}x$ by $4x^4$.

20. $a^{\frac{1}{2}}b^{\frac{1}{2}} - b^{\frac{1}{2}}c^{\frac{1}{2}} - c^{\frac{1}{2}}a^{\frac{1}{2}} - a^{\frac{1}{2}}b^2 - b^{\frac{1}{2}}c^2$ by $-ab^2c$.

Simplify :—

21. $3b(a-b) + 4c(b-c)$.

22. $\frac{z}{2}(x-2y) - \frac{x}{3}(y+3z)$.

23. $a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3)$.

24. $a(a+b-c) + b(b+c-a) + c(c+a-b)$.

25. $2(ab+bc+ca) - a(b+c-a) - b(c+a-b) - c(a+b-c)$.

26. $\frac{c}{2}(a+2b-3c) - \frac{a}{3}(b+2c-3a) - \frac{b}{4}(c+2a-3b)$.

27. $x^3(x-4) - 2x^2(x-3) - 3x(2x+5) + 4(3x+6)$.

28. $3a^3(a-2) - 2a^2(a^2-2a+3) + 5a(a^3 - 2a^2 + 2a - 4)$
$\qquad - 4(a^4 - a^3 + a^2 - a + 1)$.

29. $x^4(x-7) - 2x^2(x^2 - 3x + 5) + 3x(x^3 - 4x + 8)$
$\qquad - 5(x^3 - 6x^2 + 16a - 20)$.

30. If $P = ax^2 + bx - c$, and $Q = cy^2 - by + a$, find the value of $2Py - 3Qx$.

31. If $A = 2x + y - z$, $B = 2y + z - x$, and $C = 2z + x - y$, find the value of $A + 2B - C$.

32. If $X = 3px^2 - qx - 4r$, $Y = 16p + rx - 3q^2$, and $Z = -px + 3rx^2 - 12q$, find the value of $pX - qY + rZ$.

47. We have shown in Art. 44 that

$$(b+c)a = ba + ca.$$

Putting $l+m$ for a, we have

$$(b+c)(l+m) = b(l+m) + c(l+m)$$
$$= bl + bm + cl + cm \ldots\ldots (1)$$

Similarly, $(b+c)(l-m) = bl - bm + cl - cm \ldots\ldots (2)$

Again $(b-c)(l+m) = b(l+m) - c(l+m)$
$$= bl + bm - (cl + cm)$$
$$= bl + bm - cl - cm \ldots\ldots (3)$$

Add $(b-c)(l-m) = b(l-m) - c(l-m)$
$$= bl - bm - (cl - cm)$$
$$= bl - bm - cl + cm \ldots\ldots\ldots (4)$$

From (1), (2), (3) and (4), we infer the following rule for multiplying together two compound expressions :—

Rule.—"*Multiply each term of the first expression by each term of the second ; then connect the several products together by the rules of addition.*"

Note.—When the terms multiplied together have *like* signs, prefix to the product the sign *plus* ; when *unlike*, prefix *minus*.

We shall work some examples where both multiplicand and multiplier are binomials.

Ex. 1. Multiply $a+b$ by $c+d$ and $a-b$ by $c-d$.

$a+b$

$c+d$

$(a+b)c = ac+bc$ .....(Multiplicand $\times c$)

$(a+b)\,d = \quad +ad+bd$ .....(" $\times d$)

Product $= ac+bc+ad+bd.$

Again, $a-b$

$c-d$

$(a-b)\,c = ac-bc$ .....(Multiplicand $\times c$)

$-d(a-b = \quad -ad+bd$ (" $\times d$)

Product $= ac-bc-ad+bd.$

Ex. 2. Multiply—i. **a+b by a+b.**
ii. **a—b by a—b.**
iii. **a+b by a−b**

(i) $a+b$

$a+b$

Multiplicand $\times a = a^2+ab$

" $\times b = \quad +ab+b^2$

$\therefore$ **$(a+b)(a+b) = a^2+2ab+b^2$** .....(by addition).

(ii) $a-b$

$a-b$

Multiplicand $\times a = a^2-ab$

" $\times(-b) = \quad -ab+b^2$

$\therefore$ **$(a-b)(a-b) = a^2-2ab+b^2$** .....(by addition).

(iii) $a+b$

$a-b$

Multiplicand $\times a = a^2+ab$

" $\times(-b) = -ab-b^2$

$\therefore$ $(a+b)(a-b) = a^2 - b^2$.....(by addition).

Multiply each term of the multiplicand by each term of the multiplier, beginning on the left. The results in these examples are important and the student must pay special attention to them.

Ex. 3. Multiply $a+4$ by $a+6$.

$a+4$

$a+6$

a^2+4a

$+6a+24$

$a^2+10a+24$ .....(by addition).

In this and the preceding example, we place the second result one place to the right so that like terms may stand in the same vertical column.

Ex. 4. Multiply $x-8$ by $x-7$.

$x-8$

$x-7$

x^2-8x

$-7x+56$

$x^2-15x+56$..... (by addition).

Ex. 5. Multiply $(x-8)$ by $x+2$.

$(x-8)(x+2) = x(x+2) - 8(x+2)$

$= x^2+2x-8x-16$

$= x^2-6x-16$.

From the above three examples the following are noticed :—

1. The product consists of three terms.

2. The first is the product of the first terms, *i. e.*, the square of the common term in this case.

3. The second is the common term with a co-efficient which is the sum of the second terms in the expressions (the proper sign being taken.)

4. The third is the product of the second terms.

Or thus,

$$\begin{aligned}&(2x+3y)\ (4x-5y)\\ &=4x\ (2x+3y)-5y\ (2x+3y)\\ &=8x^2+12xy-10xy-15y^2.\\ &=8x^2+2xy-15y^2.\end{aligned}$$

EXERCISE 15.

Write down the products of the following by inspection:—

1.	$(x+4)\ (x+5)$.	2.	$(x+4)\ (x+1)$.
3.	$(a+9)\ (a+11)$	4.	$(y+3)\ y+15)$.
5.	$(y+5)\ (y+11)$.	6.	$(y+11)\ (y-3)$.
7.	$(y+2)\ (y-3)$.	8.	$(y-5)\ (y+6)$.
9.	$(x+12)\ (x-10)$.	10.	$(x-10)\ (x+9)$.
11.	$(x-4)\ (x-5)$.	12.	$(y-10)\ (y-5)$.
13.	$(y-4)\ (y-4)$.	14.	$(y+4)\ (y-4)$.
15.	$(x+y)\ (x-y)$.	16.	$(x+2a)\ (x+3a)$.
17.	$(x+a)\ (x-3a)$.	18.	$(x+2b)\ (x-b)$.
19.	$(x+2c)\ (x-c)$.	20.	$(x+c)\ (x+c)$.
21.	$(2+b)\ (2a-b)$.	22.	$(2x+3)\ (2x-11)$.
23.	$(2y-3)\ (2y-1)$.	24.	$(2a+1)\ (2a-1)$.

Ex. 6. Multiply $2x+3y$ by $4x-5y$.

$$\begin{array}{r} 2x+3y \\ 4x-5y \\ \hline 8x^2+12xy \\ \quad -10xy-15y^2 \\ \hline 8x^2+2xy-15y^2 \end{array}$$ .....(by addition).

EXERCISE 16.

Multiply :—

1.	$2m-3n$ by $m-n$.	2.	$3x-5$ by $4x-3$.
3.	$2a-4$ by $-3a+4$.	4.	$y+a$ by $-y+a$.
5.	$x-4y$ by $-x+3y$.	6.	$2x+3y$ by $2x-3y$.
7.	$2a+3c$ by $-2a-3c$.	8.	$x-15$ by $-2x+3$.
9.	$4-3c$ by $c-5$.	10.	$2a+x$ by $a+2x$.
11.	$7a-3c$ by $4c-3a$.	12.	$4y-3a$ by $2a+3y$.
13.	$-7x+2$ by $2+7x$.	14.	$4m-3n$ by $7n-5m$.

15. $ax+2b$ by $3b-2ax$.
16. $ac-3xy$ by $3ac-xy$.
17. $2mn-3a$ by $3mn-2a$.
18. $bc+ad$ by $bc-ad$.
19. $2xy-3ab$ by $3xy+2ab$.

Simplify :—

20. $(ax+by)(cz-ay)-(ay-bz)(cx+az)$.
21. $(\frac{1}{3}x-\frac{1}{3}y)(\frac{1}{4}y-\frac{1}{5}z)+(\frac{1}{4}y-\frac{1}{5}z)(\frac{1}{4}z-\frac{1}{5}x)$.
22. $(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)$.
23. $(x-2y)(3y+4z)+(y-2z)(3z+4x)+(z-2x)(3x+4y)$.
24. $(a+b)(a^2-b^2)+(b+c)(b^2-c^2)+(c+a)(c^2-a^2)$.
25. Find the cost of $(a+2b)$ things at $(a-b)$ rupees each.
26. The length of a rectangular room is $(2x+3y)$ ft. and the breadth $(x-2y)$ ft. Find its area in sq. ft.
27. Find the cost of $(2a-3)$ books at $(3a-2)$ annas each.
28. A train goes $(2x+5)$ miles an hour. Find the distance it goes in $(x-2)$ hours.
29. The strength of the Fourth Form of a school is $3a+2b$. Each boy pays $(2a+b)$ rupees per year for school fee. What is the total income from the Form ?

58. When the multiplicand and the multiplier contain more than two terms, we may employ a similar method to obtain the product.

Ex. 1. Multiply—i. a^2-ab+b^2 by $a+b$.
ii. a^2+ab+b^2 by $a-b$.

(i) a^2-ab+b^2
$a+b$
$a^3-a^2b+ab^2$
$+a^2b-ab^2+b^3$
a^3+b^3.

(ii) a^2+ab+b^2
$a-b$
$a^3+a^2b+ab^2$
$-a^2b-ab^2-b^3$
a^3-b^3.

$\therefore (a+b)(a^2-ab+b^2)=a^3+b^3$
And $(a-b)(a^2+ab+b^2)=a^3-b^3$.

Ex. 2. Multiply :—
i. $a^2+2ab+b^2$ by $a+b$.
ii $a^2-2ab+b^2$ by $a-b$.

i. $a^2+2ab+b^2$
$a+b$
$a^3+2a^2b+ab^2$
$+a^2b+2ab^2+b^3$
$a^3+3a^2b+3ab^2+b^3$.

ii. $a^2-2ab+b^2$.
$a-b$
$a^3-2a^2b+ab^2$
$-a^2b+2ab^2-b^3$
$a^3-3a^2b+3ab^2-b^3$.

$\therefore (a+b)^3 = (a^2+2ab+b^2)(a+b) = a^3+3a^2b+3ab+b^3$

And $(a-b)^3 = (a^2-2ab+b^2)(a-b) = a^3-3a^2b+3ab^2-b^3$.

Note.—The above results are important.

Ex. 3. Multiply $2x^3-3xy+4y^2$ by $4x^2-5xy-2y^2$.

$$\begin{array}{l} 2x^2-3xy+4y^2 \\ 4x^2-5xy-2y^2 \\ \hline 8x^4-12x^3y+16x^2y^2 \\ \quad -10x^3y+15x^2y^2-20xy^3 \\ \quad\quad -4x^2y^2+6xy^3-8y^4 \\ \hline 8x^4-22x^3y+27x^2y^2-14xy^3-8y^4. \end{array}$$

Ex. 4. Multiply $\frac{3}{2}x^2-ax-\frac{2}{3}a^2$ by $\frac{3}{4}x^2-\frac{1}{2}ax+\frac{1}{3}a^2$.

$$\begin{array}{l} \frac{3}{2}x^2-ax-\frac{2}{3}a^2. \\ \frac{3}{4}x^2-\frac{1}{2}ax+\frac{1}{3}a^2. \\ \hline \frac{9}{8}x^4-\frac{3}{4}x^3a-\frac{1}{2}x^2a^2 \\ \quad -\frac{3}{4}x^3a+\frac{1}{2}x^2a^2+\frac{1}{3}xa^3 \\ \quad\quad +\frac{1}{2}x^2a^2-\frac{1}{3}xa^3-\frac{2}{9}a^4 \\ \hline \frac{9}{8}x^4-\frac{3}{4}x^3a+\frac{1}{2}x^2a^2 \qquad -\frac{2}{9}a^4. \end{array}$$

EXERCISE 17.

Multiply :—

1. a^2+ab+b^2 by $a+b$.
2. a^2-ab+b^2 by $a-b$.
3. $x-2y+z$ by $x+2y-z$.
4. $a^4-a^2b^2+b^4$ by a^2+b^2.
5. $a^4+a^2b^2+b^4$ by a^2-b^2.
6. $xy+yz+zx$ by $xy-yz-zx$.
7. $x^2+y^2+z^2$ by $x-y-z$.
8. $a-3b+4c$ by $a+2b-3c$.
9. $m+n+p$ by $3m-4n-5p$.
10. $4a^2-3a+7$ by $5a^2-4a+8$.
11. $a^2-5ab+2b^2$ by $a^2+5ab-2b^2$.
12. $2x^2-4xy+y^2$ by $3x^2+5xy+2y^2$
13. x^2-2x+2 by $2x^2-4x+5$.
14. $9a^2-24ab+16b^2$ by $3a-4b$.

15. $-a^3+2a^2b^2+b^3$ by $-a+b$.
16. $-2x^3+3xy+y^3$ by $-x-y$.
17. $\frac{1}{2}a+\frac{2}{3}b-\frac{3}{4}c$ by $\frac{1}{3}a-\frac{1}{4}b$.
18. $\frac{3}{4}a^2-ab+\frac{2}{3}b^2$ by $a^2-\frac{1}{2}ab+\frac{1}{3}b^2$.
19. $\frac{3}{5}x-y-\frac{3}{4}z$ by $\frac{2}{5}x-\frac{3}{5}y$.
20. $\frac{2}{3}x^2-\frac{1}{3}xy+\frac{1}{5}y^2$ by $\frac{3}{5}x+\frac{3}{4}y$.
21. $\frac{4}{5}a^2-\frac{3}{4}a+\frac{1}{2}$ by $\frac{3}{2}a^2-2a+1$.
22. $\frac{5}{3}x^2-x+\frac{2}{5}$ by $\frac{1}{2}x^2-\frac{2}{5}x+1$.
23. $\frac{1}{3}a^2+\frac{2}{5}ax+\frac{4}{3}x^2$ by $\frac{3}{4}a^2-\frac{3}{2}ax+\frac{6}{5}x^2$.
24. $m^2-\frac{1}{2}mn+\frac{1}{3}n^2$ by $\frac{1}{2}m^2-mn+\frac{3}{2}n^2$.
25. $-\frac{1}{2}x^2-\frac{3}{4}bx+\frac{1}{4}b^2$ by $-\frac{2}{3}x^2+\frac{1}{3}bx+b^2$.

Simplify :—

26. $(a+2b-c)(b+2c-a)+(c+2a-b)(a+2b-c)$.
27. $(a-b)(a+b-c)+(b-c)(b+c-a)(c-a)(c+a-b)$.
28. If $X=3a-2b$, and $Y=2a-3b$, find the value of $(2X-3Y)(3X-2Y)$.
29. If $A=x+2y-3z$, $B=y+2z-3x$, and $C=z+2x-3y$, find the value of $(A+2B-3C)(B+2C-3A)$.
30. If $P=l-m+n$, $Q=m-n+l$, $R=n-l+m$, find the value of $(P-Q)(Q-R)+(Q-R)(R-P)+(R-P)(P-Q)$.

49. When one expression is to be multiplied by another, it is convenient to arrange boththe multiplicand and the multiplier according to *descending* or *ascending* powers of some letter common to them and then proceed with the multiplication.

The expression $2a+3a^3-6a^2+a^4+a^5+2$ when arranged according to *descending* powers of a is $a^5+a^4+3a^3-6a^2+2a+2$; and when arranged according to *ascending* powers of a is $2+2a-6a^2+3a^3+a^4+a^5$.

Ex. 1. Multiply a^2+b^2+ab by a^2+b^2-ab.

Arranging the two expressions according to the descending powers of a or the ascending powers of b, we have,

$$\begin{array}{l} a^2+ab+b^2 \\ a^2-ab+b^2 \\ \hline a^4+a^3b+a^2b^2 \\ \quad -a^3b-a^2b^2-ab^3 \\ \qquad +a^2b^2+ab^3+b^4 \\ \hline a^4 \qquad +a^2b^2 \qquad +b^4 \end{array}$$

Ex. 2. Multiply $a^2+b^2+c^2-ab-ac-bc$ by $a+b+c$.

$$a^2-ab-ac+b^2-bc+c^2$$
$$a+b+c$$
$$a^3-a^2b-a^2c+ab^2-abc+ac^2$$
$$+a^2b \quad -ab^2-abc+b^3 \quad -b^2c+bc^2$$
$$+a^2c \quad -abc-ac^2 \quad +b^2c-bc^2+c^3.$$
$$a^3 \quad -3abc \quad +b^3 \quad +c^3.$$

$\therefore (a+b+c)(a^2+b^2+c^2-ab-ac-bc)=a^3+b^3+c^3-3abc.$

Note—The multiplicand and the multiplier are arranged according to *descending* powers of a and we place the *like* terms in the same vertical column. The result is very important.

Ex. 3. Multiply ax^2+bx+c by cx^2+ax+b.

$$ax^2+bx+c$$
$$cx^2+ax+b$$
$$acx^4+bcx^3+c^2x^2$$
$$+a^2x^3+abx^2+acx$$
$$+abx^2+b^2x+bc$$

Product $=acx^4+(cb+a^2)x^3+(c^2+2ab)x^2+(ac+b^2)x+bc$

EXERCISE 18.

Multiply :—

1. a^2-a+1 by a^2+a+1.
2. $x^2-2xy+y^2$ by $x^2+2xy+y^2$.
3. $5x^3+4x-3x^2+1$ by $2x^2-3x+2$.
4. $x^2+y^2+z^2-xy-yz-zx$ by $x+y+z$.
5. x^3+x^2+x+1 by $-1+x-x^2+x^3$.
6. $x^3-x^2y+xy^2-y^3$ by $x^3+x^2y+xy^2+y^3$
7. $3a^3+2a+a^3+1$ by a^2-2a+1.
8. $a^4-a^3b+a^2b^2+b^4-ab^3$ by a^2-ab+b^2.
9. $c^3-1+3c-3c^2$ by $1-2c+c^2$.
10. $a^3+a^2b+b^2+ab^2$ by $a-b$.
11. $-2+3y-4y^3+3y^2$ by y^3-y+2y^2+1.
12. $a^3x^3-3ax+4a^2x^2+5$ by $2ax+3$.
13. $b^3-b^2x+bx^2+x^3$ by x^2-bx+b^2.
14. $2y^3-3y^2+4y-5$ by $-2-3y+4y^2-y^3$.
15. px^2+qx+r by rx^2+px+q.

16. px^3+qx^2+rx+s by ax^2-bx-c.

17. $a^3-x^3+x^2-3a^2x$ by a^2+x^2-2ax.

18. $1+2x+x^4+2x^3+3x^2$ by $1+x^2-2x$.

19. $b^4+b^2a^2+ba^3+b^3a+a^4$ by $a^4+b^4+b^2a^2-b^3a-ba^3$.

20. $a^4-5ab+3b^2a+4a^2b-b^4$ by $a^4+3b^2a+5ab-4a^2b+b^4$.

21. $\frac{x}{12}-\frac{5x^2}{3}+\frac{x^3}{2}+3$ by $3+\frac{x^3}{2}-x$.

22. $\frac{3x^3}{2}+3x^2-\frac{5x}{4}-\frac{23}{2}$ by $\frac{x^2}{6}-\frac{x}{3}+\frac{1}{2}$

22. Multiply the sum of $3x^2-5xy$ and $2xy-y^2$ by the excess of $3x^2+y^2$ over $2y^2+3xy$.

24. Multiply the remainder after subtracting $1-3x$ from x^2-2x-1 by the sum of x^2-1+2x and $4-3x$.

Simplify :—

25. $(a+b)[a^3-b\{a^2-b(a-b)\}]$.

26. $6\{a-\frac{3}{2}(b-\frac{2}{3})\}\{\frac{1}{3}(2a+1)+(b+1)\}$.

27. $(x-y)[3z-x\{3y-(2z-x+4y)-(2x-3z)(z+y)\}]$.

28. $(a+b)(2b-3c)-[a(b-c)-2b\{(a-c)(a+c)-b(c-a)-c(a-b)\}-a+b-2c)(b+c)]$.

29. $3(b-c)(b^2c-bc^2)+5(c-a)(c^2a-ca^2)+7(a-b)(a^2b-ab^2)$.

30. $2(b+c)(b^2a-bca+ac^2)-3(c+a)(c^2b-cab+ba^2)+4(a+b)(a^2c-abc+cb^2)$.

31. If $A=x^2+xy-y^2$, $B=y^2+yz-z^2$, and $C=z^2+zx-x^2$, find the values of—

(i) $A(B-C)-B(C-A)$, (ii) $(Ax+By)(Cy+Az)$.

32. If $A=x^2-y^2$, $B=y^2-z^2$ and $C=z^2-x^2$, find the value of $A(B+C)+B(C+A)+C(A+B)$.

33. If $V=3a+2b-4c$, $X=-a-7b+5c$, $Y=18a+5b-2c$, and $Z=11a-3b+7c$.

Evaluate

$$V(a+b)-b(X+Y)+(b-c)Z.$$

50. The product of three or more expressions is called **their continued product**.

Ex. 1. Find the continued product of $x+1$, $x-1$, and $x+2$.

$$\begin{array}{r} x+1 \\ x-1 \\ \hline x^2+x \\ -x+1 \\ \hline x^2 \quad -1 \end{array} \qquad \begin{array}{r} x^2-1 \\ x+2 \\ \hline x^3-x \\ +2x^2-2 \\ \hline x^3+2x^2-x-2. \end{array}$$

Ex. 2. Find the continued product of $\mathbf{x+a}$, $\mathbf{x+b}$ **and** $\mathbf{x+c}$.

$$\begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \\ +bx+ab \\ \hline x^2+ax+bx+ab. \end{array} \qquad \begin{array}{r} x^2+ax+bx+ab \\ x+c \\ \hline x^3+ax^2+bx^2+abx \\ +cx^2+acx+bcx+abc. \\ \hline x^3+ax^2+bx^2+cx^2+abx+acx+bcx \\ +abc. \end{array}$$

$\therefore$ $\mathbf{(x+a)(x+b)(x+c)}$

$$\mathbf{=x^3+x^2(a+b+c)+x(ab+ac+bc)+abc.}$$

Ex. 3. Find the continued product of $\mathbf{x-a}$, $\mathbf{x-b}$ **and** $\mathbf{x-c}$.

$$\begin{array}{r} x-a \\ x-b \\ \hline x^2-ax \\ -bx+ab \\ \hline x^2-ax-bx+ab. \end{array} \qquad \begin{array}{r} x^2-ax-bx+ab \\ x-c \\ \hline x^3-ax^2-bx^2+abx \\ -cx^2+acx+bcx-abc \\ \hline x^3-ax^2-bx^2-cx^2+abx+aex+bcx \\ -abc. \end{array}$$

$\therefore$ $\mathbf{(x-a)(x-b)(x-c)}$

$$\mathbf{=x^3-x^2(a+b+c)+x(ab+ac+bc)-abc.}$$

Note.—This product may be deduced from that, of $x+a$ $x+b$, and $x+c$ by putting—a for a,—b, for b, and—c for c.

EXERCISE 19.

Find the continued product of—

1. $a+b$, $a-b$, and a^2-b^2.
2. $x-a$, $x+b$, and $x-c$.
3. $a+b$, $b+c$, and $c+a$.
4. $a-b$, $b-c$ and $c-a$.
5. $x+3$, $x+7$, and $x-5$.
6. $2x+3$, $x-7$, and $3x-4$.
7. $y+6$, $2y-3$, and $3-4y$.
8. $4-c$, $3-2c$, and $2-c$.
9. $x+y$, $x-y$, x^2+y^2, and x^4+y^4.

10. $x+2y$, $x-2y$, and x^2+4y^2.
11. a^2+ab+b^2, a^2-ab+b^2, and $a^4-a^2b^2+b^4$.
12. x^2+3x 1, x^2-3x+1, and x^4+7x^2+1.
13. $2b+3c$, $3b-2c$, and a $-b$.
14. $x-y-z$, $x-y+z$, and $y+z-x$.
15. $x-2c$, $x+3c$, and $x-4c$.
16. $a-2x$, $2x-a$, and $2x^2+a^2$.
17. a^2-2a+2, a^2+2a+2, and a^4-4.
18. $x^2-2bx-2b^2$, $x^2+2bx+2b^2$, and x^4-4b^4.
19. $x+a-b$, $x+b-c$, and $x+c-a$.
20. a^2-ab+b^2, $a-b$, a^2+ab+b^2 and $a+b$.
21. $a+b+c$, $b+c-a$, $c+a-b$, and $a+b-c$.
22. $1-x+x^2$, $1-x^2+x^4$, $1-x^4+x^8$, and $1+x+x^2$.

CHAPTER V.

DIVISION

51. Division is the inverse of Multiplication. In Multiplication we determine the product of two given factors ; in Division we have the product and one of the factors given, and our object is to determine the other factor.

When one quantity is divided by another, the former is called the **dividend**, and the latter the **divisor** ; the result is called the **quotient**.

Dividend = Quotient × Divisor.

∴ Quotient = Dividend ÷ Divisor

52. Rule of Signs.—"*When the dividend and the divisor have the same sign, the quotient is positive, and when they have different signs, the quotient is negative.*"

Thus $ab \div a = \frac{ab}{a} = \frac{a \times b}{a} = +b$ .....(1)

$-ab \div a = \frac{-ab}{a} = \frac{a \times (-b)}{a} = -b$ .....(2)

$ab \div -a = \frac{ab}{-a} = \frac{-a \times (-b)}{-a} = -b$ .....(3)

$-ab \div -a = \frac{-ab}{-a} = \frac{-a \times (+b)}{-a} = +b$ .....(4)

From the above results, we infer that in Division as well as in Multiplication, *like signs produce* +, *and unlike signs* −.

53. There are three cases in Division.

First case. When the divisor and the dividend are *both simple quantities* or *monomials.*

Rule—"*Write the divisor under the dividend in the form of a fraction and divide the terms of the fraction by all the factors which are common to both and prefix to the result the proper sign.*"

Ex 1. Divide $16x^2y^3$ by $-4xy^2$.

The dividend $= 4 \times 4 \times x \times x \times y \times y \times y$.

The divisor $= -4 \times x \times y \times y$.

∴ The quotient $= \frac{4 \times 4 \times x \times x \times y \times y \times y}{-4 \times x \times y \times y} = -4xy$.

Ex. 2. Divide $-14m^2n^2p$ by $-7mp$.

The dividend $= -7\times 2\times m\times m\times n\times n\times p$.

The divisor $= -7\times m\times p$.

$\therefore$ The quotient $= \dfrac{-7\times 2\times m\times m\times n\times n\times p}{-7\times m\times p}$

$= 2m\times n\times n = 2mn^2$.

Ex. 3. Divide a^5 by a^3.

$a^5 = a\times a\times a\times a\times a$.

$a^3 = a\times a\times a$.

$\therefore$ The quotient $= \dfrac{a\times a\times a\times a\times a}{a\times a\times a} = a\times a = a^2 = a^{5-3}$

Hence we get the following rule.

Rule.—"*When the dividend and the divisor are powers of the same letter, subtract the index of the power in the divisor from the index of the power in the dividend.*"

Generally, when m and n are positive integers.

$\dfrac{\mathbf{a}^m}{\mathbf{a}^n} = \mathbf{a}^{m-n}$ **when** $\mathbf{m} > \mathbf{n}$

For $\dfrac{a^m}{a^n} = \dfrac{a\times a\times a\ldots\ldots\text{to } m \text{ factors}}{a\times a\times a\ldots\ldots\text{to } n \text{ factors}}$

$= a\times a\times a\ldots\ldots\text{to } (m-n) \text{ factors}$

$= a^{m-n}$.

Also $\dfrac{\mathbf{a}^m}{\mathbf{a}^n} = \dfrac{\mathbf{1}}{\mathbf{a}^{n-m}}$, **when** $\mathbf{n} > \mathbf{m}$.

For $\dfrac{a^m}{a^n} = \dfrac{a\times a\times a\ldots\ldots\text{to } m \text{ factors}}{a\times a\times a\ldots\ldots\text{to } n \text{ factors}}$

$= \dfrac{1}{a\times a\times a\ldots\ldots \text{ to } (n-m) \text{ factors}}$

$= \dfrac{1}{a^{n-m}}$.

To prove that $\mathbf{a}^0 = 1$.

In $\dfrac{a^m}{a^n} = a^{m-n}$, let $m = n$. Then $\dfrac{a^n}{a^n} = a^{n-n}$; or $1 = a^0$.

Note.—Any expression raised to the power of 0 = 1.

Ex. 4. Divide $4a^5b^3$ by $-8a^2b^2$.

Quotient $= \dfrac{4a^5b^3}{-8a^2b^2} = -\tfrac{1}{2}a^{5-2}b^{3-2} = -\tfrac{1}{2}a^3b$.

Ex. 5. Divide $-\frac{4}{3}x^7y^6z^5$ by $-\frac{1}{4}x^4y^2z^3$.

$$\text{Quotient} = \frac{-\frac{4}{3}x^7y^6z^5}{-\frac{1}{4}x^4y^2z^3} = +4x^{7-4}y^{6-2}z^{5-3} = 4x^3y^4z^2$$

EXERCISE 20.

Divide :—

1. ax by $-a$.
2. abc by $-b$.
3. $3a^3b^3x$ by $-abx$.
4. $-14a^3b^2c^3$ by $28abc^2$.
5. $5x^7$ by $3x^2$.
6. $6y^7x^7$ by $-9y^2x^4$.
7. $-25x^7y^8z^9$ by $-45x^6y^4z^8$.
8. $48a^{12}b^8c^7$ by $-64a^2b^3c$.
9. $120a^7x^7y^2$ by $-24x^5y^2$.
10. $15m^2np$ by $-3npm^2$.
11. $-25p^7n^4m$ by $-50p^3n^2m$.
12. $63x^2y^8z^{15}$ by $-72z^4y^4x^2$.
13. $28m^7n^6p^5q^4$ by $-70q^3p^4n^5m^6$.
14. $111x^2y^3z^4a^{10}b^{11}$ by $-37a^3b^4x^2y^3z^2$.
15. $-81m^3x^3z^{10}b^6$ by $72bz^2xm$.
16. $a^3(b-c)^2$ by $a^2(b-c)^3$.

31. *Second case.* When the *divisor alone is a simple quantity.*

Rule.—"*Divide each term of the dividend by the divisor and add the quotients.*"

Ex. 1. Divide $ab+ac$ by $-a$,

$$\text{Quotient} = \frac{ab}{-a} + \frac{ac}{-a} = -b-c.$$

Ex. 2. Divide $3x^2+4xy-5x^2y^2$ by $-2x$.

$$\text{Quotient} = \frac{3x^2}{-2x} + \frac{4xy}{-2x} + \frac{-5x^2y^2}{-2x}$$
$$= -\tfrac{3}{2}x - 2y + \tfrac{5}{2}xy^2.$$

Ex. 3. Divide $45a^4b^4-35a^3b^3+15a^2b^4$ by $5a^2b^2$.

$$\text{Quotient} = \frac{45a^4b^4}{5a^2b^2} + \frac{-35a^3b^3}{5a^2b^2} + \frac{15a^2b^4}{5a^2b^2}$$
$$= 9a^2b^2 - 7ab + 3b^2.$$

Ex. 4. Divide $4x^2y^2-5xy+\frac{3}{2}y^2$ by $-\frac{1}{2}y$.

$$\text{Quotient} = \frac{4x^2y^2}{-\frac{1}{2}y} + \frac{-5xy}{-\frac{1}{2}y} + \frac{\frac{3}{2}y^2}{-\frac{1}{2}y}$$
$$= -8x^2y + 10x - 3y.$$

EXERCISE 21.

Divide—

1. x^3-3x^2y by $-x^2$.
2. $a^3-3a^2b+3ab^2$ by $-a$.
3. $x^7-x^3+2x^2$ by x^2.
4. $a^4x^4-a^6x^3$ by $-a^3x^3$.
5. $abc^2+a^2bc+b^2ac$ by $-abc$.
6. $9a^3b^2+27b^3a^2-81a^2b^3$ by $-27a^2b^2$.
7. $9a^7+11a^5-12a^4b^2-13a^3b^3$ by $-2a^3$.
8. $-34x^6y+17x^5y^5-51x^4y^6$ by $-\frac{1}{2}7x^3y$.
9. $-\frac{1}{2}a^2b^2+\frac{1}{4}a^3b-\frac{1}{8}a^4b^2$ by $-2ab^2$.
10. $3a^2b^4-\frac{3}{8}ab^3+\frac{1}{3}a^3b^2$ by $-\frac{1}{2}ab^2$.
11. $-\frac{3}{2}m^2+\frac{3}{8}mn-\frac{3}{4}m^2n^2$ by $2m$.
12. $b^7-\frac{1}{2}b^6+\frac{3}{8}b^5-\frac{3}{4}b^3$ by $-\frac{1}{8}b^2$.
13. $b^3c^3-\frac{5}{2}b^4c^2+\frac{4}{3}b^3c^4$ by $-\frac{1}{2}b^2c^2$.
14. $-\frac{7}{3}a^4x^4-3a^5x^2+\frac{7}{6}a^3x^7$ by $\frac{7}{3}x^2a^3$.
15. $-\frac{4}{3}m^4n^4p^3+\frac{7}{3}m^2n^3p^4-\frac{1}{3}\frac{4}{3}m^4n^2p^5$ by $-\frac{4}{7}mnp$.

55. *Third case.* When both the divisor and the dividend *are compound expressions.*

Rule.—(*a*) *Arrange the divisor and the dividend according to descending or ascending powers of some common letter.*

(*b*) *Divide the first term of the dividend by the first term of the divisor, and put the quotient obtained as the first term of the required quotient.*

(*c*) *Multiply the whole divisor by this quotient and put the product under the dividend.*

(*d*) *Subtract the product from the dividend and bring down only as many terms as are necessary.*

(*e*) *Repeat the operation using each new remainder as dividend, till all the terms in the dividend are exhausted and the remainder vanishes.*

(*f*) *If the remainder does not vanish, then set down as the last term of the quotient the remainder over the divisor in the form of a fraction.*

We shall first illustrate the rule by working some examples where the divisor is a *binomial.*

Ex. 1. Divide $x^2+7x+10$ by $x+5$.

Divisor. *Dividend.* *Quotient.*

$x+5)\ x^2+7x+10\ (x+2$

x^2+5x

$2x+10$

$2x+10$

0

We first arrange the dividend and the divisor in *descending* powers of x.

Then we divide x^2 the first term of the dividend by x, the first term of the divisor. the quotient is x. We multiply the divisor $x+5$ by x and put the product x^2+5x under the dividend (as shown in the working). We subtract this product x^2+5x from the dividend $x^2+7x+10$, and the remainder is, $2x+10$.

The next term of the quotient $(+2)$ is obtained by dividing $2x$, the first term of the remainder, by x, the first term of the divisor. We then multiply the divisor $x+5$ by 2, and put the product under the remainder (as shown in the working).

The quotient required is $x+2$.

Note.—*The reason for the above rule is this* :

The dividend is split up into a number of parts, each of which contains the divisor exactly. The sum of all the partial quotients forms the complete quotient. In the example worked out, the dividend $x^2+7x+10$ is separated into two parts x^2+5x, $2x+10$, *i.e.*, $x(x+5)$ and $2(x+5)$ and each of these is divided by $x+5$. The partial quotients are x and $+2$. The sum of these, *i.e.*, $x+2$ is the complete quotient.

Ex. 2. Divide $21a^2-43ab+20b^2$ by $3a-4b$.

Divisor. *Dividend.* *Quotient.*

$$3a-4b)\ 21a^2-43ab+20b^2\ (7a-5b$$
$$21a^2-28ab.$$
$$-15ab+20b^2$$
$$-15ab+20b^2$$

Ex. 3. Divide $-1-3x^2+x^3+3x$ by $x-1$.

We arrange the dividend according to *descending* powers of x.

Divisor. *Dividend.* *Quotient.*

$$x-1)\ x^3-3x^2+3x-1\ (x^2-2x+1.$$
$$x^3-x^2$$
$$-2x^2+3x$$
$$-2x^2+2x$$
$$x-1$$
$$x-1$$
$$0$$

Ex. 4. Divide $24x^3 - 98x^2y + 133xy^2 - 60y^3$ by $4x - 5y$.

Divisor. *Dividend.* *Quotient.*

$$
\begin{array}{l|l}
4x - 5y) & 24x^3 - 98x^2y + 133xy^2 - 60y^3 \ (6x^2 - 17xy + 12y^2 \\
\cline{2-2}
 & \underline{24x^3 - 30x^2y} \\
 & \quad -68x^2y + 133xy^2 \\
 & \quad \underline{-68x^2y + 85xy^2} \\
 & \qquad\qquad 48xy^2 - 60y^3 \\
 & \qquad\qquad \underline{48xy^2 - 60y^3} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

EXERCISE 22.

Divide—

1. $a^2 + 4a +$ by $a + 2$.
2. $a^2 - 6a + 5$ by $a - 1$.
3. $x^2 - 17x + 60$ by $x - 5$.
4. $x^2 + x - 56$ by $x + 8$.
5. $x^2 - 47x + 550$ by $x - 25$.
6. $6x^2 - 13x + 6$ by $2x - 3$.
7. $8x^2 - 22x + 15$ by $4x - 5$.
8. $3a^2 + 34a + 11$ by $3a + 1$.
9. $6x^2 - 7x - 3$ by $3x + 1$.
10. $10x^2 - 27x - 28$ by $2x - 7$.
11. $24x^2 - 65x + 21$ by $8x - 7$.
12. $15a^2 - 26ab + 8b^2$ by $3a - 4b$.
13. $6x^2 + 7xy - 20y^2$ by $2x + 5y$.
14. $21b^2 - 53bc + 30c^2$ by $7b - 6c$.
15. $63a^2 + 67ax + 14x^2$ by $9a + 7x$.
16. $99x^2 + 4xy - 4y^2$ by $11x - 2y$.
17. $44a^2 - 39ac - 135c^2$ by $4a - 9c$.
18. $7x^2 + 96xy - 28y^2$ by $7x - 2y$.
19. $100a^2 - 3b^2 - 13ab$ by $25a + 3b$.
20. $3x + 3x^2 + x^3 + 1$ by $x + 1$.
21. $8x^3 - 1 - 12x^2 + 6x$ by $2x - 1$.
22. $x^3 - 8x^2 + 12x$ by $x - 6$.
23. $6x^3 + x^2 - 2x$ by $3x + 2$.
24. $x^3 - 6 - 6x^2 + 11x$ by $x - 3$.
25. $27x^3 - 12x^2 - 25x + 10$ by $x - 1$.
26. $4x^3 + 5x^2 - x - 2$ by $x + 1$.
27. $8a^3 + 6a^2b - 3ab^2 - b^3$ by $a + b$.
28. $16x^3 - 46x^2 + 39x - 9$ by $8x - 3$.
29. $a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$ by $2b - a$.
30. $x^3 - y^3 + x^2y - xy^2$ by $x - y$.

31. $x^3+y^3-x^2y-xy^2$ by $x+y$.
32. $4x^3+7x^2-3x-15$ by $4x-5$.
33. $a^4-b^4+2a^3b+2ab^3$ by a^2+b^2.
34. $a^4+b^4-2a^2b^2+2a^3b-2ab^3$ by a^2-b^2.
35. $81a^4-216a^3b+216a^2b^2-96ab^3+16b^4$ by $2b-3a$.

36. The rule enunciated in Art. 55 is applicable to cases where the divisor consists of *more than two terms*; and both the divisor and the quotient may be written on one side.

Ex. 1. Divide $x^6-2x^5a^3+a^6$ by $x^2-2xa+a^2$.

$$
\begin{array}{l|l}
x^6-2x^5a^3+a^6 & x^2-2xa+a^2 \\
\underline{x^6-2x^5a+x^4a^2} & \overline{x^4+2x^3a+3x^2a^2+2xa^3+a^4} \\
2x^5a-x^4a^2-2x^3a^3 & \\
\underline{2x^5a-4x^4a^2+2x^3a^3} & \\
3x^4a^2-4x^3a^3 & \\
\underline{3x^4a^2-6x^3a^3+3x^2a^4} & \\
2x^3a^3-3x^2a^4 & \\
\underline{2x^3a^3-4x^2a^4+2xa^5} & \\
x^2a^4-2xa^5+a^6 & \\
\underline{x^2a^4-2xa^5+a^6} &
\end{array}
$$

Ex. 2. Divide x^4+64 by x^2+4x+8.

$$
\begin{array}{l|l}
x^4+64 & x^2+4x+8 \\
\underline{x^4+4x^3+8x^2} & \overline{x^2-4x+8} \\
-4x^3-8x^2+64 & \\
\underline{-4x^3-16x^2-32x} & \\
8x^2+32x+64 & \\
\underline{8x^2+32x+64} &
\end{array}
$$

Ex. 3. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$
\begin{array}{l|l}
a^3-3abc+b^3+c^3 & a+b+c \\
\underline{a^3+a^2b+a^2c} & \overline{a^2-ab-ac+b^2-bc+c^2.} \\
-a^2b-a^2c-3abc & \\
\underline{-a^2b-ab^2-abc} & \\
-a^2c+ab^2-2abc & \\
\underline{-a^2c-abc-ac^2} & \\
ab^2-abc+ac^2+b^3 & \\
\underline{ab^2+b^3+b^2c} & \\
-abc+ac^2-b^2c & \\
\underline{-abc-b^2c-bc^2} & \\
ac^2+bc^2+c^3 & \\
\underline{ac^2+bc^2+c^3} &
\end{array}
$$

The result of this example is important. Compare this with **Ex. 2**, worked out in Art. 49.

Ex. 4. Divide $16x^4+36x^2+6x+86$ by $4x^2+6x+9$.

$$\begin{array}{l|l} 16x^4+36x^2+6x+86 & 4x^2+6x+9 \\ \underline{16x^4+24x^3+36x^2} & 4x^2-6x+9 \\ -24x^3+6x+86 & \\ \underline{-24x^3-36x^2-54x} & \\ 36x^2+60x+86 & \\ \underline{36x^2+54x+81} & \\ 6x+5 \text{.....Remainder.} & \end{array}$$

Thus the quotient is $4x^2-6x+9+\dfrac{6x+5}{4x^2+6x+9}$.

Note—When the division is not exact, the work should be carried on until the remainder is of lower dimensions than the divisor.

EXERCISE 23.

Divide—

1. x^3-3x^2+3x-1 by x^2-2x+1.
2. $3x^3-10x^2+x+6$ by x^2-4x+3.
3. x^6-a^6 by $x^3-2ax^2+2a^2x-a^3$.
4. x^4+64 by x^2-4x+8.
5. x^4-256 by $x^3+4x^2+16x+64$.
6. $x^7-13x-30$ by x^2-2x+3.
7. $1-5x+4x^5$ by $1-2x+x^2$.
8. $3x^5-10x^4y+16x^3y^2-12x^2y^3+xy^4+2y^5$ by $x^2-2xy+y^2$.
9. $21x^5-2x^4-70x^3+23x^2+33x+27$ by $7x^2+4x-9$.
10. $1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{15}$ by $1-x^5+x^6$.
11. $a^3-8b^3-27c^3-18abc$ by $a-2b-3c$.
12. $x^3-y^3+z^3+3xyz$ by $x-y+z$.
13. $m^3+n^3+3mn-1$ by $m+n-1$.
14. x^3+x^2+x-3 by x^2+2x+3.
15. x^3+3x^2+5x+3 by x^2+2x+3.
16. $4x^4-9x^2+6x-1$ by $1-3x+2x^2$.
17. $3x^5-10x^3+15x+8$ by x^3+3x^2+3x+1.
18. $x^5-2x^4-6x^3+4x^2+13x+6$ by $1+3x+3x^2+x^3$.
19. x^5+11x^3-54 by x^2+2x+3.
20. $6x^2+21xy+18y^2+28z^2+45yz+26xz$ by $2x+3y+4z$.
21. $6a^2+13ab-5ac-5b^2+13bc-6c^2$ by $3a-b+2c$.

22. $10a^2 - 8ab - 11ac - 24b^2 - 58bc - 35c^2$ by $2a - 4b - 5c$.

23. $14x^2 + 33xy + 46x + 18y^2 + 42y + 12$ by $2x + 3y + 6$.

24. $6a^2 + 11ab - 13a - 16b^2 + 34b - 28$ by $2a + 5b - 7$.

25. $x^9 - y^9$ by $x^3 + xy + y^2$.

26. $x^9 + y^9$ by $x^2 - xy + y^2$.

27. $x^3 + y^3 + 3x^2y + 3xy^2 + 1$ by $x + y + 1$.

28. $x^8 - y^8$ by $x^3 + x^2y + y^2x + y^3$.

29. $x^7 + 2187$ by $x + 3$.

30. $1 - a^3 - 8b^3 - 6ab$ by $1 - a - 2b$.

31. $x^{12} - y^{12}$ by $x^4 - y^4$.

32. $a^8 + 36a^3 - 4a^2 + 8a - 17$ by $a^4 + 2a^3 - 1$.

33. $a^4 + \frac{1}{4}a^2 + \frac{1}{16}$ by $a^2 - \frac{1}{2}a + \frac{1}{4}$.

34. $\frac{1}{8}x^3 - \frac{1}{2}x^2y + \frac{11}{18}xy^2 - \frac{2}{9}y^3$ by $\frac{1}{2}x - y$.

35. $2a^5 - 11a^3 - 9$ by $a^2 + 2a + 3$.

36. $36a^2 + \frac{1}{9}b^2 + \frac{1}{4} - 4ab - 6a + \frac{1}{3}b$ by $6a - \frac{1}{3}b - \frac{1}{2}$.

37. $4x^2 + \frac{1}{4}y^2 + \frac{1}{9} - 2xy + \frac{1}{3}x - \frac{1}{3}y$ by $2x - \frac{1}{2}y + \frac{1}{3}$.

38. $\frac{2}{15}x^4 - \frac{3}{4}x^3 - \frac{7}{4}x^2 + \frac{4}{3}x + \frac{1}{9}$ by $\frac{3}{2}x^2 - x - \frac{8}{3}$.

39. $\frac{1}{8}a^3 + b^3 - \frac{1}{27} + \frac{1}{2}ab$ by $\frac{1}{2}a + b - \frac{1}{3}$.

40. $\frac{1}{4}a^4 + \frac{1}{12}b^4 - \frac{1}{3}a^3b + \frac{19}{48}a^2b^2 - \frac{17}{72}ab^3$ by $\frac{1}{2}a^2 + \frac{1}{4}b^2 - \frac{1}{3}ab$.

EXAMINATION PAPERS.

First Series on Chapters I—V.

1.

1. If $a=\cdot05$, $b=\cdot06$, $c=-\cdot11$, find the value of (1) $a^2-b^2-2bc-c^2$ and (2) $a^3+b^3+c^3-3abc$.

2. Simplify:—

$2\left[4x-\left\{2y+(2x-y)-(x+y)\right\}\right]$.

3. Sum up $3a^2-4ab+7a^2b^2+2ab^3$, $2ab-2b^2-a^2b^2+4a^2b$, $ab-2a^3+2b^3+a^2b^2$.

4. Multiply (1) $2x^3-3x^2+4x-5$ by $7x+9$ (2) $\frac{2}{5}a^3-\frac{4}{5}a^2x-\frac{1}{5}x^3$ by $\frac{3}{5}a-2x$.

5. Prove that $a-(b+c)=a-b-c$.

6. Find the cost of $(a-2b)$ books at $(2a-b)$ annas each.

2.

1. Simplify:—$(2b+c)(b-c)+(2c+a)(c-a)+2a+b)(a-b)$, and find the value of the result when $a=1$, $b=2$ and $c=3$.

2. Remove the brackets from and find the value of:—

$a-[2a-3b-\left\{4a-5b-6c-(7a-8b-9c)\right\}]$.

3. Find the continued product of a^2+2a+2, a^2-2a+2, a^2-4

4. 1. Prove that $a^m+a^n=a^{m-n}$, when m and n are positive integers.

2. Prove that $a-(b-c)=a-b+c$.

5. Find what values $\frac{3x-5}{2}$ will take when x takes the following values:—2, 0, −3, −1.

6. Divide $a^3-b^3+c^3+3abc$ by $a-b+c$.

3.

1. Add together $a+b-2c$, $b+c-2a$ and $c+a-2b$, and subtract the result from $a+b+c$.

2. If $a=1$, $b=2$, $c=-\frac{1}{2}$, $d=0$ find the value of $\frac{a-b+c}{a-b-c}-\frac{ad-bc}{bd+ac}$

$-\sqrt{\left(\frac{b^3}{a^3}-\frac{a^3}{c^3}\right)}$.

3. Simplify:—$(a+b-c)(b+c-a)(c+a-b)$.

4. The length of a rectangular room is $(2x+3y)$ ft. and the breadth is $(x+2y)$ ft. find the area in sq. yds.

5. Divide $x^3-6x^2+11x-6$ by $x-2$.

6. Prepare a table showing the values of $2x^2-x+3$ corresponding to the values of x:—0, -1, 2, -4.

4.

1. If $a=3$, $c=-4$, find the values of :—

$$21\left(\frac{2a-4c}{3}\right)-12\left[\frac{2a-c}{6}-\left\{\frac{5a-7c}{4}-\frac{1}{3}\left(\frac{a}{2}-\frac{5a-c}{4}\right)\right\}\right].$$

2. The product of two expressions is $10x^4+14x^3+11x^2+15x-3$ and one of them is $2x^2+4x-1$; find the other.

3. Divide $1-x$ by $1+x$ to 5 terms.

4. Find the square of $a+b+c$ and subtract the result from zero.

5. Show that $x^3+6x^2+11x+6=0$ when $x=-1$, -2, -3.

6. Find the value of $2x^2-3x+1$ when x has the values 1, 3, 5.

5.

1. Show that $x^2+y^2+z^2+xy+yz+zx$ has the same values when $x=2$, $y=3$, $z=4$ as it has when $x=-2$, $y=-3$ and $z=-4$.

2. If $a=\cdot02$, $b=\cdot08$, $c=\cdot1$, find the value of $a^3+b^3+3abc-c^3$.

3. The product of two expressions is $(x+2y)^3+(3x+z)^3$ and one of them is $4x+2y+z$; find the other.

4. If $a=1$, $b=2$ and $c=3$, show that $\frac{ab+ca}{ba+ac}+\frac{aa+bb}{cc+abc}-\frac{1}{a^c b^b c^a}$ $=\frac{20}{ab+bb+cb}$.

5. If $A=2x+y-z$, $B=2y+z-x$ and $C=2z+x-y$, find the value of $A+B-2C$.

6. Simplify:—$(a+b)^2-(a-b)^2$.

6.

1. $ax^3+bx+cx^3+5+cx-bx^2-x^3$. Collect the coefficients of the powers of x and insert them in brackets.

2. Multiply $a^5-3a^4x+9a^3x^2-27a^2x^3+81ax^4-243x^5$ by $a+2x$ and divide the product by $a-3x$.

3. Expand $(a-2b+c)^2$.

4. If $y = -3x^2 + 2x - 5$, find the values of y when x has the values $1, -2, 0, 2, 3$.

5. If $x = \frac{1}{4}$, $y = \frac{1}{2}$, $z = \frac{3}{4}$, find the value of $2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4$.

6. Simplify:—

$$[-\{-1-(-1-\overline{1-1})-\overline{1-1}\}-\overline{1-1}].$$

7.

1. Multiply $3a^2 + 2ab - b^2$ by $a^2 - 2ab + b^2$ and divide the result by $a + b$.

2. Substitute $a - 2$ for x in $3x^2 + 2x + 1$ and arrange it in descending powers of x.

3. $y = \dfrac{5 - 4x}{3}$, find the values of y for the following values of x:—$\frac{1}{2}$, 0, $\frac{1}{3}$, $-\frac{1}{2}$.

4. Simplify:—

$$(4x+5)(4x-5) - (5x+4)^2 - (5x-4)(5x+4).$$

5. The area of a rectangular room is $(2x^2 + x - 1)$ sq. ft. and its length is $(2x - 1)$ ft. Find the breadth in yards.

6. Express x pounds y shillings as pence.

8.

1. If $y = 5 + x - 4x^2$, find the values of y when x has the values $-\frac{1}{2}$, 1, -2.

2. Divide $x^3 + 3x^2 + 5x + 3$ by $x^2 + 2x + 3$.

3. Substitute $a - 3$ for x in $4x^2 - 5x - 1$ and arrange the result in ascending powers of x.

4. Find the continued product of $a^2 - ab + b^2$, $a^2 + ab + b^2$, $a^4 - b^4$.

5. Simplify:—

$$(3x-2)(2x-3) + (2+3x)(3+2x).$$

6. Show that $(x+y)^2 - 2(x+y)(x-y) + (x-y)^2 = 4y^2$.

CHAPTER VI.

SIMPLE EQUATIONS.

57. There is a certain number. If 5 is added to twice that number the result is 35. What is that number ?

In answering this question the boy performs the following working mentally :—

35 is the final result. As this is obtained after adding 5 to twice the original number, twice the original number is got by subtracting 5 from 35. This 30 is two times the original number. Therefore the original number is got by dividing 30 by 2. The original number is therefore 15.

The original number may be supposed to be x.

Twice the number $= 2x$.

If 5 is added to this, we get $2x+5$.

The final result is $2x+5$, but by the question it is given to be 35, Therefore $2x+5$ and 35 are equal, this is denoted by writing $2x+5$ and 35 with a sign of equality $=$ between them as –

$$2x+5=35$$

We are now stating that the two quantities $2x+5$ and 35 are equal.

$2x+5=35$ is a statement of equality between the two expressions.

A statement that two expressions are equal is called an *equation*.

We know that when $x+4$ is multiplied by $x-4$ the product is x^2-16. This is generally written as—

$$(x+4)\,(x-4)=x^2-16.$$

This is also a statement that two expressions are equal. This may also be called an equation but a point of difference can be noticed between the truth of the two statements.

In the second case we have $(x+4)\,(x-4)=x^2-16$. The letter x may stand for any number. Even then the statement is true. For example.

Suppose $x=7$, 10, 8, we see that our statement "$(x+4)\,x-4)$ is equal to x^2-16" is still true as :—

(i) $(7+4)\,(7-4)=33$
$7^2-16=33.$

(ii) $(10+4)\,(10-4)=84.$
$10^2-16=84.$

(iii) $(8+4)\,(8+4)=48$
$8^2-16=48.$

Therefore the truth of this statement does not depend upon the value of x or the letter involved.

The statement is true for all values of the involved letter or letters, in other words $(x+4)\,(x-4)$ is only transformed as x^2-16 by the process of multiplication.

This statement of equality or equation is unconditional and does not depend on any particular value of x.

This is generally called an identical equation or simply an identity.

Again taking the equation

$2x+5=35$, we find we cannot give any value to x in order that $2x+5$, may be equal to 35. We can see that this statement is not true for any value of x, as in the previous case. If 3 is written for x, $2x+5$ becomes 11, but not 35. We also saw that this equation is equivalent to the arithmetical problem given above, and that there is only one answer to that question or only one number. So the statement that $2x+5=35$ is true only for one value of x, *i.e.*, when $x=15$

The statement $2x+5=35$ becomes an identity only when $x=15$

This equation is only a conditional identity.

This is known as an equation.

When the statement of equality is always true we use the sign $\equiv$ and when it is true under certain conditions, we use the sign $=$ to denote equality.

In the case of the equation when we are able to find the value of x, so that $2x+5=35$ may become an identity, we are said to *solve* the equation. That value of x is said to *satisfy* the equation. This value of x which satisfies the equation is called the *root* of the equation or *solution* of the equation. x is called the unknown quantity or the unknown. The two expressions connected by sign $=$ are called the sides.

The right hand side expression is for convenience denoted by R. H. S. and the left hand side expression by L. H. S.

If only the first power of the unknown quantity is involved in an equation, it is called a *Simple Equation.*

58. To solve the equation $2x+5=35$.

This means that we should find a certain value of x which when substituted for x, satisfies the equation or makes the equation an identity.

The process of solution is similar to the arithmetical solution of the original problem given at the beginning.

The No. 5 is added to $2x$ and therefore $2x+5$ is got and this is 35.

Therefore if 5 is taken away from $2x+5$ and also from 35, the result 30 should be equal to $2x$.

We find the value of $2x$ to be 30. To obtain x from this relation, we should divide $2x$ by 2 and also 30 by 2. Thus we get $x=15$.

This is represented as :—

$$2x+5=35.$$
$$\therefore 2x+5-5=35-5.$$
$$2x=30.$$
$$x=\tfrac{30}{2}=15.$$

Verification :—

$$2\times 15+5\equiv 35.$$

On obtaining the value of x, it must be verified whether this value makes the equation an identity or not.

59. It is seen then that the process depends upon the following important axioms.

(1) If same quantity is added to each side of an equation, the two sides are still equal.

(2) If same quantity is subtracted from each side, the two sides are still equal.

(3) If same quantity multiplies each side, the two sides are still equal.

(4) If same quantity divides each side, the two sides are still equal.

Ex. 2 :—Solve

$3x-11=25$.. .....A.

Adding 11 to both sides, we have

$3x-11+11=25+11$.

$3x=25+11$.....B.

$3x=36$.

Dividing each side by 3, we have

$x=12$.

Verification:— $3\times12-11\equiv25$.

In the course of the above solution, we find that 11 which appeared on the right hand side of the equation A appears on L. H. S. of B, but with its sign changed.

We, therefore, notice that a quantity can be made to change from one side of the equation to the other, by changing its sign. When a quantity is thus changed we are said to transpose it.

Ex. 3:— Solve

$3x+4=x+22$.

Transposing 4 to the other side, we have

$3x=x+22-4$.

Transposing x to the other side we have

$3x-x=22-4$.

$2x=18$.

$x=9$.

Verification:—

R. H. S. $=3\times9+4=31$.

L. H. S. $=9+22=31$.

The transpositions above may be made in one step. It should be the aim of the student to bring all the terms containing the unknown quantity to one side and all the known quantities to the other side and finally divest x the unknown quantity from all co-efficients other than unity.

Ex. 4. Solve:—

$(x+3)-(5x-2)=(2x+3)+(7-x)$

Simplifying we have

$x+3-5x+2=2x+3+7-x$

$-4x+5=x+10$

Transposing

$$-4x-x=10-5$$
$$-5x=5$$
$$x=-1$$

Verification

L. H. S $=(-1+3)-\{(5\times-1)-2\}$
$=2+7=9.$

R. H. S $=(2\times-1+3)+(7-\quad-1)$
$=+1+8=9.$

The student should make it a point to verify the answer he obtains after solving an equation. The beginners are apt to commit the very common mistake of finding the value of x in terms of x again, which therefore, still remains unknown.

Ex. 5. $\frac{x}{3}=7\frac{1}{5}.$

Here the co-efficient of x is a fraction, *viz.*, $\frac{1}{3}$. The equation may also be written as $\frac{1}{3}x=7.$

To obtain the value of x from the above, we should get rid of the fractional co-efficient and obtain 1 in its place. This can be done by multiplying it by 3. So we have.

$$\tfrac{1}{3}x\times3=7\tfrac{1}{5}\times3.$$
$$x=\tfrac{36}{5}\times3=\tfrac{108}{5}\text{ or }21\tfrac{3}{5}.$$

Ex. 6. Solve : $\frac{3x}{4}=4\frac{3}{8}.$

The co-efficient of x is $\frac{3}{4}$. So multiply the equation by $\frac{4}{3}$ to get the value of x.

$$\therefore \tfrac{3}{4}x\times\tfrac{4}{3}=4\tfrac{3}{8}\times\tfrac{4}{3}.$$
$$x=\tfrac{35}{8}\times\tfrac{4}{3}=\tfrac{35}{6}=5\tfrac{5}{6}.$$

Ex. 7.

$(x+2)(x+3)-(x+1)^2=(x+1)(x+2)-(x-4)^2.$

Expanding each side we have

$(x^2+5x+6)-(x^2+2x+1)=x^2+3x+2-(x^2-8x+16.)$

$$x^2+5x+6-x^2-2x-1=x^2+3x+2-x^2+8x-16.$$

Simplifying

$$3x+5=11x-14.$$

Transposing :—

$$\therefore 3x - 11x = -5 - 14.$$
$$-8x = -19.$$
$$8x = 19.$$
$$x = 2\tfrac{3}{8}.$$

Verification :—

$$\text{L. H. S} = (2\tfrac{3}{8} + 2)(2\tfrac{3}{8} + 3) - 2\tfrac{3}{8} + 1)^2.$$
$$= 4\tfrac{3}{8} \times 5\tfrac{3}{8} - (3\tfrac{3}{8})^2.$$
$$= \tfrac{35}{8} \times \tfrac{43}{8} - (\tfrac{27}{8})^2.$$
$$= \frac{1505}{64} - \frac{729}{64}.$$
$$= \frac{776}{64}.$$

$$\text{R. H. S} = (2\tfrac{3}{8} + 1)(2\tfrac{3}{8} + 2) - 2\tfrac{3}{8} - 4)^2$$
$$= 3\tfrac{3}{8} \times 4\tfrac{3}{8} - (1\tfrac{5}{8})^2$$
$$= \tfrac{27}{8} \times \tfrac{35}{8} - \tfrac{169}{64}$$
$$= \frac{945}{64} - \frac{169}{64}$$
$$= \frac{776}{64}.$$

EXERCISE 24.

1. Solve orally :—

(*a*) $2x = 20$. (*b*) $3x = 15$.
(*c*) $15x = 75$. (*d*) $20x = -80$.
(*e*) $9x = -81$. (*f*) $-5x = 10$.
(*g*) $-7x = 35$. (*h*) $12x = 48$.
(*h*) $2x = \frac{1}{5}$. (*l*) $3x = \frac{21}{50}$.

Solve :—

2. $x + 5 = 15$.
3. $x + 4 = 21$.
4. $x - 3 = 10$.
5. $x - 5 = 25$.
6. $2x + 3 = 13$.
7. $3x + 5 = 23$.
8. $4x + 1 = 20$.
9. $2x - 4 = 20$.
10. $3x - 6 = 15$.
11. $10x - 1 = 99$.
12. $x + 4x = 20$.
13. $5x - x = 16$.
14. $2x + 5x = 10 + 4$.
15. $10x + 2x = 68 - 8$.

16. $2x+3=x+17$.
17. $2x+7=10+x$.
18. $7x-6=6x-4$.
19. $6x+2=5x+4$.
20. $7x-3=5x+4$.
21. $8x-5=13-7x$.
22. $8x-15=2(x-4)$.
23. $9x-3-(4x+22)=0$.
24. $7x-4-(3x-11)=0$.
25. $21x-12=4(x-3)+3x+42$.
26. $\frac{x}{5}=10$.
27. $\frac{x}{7}=3$.
28. $\frac{2x}{3}=\frac{3}{8}$.
29. $\frac{3x}{7}=1$.
30. $\frac{7x}{3}=\frac{3}{7}$.
31. $\frac{4x}{5}=2\frac{1}{5}$.
32. $\frac{x}{12}=\frac{1}{2}$.
33. $\frac{x}{3}=\frac{1}{12}$
34. $\frac{3x}{5}=1\frac{3}{5}$.
35. $\frac{4x}{15}=0$.
36. $\frac{3x}{7}=\frac{3}{7}$.
37. $34+12=50+55x-185$.
38. $2x-5(1+x)-7=0$.
39. $5(x+1)+6(x+2)=9(x+3)$.
40. $7(x-1)+3(2x+1)=4(x+5)-(x+1)$.
41. $3(x-5)-8(x+2)=7(2x-5)-5(3x+3)$.
42. $15(2x-5)-10(3x-1)=20(3x+2)-15(4x-1)$.
43. $3(5x+15)-2(5x-7)=10(x-1)-5(x-2)$.
44. $(x-2)(x-6)=x(x-7)$.
45. $x(x+2)+x(x+1)=(2x-1)(x+3)$.
46. $(3x-1)^2+(4x+2)^2=(5x+2)^2+4$.
47. $(x+1)(x+3)=(x+3)(x+4)$.
48. $(x+1)(x+2)=(x-3)(x-4)$.
49. $(x+1)(x-2)+(x-3)(x-4)=2(x-7)(x-8)+116$.
50. $(x+2)(x+1)-(x+3)(x-5)=(x+3)(x-3)-x(x+5)$.
51. $(2x+1)x-2x(x+5)=x(3x+8)-3x(x+1)$.
52. $(x+3)(x+4)+(x+1)^2=2x(x+2)+5$.
53. $2x(x+5)-(x+4)(x+1)-(x-1)(x+3)=0$.
54. $2(x+10)(x+9)-3(x+5)(x+2)+(x+1)(x+2)=0$.
55. $(x-5)(x-3)=(x-1)^2$.
56. $(x+2)(x+7)=(x+3)^2$.

57. Find the value of x which makes the expression $\frac{1}{2}(x-4)-\frac{2}{3}(x+2)$ vanish.

58. Determine the value of x so that $\frac{2}{3}(x+10)-\frac{1}{3}(x-2)$ may exceed x by unity.

59. Find the value of x so that the sum of the expressions $\frac{1}{2}(3x+1)+\frac{3}{4}$ and $\frac{3}{4}x+\frac{1}{2}(4x-1)$ may be equal to 1.

60. Determine the value of x for which $\frac{-2x+5}{2}+\frac{x-3}{4}$ exceeds unity by $\frac{1}{3}$.

61. Find the value of x which makes $(8x-5)-(x-1)$ and $(3x-5)+3$ equal.

62. Find for what value of x the expressions $(3x-5)-x$ and $4x-(x+2)$ are equal.

63. Find the value of x which satisfies the equation $x(x+3)+x=x^2-(x-3)$.

64. What value of x will make $7(x+3)-4(x-1)+(x+1)$ equal to zero.

65. Find for what value of x the expression $5(x-1)-(6x+3)(x+5)$ vanishes.

60. *Equations involving fractions* :—We shall now deal with equations when the co-efficients of the unknown quantity are fractions.

Ex. I. Solve :—$\frac{x}{6}+\frac{x}{4}=2\frac{1}{3}$.

Here there are two terms containing the unknown quantity and their co-efficients are fractions. It is these denominators that give us trouble. So our object should be, first to get over this difficulty and make the co-efficients integral. In the previous examples where there is only one term involving the unknown quantity with a fractional co-efficient, we multiplied the equation by the denominators of x to get rid of it and convert the co-efficient into (one) 1. But in the present case, there are two different denominators and both must vanish by one operation. We must therefore choose a number such that the denominators 6 and 4 may disappear when the equation is multiplied by it. Such a number should therefore be a multiple of 6 and 4. There are always very many common multiples of 6 and 4,—of course the smallest or the least of these can be selected to simplify our work

Therefore the least common multiple of 6 and 4 should be taken and the equation should be multiplied by it

As the work with fractions is more difficult than with integers, we should make the denominator of the term that does not contain x also disappear by taking that also into consideration is selecting the number by which the equation is to be multiplied.

So all the denominators of the terms of the equation are 6, 4, 3 and their L. C. M. is 12.

Multiplying the equation by 12, we have

$$\frac{x}{6}\times 12+\frac{x}{4}\times 12=2\tfrac{1}{3}\times 12.$$

$$2x+3x \quad =28.$$

$$5x \quad =28.$$

$$x \quad =\frac{28}{3} \text{ or } 9\tfrac{1}{2}.$$

Ex. 2. Solve :— $\frac{x}{2}-\frac{x}{3}=\frac{x}{5}+1.$

The L. C. M. of the denominators is 30.

Multiplying each side of the equation by 30, we have $15x-10x=6x+30.$

By transposing

$$15x-10x-6x=30.$$

$$-x \quad =30.$$

$$x \quad =-30.$$

Ex. 3. Solve :— $\frac{x-1}{2}+\frac{2x+3}{3}=\frac{6x+19}{8}.$

Multiplying each side by 24, the L. C. M. of the denominators we have :—

$$24\times\frac{(x-1)}{2}+24\times\frac{(2x+3)}{3}=24\frac{(6x+19)}{8}.$$

$$\therefore 12\ (x-1)+8\ (2x+3)=3\ (6x+19).$$

$$\therefore 12x-12+16x+24=18x+57.$$

Transposing, we have,

$$12x+16x-18x=57+12-24$$

$$10x=45$$

$$x=\frac{45}{10}=4\tfrac{1}{2}.$$

Verification—

$$\text{L. H. S.} = \frac{4\frac{1}{2}-1}{2}+\frac{2\times 4\frac{1}{2}+3}{3}$$

$$= \frac{3\frac{1}{2}}{2}+4$$

$$= \frac{7}{4}+4$$

$$= \frac{23}{4} = 5\frac{3}{4}$$

$$\text{R. H. S.} = \frac{6\times 4\frac{1}{2}+19}{8}$$

$$= \frac{27+19}{8}$$

$$= \frac{46}{8} = 5\frac{3}{4}$$

This equation may be written also thus

$\frac{1}{2}(x-1)+\frac{1}{3}(2x+3)=\frac{1}{8}(6x+19)$.

Ex. 4:—$\frac{x+3}{2}-\frac{2x-5}{8}=\frac{2x+1}{3}-\frac{x-2}{6}$

Multiplying each term of the equation by 24, the L. C. M. of the denominators.

$\therefore 12(x+3)-3(2x-5)=8(2x+1)-4(x-2)$

$12x+36-6x+15=16x+8-4x+8$

Transposing,

$12x-6x-16x+4x=8+8-36-15$

$-6x=-35$

$x-\frac{35}{6}=5\frac{5}{6}$.

Ex. 5:—Solve—

$$\frac{\cdot 05-\cdot 01x}{\cdot 1}-(\cdot 03-\cdot 02x)=\cdot 03$$

Reducing the decimals to vulgar fractions, we find that the equation becomes,

$$\frac{\frac{5}{100}-\frac{1}{100}x}{\frac{1}{10}}-\left(\frac{3}{100}-\frac{2}{100}x\right)=\frac{9}{100}.$$

$$\therefore \frac{5}{10}-\frac{1}{10}x-\frac{3}{100}+\frac{2}{100}x=\frac{3}{100}.$$

Multiplying by 100, the L. C. M. of their denominators, we have—

$$50-10x-3+2x=3$$
$$-8x=3-50+3=-44$$
$$x=\frac{44}{8}=5\frac{1}{2}$$

6 The student should be warned against the very common mistake of omitting the multiplication of the one side of the equation by the L. C. M.

He must be made to clearly understand the sign—before $\frac{2x-5}{8}$, in ex. 4, affects the whole expression $2x-5$, just as it does, when it is written as $-\frac{1}{8}(2x-5)$. After the multiplying it by 24, we will have $-3(2x-5)$. In simplification, boys, without noticing this, are apt to change the sign of $2x\times 3$ which is immediately next to the negative sign and omit the change of the sign in the case of other terms. Boys should be warned against this.

Orderly arrangement and placing of the signs of equality in a column should be insisted on. They should also be cautioned against using $-$ meaninglessly before each step of the solution. Verification of the solution should be, as a rule, insisted upon in the case of beginners.

EXERCISE 25.

Solve:—

(1) $\frac{x}{5}=\frac{2}{15}$; (2) $\frac{4x}{7}=\frac{3}{14}$; (3) $\frac{x}{8}=\frac{5}{4}$

(4) $\frac{x}{4}=\frac{7}{10}$; (5) $\frac{2x}{3}=1\frac{2}{3}$; (6) $\frac{3x}{4}=1\frac{7}{8}$

(7) $\frac{5x}{8}=15$; (8) $\frac{3x}{11}=1$; (9) $\frac{x}{4}+\frac{x}{3}=4\frac{2}{3}$

(10) $\frac{x}{3}+\frac{x}{5}=1\frac{3}{5}$; (11) $\frac{x}{7}+\frac{x}{3}=10$

(12) $\frac{2x}{5}+\frac{x}{4}=1$; (13) $\frac{2x}{8}-\frac{x}{4}=2$

(14) $\frac{5x}{7}-\frac{10x}{13}=12$; (15) $\frac{x}{6}-\frac{5x}{9}=\frac{x}{6}+1$

(16) $\frac{2x}{15}=\frac{3x}{10}-\frac{x}{20}+1$;

(17) $\frac{x}{7}+4=\frac{2x}{3}-1$

(18) $\frac{13x}{2}-3=\frac{11x}{3}+2$;

(19) $\frac{1}{3}x+\frac{4}{5}x=\frac{2}{3}x-1$

(20) $\frac{2}{7}x-\frac{1}{7}=\frac{1}{3}x+\frac{1}{2}$;

(21) $\frac{5x}{4}-\frac{3x}{2}=\frac{3x}{4}+1$

(22) $\frac{3x}{20}-\frac{x}{4}=\frac{x}{2}+3\frac{1}{2}$;

(23) $\frac{2x}{3}+3\frac{1}{8}=\frac{4x}{9}+3$

(24) $\frac{x}{2}-\frac{1}{4}+\frac{x}{3}=\frac{1}{8}$;

(25) $\frac{3x}{4}+12=\frac{5x}{6}+9$;

(26) $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$.

(27) $\frac{x}{8}-\frac{2x-2}{5}=\frac{3x-4}{15}+\frac{x}{12}$.

(28) $\frac{x-1}{2}+\frac{x-2}{3}+\frac{x-3}{4}=\frac{5x-1}{6}$.

(29) $\frac{2x-6}{7}-\frac{x-4}{9}-\frac{3x}{13}=0$.

(30) $\frac{1}{4}(x-2)+\frac{1}{3}(x+2)=8-\frac{1}{2}(x-6)$.

(31) $\frac{2}{5}(x-5)-\frac{3}{7}(2x-9)=\frac{3}{4}(3x-21)-5\frac{1}{4}(x-11)$.

(32) $\frac{3}{8}(x+8)-\frac{1}{6}(2x+4)-2=0$.

(33) $\frac{9x+7}{4}-\frac{1}{2}\left(x-\frac{x-2}{7}\right)=18$.

(34) $6x+\frac{20-x}{5}=6x-2\left(\frac{5-8x}{3}+24\right)$

(35) $\frac{x+10}{3}-\frac{2}{5}(3x-4)+\frac{(3x-2)(2x-3)}{6}=x^2-\frac{2}{15}$

(36) $\frac{\cdot05-\cdot01x}{\cdot1}-(-\cdot03-\cdot02x)=\cdot03$.

(37) $\cdot03x+\cdot02=\cdot02x-\cdot06$.

(38) $\cdot07(x-10)-\cdot54=\cdot2(\cdot1-\cdot1x)-3(\cdot05-\cdot02x)$.

(39) $\cdot5x+\frac{\cdot45x-\cdot75}{\cdot6}=\frac{1\cdot2}{\cdot2}-\frac{\cdot3x-\cdot9}{\cdot2}$.

(40) $\frac{\cdot03x-\cdot01}{\cdot02}-\frac{\cdot02(x-1)}{\cdot03}=\frac{\cdot01x-\cdot03}{\cdot4}+\frac{\cdot21}{\cdot2}$.

(41) $\cdot011x+\frac{\cdot001x-\cdot125}{\cdot6}=\frac{5-x}{\cdot03}-\cdot145$.

CHAPTER VII.

SYMBOLIC REPRESENTATION AND FORMULAE.

62 To enable the student to solve arithmetical problems which may or may not admit of arithmetical solutions, he should be able to translate successfully the conditions of the problem into algebrical language. Certain problems which cannot be solved arithmetically can be very easily solved algebrically by reducing them to equivalent equations and solving them. Therefore when once the boy is able to form an equation that represents the problem, the solution becomes easy. For this purpose the student must be able to translate the conditions of a problem into algebraical language or symbols. Exercise of varied nature in this respect will enable a student to form equations which is the chief difficulty in solving problems. A few examples are given below to show how this can be done.

Ex. 1 If the sum of two numbers is a and one of them is b. Find the other number.

In such case when the student finds a difficulty he should think of a similar question in numerical numbers and notice what he does in that case.

When the sum of two numbers is 10 and one of them is 3, the number is 7. Here the operation of subtraction is suppressed. This must be explicit when we deal with algebraical letters.

Similarly b is the sum of the two numbers one of which is a

$\therefore$ The other number $= b - a$.

Ex. 2. The product of two quantities is x and one of them is y. Find the other.

By taking a numerical instance we see that the second number is got by dividing the product by the first number.

$\therefore$ The second number $= \frac{x}{y}$.

Ex. 3. A man travels at the rate of x miles per hour, how much will he travel in a hours and in what time does he travel y miles?

As said above, suppose numerical numbers in the place of the algebraical letters in the question and observe how the question is answered.

In 1 hour the man travels x miles

$\therefore a$ hours.....$a \times x$ or ax miles.

Again a man travels x miles in 1 hour.

$\therefore$ he travels 1 mile in $\frac{1}{x}$ hour.

$\therefore$ he travels y miles in $\frac{y}{x}$ hours.

Ex. 4. Find the excess of a over b.

The excess of 8 over 3 is $8-3$ or 5.

Similarly the excess of a over $b = a-b$.

Ex. 5 A boy is x years old now. What was his age a years ago and what will be his age in 3 years more?

His age a years ago was $x-a$ years.

His age in 3 years more will be $x+3$.

This can be seen by giving to the algebraical letters some numerical values and then mentally solving the problem.

Ex. 6. Find in pence the sum of a pounds, b shillings and c pence.

a pounds $= a \times 240$ or $240\,a$ pence.

b shillings $= b \times 12$ or $12\,b$ pence.

c pence $=$ c pence.

$\therefore a$ pounds $+ b$ shillings and c pence

$= (240\,a + 12\,b + c)$ pence.

EXERCISE 26.

1. The sum of two numbers is p; q is one; find the other.

2. The sum of two numbers is 25 and one of them is a. Find the other.

3. The difference of two numbers is 2 and the greater is x. Find the other?

4. The difference of two numbers is r and the smaller is 15. What is the other?

5. What is the excess of m over n?

6. The sum of three numbers is s, one number is a and another is b; what is the third?

7. What is the excess of c over $a+b$?

8. $q+p$ exceeds a number by 3. Find that number.

9. What quantity is less than $4a$ by $5c$?

10. If 15 is less than another number by x, what is that number?

11. The product of x and another number is y. Find the other number.

12. What quantity should be divided by x to get 4 as the quotient?

13. What quantity should be multiplied by 3 to get $12x$ as the product?

14. How many times is x contained in y?

15. How many time is c contained in $a+b$?

16. A boy is 10 years old: (1) what will be his age x years hence? (2) and what was it y years ago?

17. A boy is x years old. What will be his age 15 years hence?

18. My age exceeds the age of my brother by x years. If my present age is 30 years, what will be my brother's after a years?

19. A's age is x years and B's age is twice A's age and 2 years more. What is B's age?

20. B's age is x times A's age and A's age is 25 years. Find B's age.

21. v is an integer. What is the next above it and the next below it?

22. Write three consecutive numbers of which a is the smallest.

23. Write three consecutive numbers of which b is the middle one.

24. v is the greatest of four consecutive numbers. Find them.

25. $a+b$ is the smallest of four consecutive numbers. Write the four numbers.

26. There are three brothers; the age of the youngest is x years, the age of the eldest is twice that of the youngest and the age of the middle one is $\frac{1}{3}$ of that of the youngest and one year more. Express their ages.

27. A is four times as old as B. Express their ages 10 years hence if A's present age is x years and also express their ages five years ago.

28. The speed of a train is x miles per hour. How far does it travel in m hours?

29. A jutka ran a distance of x miles in 3 hours. At what rate did it run?

30. I went in a jutka which ran at a speed of 6 miles per hour to reach my destination in x hours. How far is my destination?

31. A ran 3 miles more per hour than B. If B's speed is x miles per hour, find the speed of A and the distance run by him in y hours

32. Two people start at the same time for a place and walk in opposite directions—one walks at p miles per hour and the other at q miles per hour. How far apart will they be in a hours?

33. The distance between two places A and B is 10 miles. Two persons P and Q start at the same time and run towards each other at x miles and y miles per hour respectively. Find the distance between them in 2 hours after they started.

34. Two men start together to walk in the same direction. The rate of the faster is x miles and that of the slower is y miles per hour. Find the distance between them at the end of (1) one hour, (2) 3 hours. (3) m hours after they have started.

35. In the above question, find, in how many hours after they have started, they would be c miles apart.

36. Two persons start together to walk at the rate of a and b miles per hour respectively. The distance between them at the time of starting is 4 miles and they walk in opposite directions. Find when they meet

37. Express a rupees, b annas and c pies in pies.

38. There are three bags one containing a sovereigns, the second b rupees and the third c shillings. Find the total value in pound sterling of the money in the three bags. (One Sovereign = Rs. 15).

39. Out of every dozen oranges m are good; how many good ones are there in (1) 84 oranges, (2) x oranges, (3) y dozen oranges?

40. Out of a purse containing x rupees and 4 annas I spend c pies. Find in pies the sum left.

41. A merchant mixes a lbs. of tea at 3 shillings per lb with b lbs. of tea at 4 shillings per lb. What is the price of the whole mixture and 1 lb. of the mixture.

42. If the cost of a books is 12 annas, find the cost in pies of (1) one book, (2) c books.

63. Further examples are given below :—

In the number 25, 2 is the digit in the tens' place and 5 in the units' place.

The local value of $2 = 2 \times 10$.

....$5 = 5 \times 1$,

$\therefore$ The number $2 \times 10 + 5 = 25$.

We notice that if the digits of a number are known, the number can be got by adding their *local values*.

In arithmetic the number is written by writing 2 and 5 side by side or in a particular order without using any sign of operation between them.

If 2 is the digit in the tens' place and a in the units' place we cannot write the number as $2a$ as this means $2 \times a$ or 2 times a. But the number is got by adding the local values of these thus $2 \times 10 + a = 20 + a$.

Ex. 1. If a, b, c are digits in the hundreds' place, tens' place and and units' place, respectively, find the number.

The local values should be added together

$\therefore$ the number $= 100a + 10b + c$.

Ex. 2. If a men can do a piece of work in p days, in how many days will b men do the same work?

$\therefore$ a men can do a piece of work in p days.

1 man ... .ap days.

b men..... $\frac{ap}{b}$ days.

Ex. 3. A man walks a distance of 10 miles in a hours at the rate of x miles an hour. Express this in the form of an equation.

In 1 hour the man walks x miles.

$\therefore$ in a hours.....ax miles.

But by the question the distance done by the man in a hours in the same time is 10 miles.

$\therefore \quad 10 = ax.$

Ex. 4. A's age is x years and B's age is three times A's. But four years ago B's age was four times A's. Express this in the form of an equation.

A's age $= x$ years.

B's age $= 3x$.

Four years ago A's age $= x - 4$.

and B's age $= 3x - 4$.

$\therefore \quad 3x - 4 = 4(x - 4)$.

EXERCISE 27.

1. Write down the number of two digits of which x is in tens' place and 4 in the units' place.

2. A number consists of three digits p, q and r which are in the hundreds,' tens' and units' places respectively. Find the number

3. A number consists of three digits of which x and y are those in the hundreds' and units' places, respectively. The middle digit is twice the sum of the first and last. Find the number

4. The digits in a number of 4 digits are four, consecutive numbers, the least of which is x and is in the units' place. Find the number.

5. x, y and z are the digits of a number in the hundreds' tens' and units' places, respectively. Write down the number with the digits reversed.

6. If m men are required to build a wall in t days, find how many are required to build a similar wall in p days.

7. A does a piece of work in a hours and B in b hours. Find in how many days they can together do it.

8. The average of m numbers is x, and that of n of them is y. Find the average of the remaining numbers.

9. The strength of a class is 38. The average mark in Elementary Mathematics of x of them is a, and of the whole class is b. Find that of the remaining boys.

10. The average age of one section of the first form of x boys is a years, that of the second section of y boys is b years and of the third section of z boys is c years. Find the average age of the boys of the three sections.

11. A man spends 10 shillings out of his purse containing x pounds and y shillings. One-fourth of the remaining sum is given away in charity. Find the sum in shillings left in the purse.

Express the following in the form of equations :—

12. The excess of a over b is c.

13. A man bought a camp cot for x rupees and sold it for y rupees thus gaining one-tenth of the cost price.

14. A man bought a oranges at x annas per dozen and b oranges at y annas per dozen. He sold each of them at z annas gaining thereby p annas.

15. A number x is added to 10 and the sum is multiplied by 5. The product is subtracted from 20, the difference is a.

16. The product of three consecutive numbers of which p is the least is a.

17. The sum of five consecutive numbers the greatest of which is x is y.

18. A number of two digits x and y is reversed and the product of the original number and the one inverted is 2·52.

19. x, y and z are the digits of a number in order from the hundreds' place. The number reversed is three times the original number.

20. x and y are the digits of a certain number from left to right; the number reversed exceeds the original number by d.

21. A's age is twice B's age which is x years. Ten years after, A's age will exceed B's by 10 years.

22. A boy is x years at present which is one-third of his father's age. 5 years ago he was one-fifth as old as B.

23. A and B start from the same place simultaneously to walk in opposite directions at the rate of x and y miles an hour, respectively. They are a miles apart at the end of t hours after they have started.

24. A and B start to walk in opposite directions from two places P and Q 2 miles apart at x and y miles an hour, respectively. It is found that they meet in t hours.

25. A walks at the rate of a miles per hour for m hours, and B at b miles an hour for n hours. It is found that B has then walked 3 miles more than A.

26. The excess of x over $a-b$ is equal to the sum of x and $c+d$.

27. A man spends annually at the rate of x rupees a week for 20 weeks; y rupees a week for 12 weeks and Rs. 6 a week for the remaining number of weeks, and finds that he has saved c rupees at the end of the year when his income per month is Rs. 100.

28. The difference between half and a third of a number x is greater than the sum of a fifth and a fourth of the number by y.

29. A man bought x eggs at three a penny and the same number of eggs at one-fourth of a penny each. The whole were sold at $3d$. with a gain of a pence.

USE OF FORMULAE.

64. We know that the area of a rectangle is given by the product of the number of units in the length and breadth. If A be the area l the length and b the breadth this statement is expressed as

$$A = lb. \quad (1)$$

From this relation we are able to find any one of three quantities when two are known.

Similarly we know that the area of a triangle is given by half the product of the base and height of the triangle and this is expressed as $A = \frac{1}{2}bh$ (2) where A, b and h denote the area, base and height of a triangle. Whatever may be the triangle, the above relation enables us to find out readily any of the three quantities when two are known. We have many other relations regarding the circle, cone, etc., as

$A = \pi r^2$ (3) where A and r are the area and the radius of a circle and

$V = \frac{1}{3}\pi r^2 h$ (4) where V, r and h denote the volume, radius of the base and height of a cone.

Such relations as the above are known as formulæ.

A formula may be stated as a general relation between certain quantities from which any one quantity can be found for particular values of the remaining quantities involved in the relation.

In other words a formula may be regarded as an equation when we are required to determine one of the quantities involved in it.

particular values of the remaining quantities being given. In all branches of mathematics we have to deal with formulæ very often and so a thorough knowledge of handling them is quite essential to a student of mathematics.

Ex. 1. Use formula (2), *viz*, $A = \frac{1}{2} bh$ to find the base of a triangle whose area is 40 sq. in. and the base is 10 in.

$$A = \tfrac{1}{2} bh.$$

Here $A = 40$ and $b = 10$. Substituting these values in the above formula we have $40 = \frac{1}{2} \times 10h$.

Now this becomes an equation, h being the unknown quantity

$$40 = 5h$$

$$\therefore \quad h = 8$$

$\therefore$ The required height $= 8$ in.

Ex 2. Find the height of a cone whose volume is 462 *c. in* and the radius of the base is 7 in.

We have the following formula for the volume of a cone.

$$V = \tfrac{1}{3} \pi r^2 h.$$

Here $V = 462$ and $r = 7$, substituting these values in the above formula we get an equation, h becoming the unknown quantity thus

$$462 = \tfrac{1}{3} \times \frac{22}{7} \times 7h$$

$$462 = \frac{154}{3} h$$

$$\therefore \quad h = \frac{462 \times 3}{154} = 9$$

$\therefore$ The required height $= 9$ in.

The spaces traversed by a moving body with a velocity of v ft. per second in t seconds is given by the formula $s = vt$ (5).

This can be used to find s, v or t where the remaining two quantities are given.

Ex. 3. Find the distance travelled in 3 hours, by a train whose speed or velocity is 60 ft. per second.

The above formula $s = vt$.

$\therefore$ Substituting these values we have

$$S = 60 \times 3 \times 60 \times 60$$

$$= 648000 \text{ ft.}$$

$$= \frac{648000}{1760 \times 3} \text{ miles.}$$

122 miles 1280 yds.

EXERCISE 28.

1. Use the formula $A = \frac{1}{2}bh$ the area of a triangle where A, b, h, denote respectively the area, base and height of a triangle to find

(i) the area, when the base is 10 in. and height is 15 in.

(ii) the base, when the area is 120 sq. ft., and height is 60 ft.

(iii) the height, when the area is 100 sq. ft and the base is 4 ft.

2. Use formula (3) given above to find.

(i) the area of a circle whose radius is 4."

(ii) the radius of a circle whose area is 154 sq. ft

(iii) the radius of a circle whose area is 120 sq cm

Correct to the nearest tenth of a c. m.

$(\pi = \frac{22}{7})$.

3. Use the formula $c = 2\pi r$ when c and r denote the circumference and radius of a circle to find

(i) c, when $r = 42$ in $(\pi = \frac{22}{7})$.

(ii) r, when $c = 94\frac{2}{7}$ ft. $(\pi = \frac{22}{7})$.

(iii) r, when $c = 65$ ft. correct to the decimal place when $\pi = 3{\cdot}14$

4. Use the formula $s = 4\pi r^2$ to find

(i) s, when $r = 2{\cdot}1$ in $(\pi = 3{\cdot}14)$.

(ii) r, when $s = 400$ sq. in. $(\pi = 3{\cdot}14)$ correct to the nearest inch.

5. Use formula (4) to find

(i) V when $r = 14''$ and $h = 24''$.

(ii) h when V = 5000 cubic feet and $h = 10$ ft.

(iii) r in feet when $v = 10{,}000$ c. yds. and $h = 20$ yds.

(iv) r in yards when V = 2,000 c. ft. and $h = 5$ ft.

(v) V in c inches when $r = 2$ ft. and $h = 3$ ft.

Answer correct to 1 decimal place.

6. Use formula (5) given above to find

(i) s when $v = 40$ ft. and $t = 30$ seconds.

(ii) s in miles when $v = 6$ miles $t = 20$ minutes.

(iii) v in feet a minute when $s = 300$ yds. and $t = 15$ ft.

(iv) v in feet per second when $s = 10$ miles and $t = 1$ hour.

(v) t in seconds when $v = 20$ miles an hour and $s = \frac{1}{2}$ a mile

7. If, when a number N is divided by D, we get the quotient Q and remainder R, we have the formula

$$N = Q \times D + R.$$

Use this to find

(i) N when $Q=25$, $D=17$ and $R=13$.

(ii) the number which when divided by 17·2 gives the quotient 13·1 and a remainder 10.

If a be the first term of a succession of n natural numbers and l the last term, the sum of all these numbers is obtained from the formula $s=\frac{n}{2}(a+l)$.

8. Use this formula to find

(i) the sum of the first 20 natural numbers.

(ii) the sum of the first 100 ,,

(iii) the sum of the 19 numbers 5, 6, 7, ... 23

(iv) the sum of the 100 numbers beginning with 27 and ending with 126.

If we consider the series of numbers 4, 7, 10, 13, 16, 19 we find that the difference between any two consecutive numbers is 3, *i. e.*, the difference is common throughout; each of the numbers in the series is called a term. The sum of all the terms is obtained by the formula.

$$s=\frac{n}{2}\left\{2a+(n-1)d\right\} \text{ where}$$

s is the sum, n the number of terms, a the first number and d the common difference.

9. Use the above formula to find

(i) the sum of the first 20 natural odd numbers.

(ii) the sum of the first 40 natural even numbers.

(iii) the sum of the first 93 natural numbers.

(iv) the sum of the series 10, 14, 18 . . . to 8 terms.

(v) the sum of the series 2, 7, 12, ... ,, 10 terms.

(vi) the sum of the series 3, 6, 9, ... ,, 21 terms.

10. Use the same formula to find.

(i) a when $s=460$, $n=20$ and $d=2$.

(ii) d when $s=875$, $n=25$ and $a=5$.

If a body falls under gravity g, the space s through which it falls in t seconds is given by the formula.

$$s=\tfrac{1}{2}gt^2 \text{ where } g=32$$

11. Use this formula to find the height from which a body falls if it strike the ground in 15 seconds.

12. Find the time a body takes to fall to the ground from a height of 1,936 ft.

13. The formula $I=\frac{Ptr}{100}$ gives the relation between the interest I, on the principal P at the rate of r per cent. per annum in t years.

Use this formula to find

(i) the interest on £365 for 3 years at 5 per cent. per annum

(ii) the interest on Rs. 475 for 4 years at 3 per cent. per annum

(iii) P in pounds when $I=$ £11-12, $r=4$, $t=2$.

(iv) P when $I=$ Rs. 28-14 as. $r=3\frac{1}{2}$, $t=3$.

(v) r when $I=$ £16-5s, $t=2$ and $P=325$.

14. The area of the four walls of a room is given by the formula $A=2H(L+B)$ where A denotes the area, H the height, L the length and B the breadth. Use this to find

(i) the area of the four walls of a room whose length is 18ft. breadth 15ft., height 12 ft.

(ii) the height when $A=1440$ sq. ft. $L=24$ ft. $B=16$ft.

(iii) the perimeter of the floor when $A=1240$ sq. yds. and $H=31$ yds.

CHAPTER VIII.

PROBLEMS ON SIMPLE EQUATIONS.

65 In this chapter the solution of problems will be dealt with. As said in the previous chapter, problems which may or may not admit of easy arithmetical solution may be readily solved algebraically. This may be done by framing an algebraical equation equivalent to the arithmetical problem and solving it.

66. A correct equation can be obtained only when the student is able to translate the conditions of an arithmetical problem accurately into the algebraical symbols or shorthand with the help of the principles treated in the previous chapters.

No general rule can however be given to enable a student to solve problems ; he should only depend upon his thorough acquaintance with the principles of the previous chapters. The following hints may however be found helpful.

(1) Represent the quantity to be determined by x. (It must be clearly stated what x is chosen to represent).

(2) Express the conditions of the problem in algebraical language, using x whenever the unknown quantity enters.

(3) Obtain two statements that are numerically equal from the conditions of the given question and translate this statement of equality into an equation.

(4) If in the problem one-half, one-third, one-fourth, etc., par of the unknown quantity is to be taken, then represent the unknown quantity by $2x$, $3x$, $4x$, etc., so as to avoid introduction of fractions into equations.

67. We shall work out some examples to serve as illustrations.

Ex. 1 There is a certain number, it is multiplied by 7 and to the product 20 is added, the result is 167. Find the number.

Let x represent the number.

When this is multiplied by 7 we get $7x$.

To the product 20 is added by the question.

So we have $7x + 20$.

This result is 167 by the question. We have, therefore, the equation.

$$7x+20=167$$
$$\therefore 7x = 167-20=147$$
$$x = 21$$

(Verification) $21\times 7+20=167$.

Ex. 2. The sum of two numbers is 28 and their difference is 4. Find the numbers.

Let x represent the greater number. Then according to the first condition, the other number is $28-x$.

By the second condition the difference *i.e.*, $x-(28-x)$ is 4.

So the equation is

$$x-(28-x)=4.$$
$$\therefore x-28+x=4.$$
$$2x = 4+28=32$$
$$x = 16$$

∴ The greater number $=16$,
and the smaller $=12$.

Or, a different equation can be framed and solved.

By the second condition the smaller number $=x-4$.

By the first condition the sum *i.e.*, $x+(x-4)$ is 28.

So the equation is

$$x+(x-4)=28$$
$$x+x-4=28$$
$$2x-4=28$$
$$2x=32$$
$$x=16$$

i.e. the greater number $=16$.
and the smaller ,, $=12$.

In either way we get at the correct answers. The student must make it a point to verify the correctness by substituting the answer obtained in the problem and see if the conditions given in the problem are satisfied. Time spent on verification is not at all time lost or wasted.

Ex. 3. The sum of two numbers is 30, the greater exceeds three times the less by 2 ; find the numbers.

Let x = the greater number ; then $30-x$ = the less.

It is given that x exceeds $3(30-x)$ by 2.

Hence we have

$$x - 3(30 - x) = 2$$
$$\therefore x - 90 + 3x = 2$$
$$\therefore 4x = 92$$
$$\therefore x = 23 \text{ (the greater).}$$
$$\text{and } 30 - x = 7 \text{ (the less.)}$$

Verification : 23 exceeds 3×7 by 2.

Ex 4. A father's age is twice as great as that of his son, but 10 years ago it was three times as great ; find the age of each.

Let x = son's age in years.

Then $2x$ = father's age in years.

Their ages 10 years ago were respectively

$$x - 10 \text{ and } 2x - 10.$$

Then by the question

$$2x - 10 = 3(x - 10)$$
$$\therefore 2x - 10 = 3x - 30$$
$$2x - 3x = 10 - 30$$
$$-x = -20$$
$$x = 20$$
$$\therefore 2x = 40.$$

$\therefore$ The father's age = 40, and the son's, 20.

Verification

Ten years ago their ages were 30 and 10.

i.e., the father's age was three times the son's age.

Ex. 5 Divide 19 guineas among A, B and C, so that A shall have twice as much as B, and C 5*s.* less than B.

Let x = B's share in shillings

then $2x$ = A's share . . . .

$x - 5$ = C's

By the question $x + 2x + x - 5 = 19 \times 21$

$$\therefore 4x - 5 = 399.$$
$$\therefore 4x = 404$$
$$x = 101s.$$

B's share = 101 shillings or £5—1*s.*

A's ,, = £ 10—2*s.*

C's ,, = £ 4—16*s.*

Verification :

$\equiv$ The 3 shares amount to £ 19-19*s* or guineas.

In this question we supposed x to represent the number of shillings ; therefore, x, $2x$, $x-5$ represent the number of shillings.

The sum of the three shares *i. e.*, $4x-5$ stands to represent the number of shillings. So it is wrong to equate this to 19

We should not say that

$4x-5=19$ as the expression on the left hand side denotes the number of shillings while 19 denotes the number of guineas.

$\therefore 4x-5=19\times 21$ is the correct equation as both the expressions denote equal numbers of common units.

Care must be taken to express every term in the same unit. It is only this omission that leads a beginner to incorrect equations and hence to inaccurate answers.

EXERCISE 29.

1. One number is four times another and their difference is 21. Find the numbers.

2. Find the number which exceeds 200 by the quantity by which 290 exceeds it.

3. Find the number which being multiplied by 15 and having 35 added to the product gives 215.

4. One-third of a number is as much less than 150 as one-half of the number is greater than 25. Find the number.

5. Find two numbers the difference of which is 14, and the greater number being divided by 4 is twice the smaller.

6. Find a number from which if 6 be taken and the remainder multiplied by 6 and then 6 added to the product, this sum divided by 3 shall give the required number.

7. I thought of a number, doubled it, then added 5. The result multiplied by 4, gave 84. Find the number.

8. Divide 150 into two parts so that three times the less may be equal to twice the greater.

9. Divide 70 into two parts such that three times the greater exceeds 60 by as much as twice the smaller is less than 118

10. Divide Rs. 90 between A and B so that twice A's share increased by Rs. 10 may be equal to twice B's diminished by Rs 10

11. Divide 15 guineas among A, B, & C such that A's share may be twice B's plus 15 shillings and B's share may be three times C's diminished by 10 shillings.

12. Find three consecutive numbers whose sum is 27.

13. Find five consecutive numbers whose sum is 35.

14. Find four consecutive numbers such that the sum of twice the smallest and the greatest may be equal to the sum of the remaining two increased by 12.

15. Find three consecutive even numbers such that the sum of three times the least and twice the greatest may be equal to two times the middle diminished by 76.

16. Find four consecutive odd numbers such that thrice the smallest number may be equal to the greatest.

17. Find three consecutive odd integers such that the square of the greatest exceeds the product of the remaining two by 46.

18. Find four numbers which increase by 5 such that the product of the first and the third may be equal to the product of the second and the fourth diminished by 95.

19. The ages of two men differ by 10 years; and 15 years ago the elder was just twice as old as the younger. Find their ages.

20. A father's age is four times that of his eldest son and five times that of his younger son; when the elder son has lived to three times his present age, the father's age will exceed twice that of his younger son by 3 years. Find their present ages.

21. A number consists of two digits whose sum is 7; five times the left digit is equal to two times the other digit. Find the number.

22. A number consists of two digits whose difference is 1; the smaller being in the tens' place. Five times the sum of the digits is equal to the number increased by 1. Find the number.

23. A number consists of two digits whose difference is 2, the greater being in the tens' place. If 18 be subtracted from the number, the result is the number reversed increased by the smaller digit. Find the number.

24. A number consists of two digits whose sum is 5; if 10 times the digit in the tens' place be added to 4 times the digit in the units' place the number will be inverted. What is the number?

25. The sum of an odd number and the third odd number after it is 16. Find the numbers.

26. Divide 180 into two parts such that $\frac{1}{4}$ of one part exceeds $\frac{1}{5}$ of the other by 9.

27. The sum of the third and fourth parts of a number is less than two-thirds of it by 4. Find the number.

28. The difference between two numbers is 5 and that of their squares is 115. Find the numbers.

29. The difference between the ninth and the sixth parts of a number is equal to the excess of half the number over 24. Find the number.

30. *A*'s age exceeds *B*'s age by 4 years and four-fifths. *A*'s age exceeds half of *B*'s age by 5 years. Find their ages.

68. We shall work below some more examples to serve as further illustrations.

Ex. 1. *A* and *B* have between them £22; After *A* gives *B* £3, two-thirds of *A*'s money is less than half of *B*'s money by 10s. How much has each at first?

Let x represent the number of pounds which *A* has at first.

Then *B*'s money $= (22 - x)$ pounds.

After £3 is given to *B* by *A*, his money $= x - 3$

B's . . . $= 22 - x + 3$

$= 25 - x$

$\therefore \frac{2}{3}(x - 3) = \frac{1}{2}(25 - x) - \frac{1}{2}$

Clearing fractions by multiplying the equation by the L. C. M. of the denominators we get

$4(x - 3) = 3(25 - x) - 3$

$4x - 12 = 75 - 3x - 3$

$4x + 3x = 75 - 3 + 12$

$7x = 84$

$x = 12$

$\therefore$ *A*'s money = £12

B's . . = £10.

Verification:—After *A* has given to B £ 3

A will have . . . £ 9

B . . . . . £ 13

$\therefore \frac{2}{3}$ of £ 9 . . . . = £6

$\frac{1}{2} \times 13 - \frac{1}{2}$ . . . . = £6

Note. In framing the equation we have written

$$\tfrac{3}{5}(x-3)=\tfrac{1}{2}(25-x)-\tfrac{1}{2}.$$

but not as $\frac{1}{2}25-x)-10$, 10 shillings is expressed as a fraction of a pound as all other quantities are expressed in £s. Care should therefore be taken to express all quantities in the same unit or denomination. It is here that beginners generally fail and are consequently led to confusion.

Ex. 2. A purse contains £1 13s 9d. made up of pence, shillings and half-crowns, but the number of half crowns is one-third the number of pence but only one-fourth the number of shillings. Find the number of coins of each sort.

(It is convenient to denote the number of half-crowns by x. In such questions the student should see which quantity can conveniently be represented by x.)

Let x = the number of half-crowns.

Then $3x$ = the number of pence.

and $4x$ = the number of shillings.

Expressing all these in the same denomination, *i.e.*, in shillings we have.

x half-crowns $= x \times 2\frac{1}{2} = \frac{5}{2}x$ shillings

$3x$ pence $= \frac{3x}{12} = \frac{1}{4}x$,,

$4x$ shillings $= 4x$,,

∴ The total value in shillings $= \frac{5}{2}x+\frac{1}{4}x+4x$.

But by the question the total value of the money in the purse $\}$ = £1. 13s. 9d.

Expressing this also in shillings we have it $\}$ = $33\frac{3}{4}$ shillings

∴ We have $\frac{5}{2}x+\frac{1}{4}x+4x=\frac{135}{4}$.

Clearing the equations of fractions we have

$$10x+x+16x=135$$

$$27x=135$$

$$x=5$$

∴ The number of half-crowns = 5.

,, ,, pence = 15.

,, ,, shillings = 20

Verification :—

5 half-crowns $= 5 \times 2\frac{1}{2} = 12\frac{1}{2}s.$

15 pence $= \frac{15}{12}s = 1\frac{1}{4}s,$

20 shillings $= 20s = 20s.$

$\therefore$ The total value $= 33\frac{3}{4}$ shillings.

Ex 3. A and B start at the same time to walk in opposite directions a distance of 15 miles between them and meet each other in 3 hours. If A walks at the rate of one mile more per hour than B, find their rates.

Let x = the number of miles A walks in one hour

$x-1=$ . . . . . B . . . . . .

In 3 hrs. A goes $3 \times x$ or $3x$ miles.

,, B ,, $3(x-1)=(3x-3)$ miles

They meet each other in 3 hours, *i. e.*, they walk together the distance between them *i.e.*, 15 miles.

$\therefore 3x+3(x-1)=15$

$3x+3x-3=15$

$6x-3=15$

$\therefore 6x=18$

$x=3$

$\therefore$ *A's* rate $=3$ miles per hour.

B's rate $=2$ miles per hour.

Verification :—In 3 hours. A walks 3×3 or 9 miles.

and B.....3×2 or 6 miles.

$\therefore$ They both walk 15 miles, hence they meet.

Ex. 4. A room is a square; but if the length is increased by 4 feet and the breadth diminished by 3, the area of the room remains the same. Find the area of the room.

(We find it is convenient to denote the said of the square by x, we can find the area when this is known. It is to be noticed that though the quantity required is the area, x is made to represent the length of the square. Sometimes we have to make x stand to represent a quantity from the quantity required in the question, and from which the required quantity can be readily found.)

Let x denote the length of the square in feet.

This is increased by 4 ft. $= x+4$.

Breadth is diminished by 3 ft. $= x-3$.

The statement is made in the question regarding the area of the room before and after the changes are made. So the area should be found out in both cases and equated.

$$x^2 = (x+4)(x-3)$$
$$x^2 = x^2 + x - 12.$$
$$\therefore -x = -12$$
$$x = 12.$$

∴ The length of the square = 12 ft.
∴ The area = 144 sq. ft.

Verification :—

The length after it is increased = 16 ft.
Breadth.....diminished = 9 ft.
Area .... = 144 sq. ft.
The area of the square..... = 144 sq. ft.

EXERCISE 30.

1. How many trees are there in a garden containing one-fifth mango trees, three-sevenths apple-trees and twenty-six trees of various other kinds ?

2. A house and a garden cost Rs. 650 and the price of the garden is equal to $\frac{1}{12}$ of the price of the house. Find the price of each.

3. Two persons A and B own a flock of sheep. They agree to divide its value. A takes 72 sheep while B takes 92 sheep and pays A £35. Required the value of a sheep.

4. Divide 19 guineas among 3 persons, so that the first shall have twice as much as the second, and the third 5s. less than the second.

5. There is a certain number composed of two figures of which the figure in the units' place is 3 times that in the tens', and if 36 be added to the number, the sum is expressed by the same digits reversed. What is the number ?

6. A number consists of two digits, the sum of the digits is 8; and if the left digit be diminished by 2 it will be equal to $\frac{1}{7}$ of the number. Find the number.

7. A post is a fourth of its length in the mud, a third of its length in the water and 10 ft. above the water. What is its length ?

8. A is twice as old as B and four years older than C. The sum of the ages of A, B and C is 96 years. Find B's age.

9. From two towns 561 miles apart two men start, one from each, at the same time; one goes 24 and the other 27 miles a day: in how many days will they meet?

10. Two trains which travel at the rate of 24 miles and 20 miles an hour respectively start at the same time from two places P and Q and run towards each other. If they meet in 2 hours 15 minutes after they start, find the distance between P and Q.

11. A man whose age is 40 has a son who is 9 years old. When will the father be twice as old as his son?

12. Divide Rs. 15,200 among A, B and C so that B shall have Rs. 1,000 more than A, and C Rs. 2,700 more than B.

13. Two men start at the same time from two places A and B, 49 miles apart. One walks at the rate of 4 miles and meets the other coming in the opposite direction in 7 hours. Find the rate of the other.

14. A sum of £5 6*s*. is made up of sovereigns, half-crowns, and six pences. There are twice as many half-crowns as there are sovereigns and thrice as many six-pences as there are sovereigns. Find the number of coins of each kind.

15. A purse contains rupees, two anna coins and one anna coins amounting in value to Rs. 29-6 as. The number of rupees is one-third the number of two anna coins and two-thirds of the number of one anna coins. Find the number of coins of each sort.

16. A person has 45 coins consisting of sovereigns and crowns amounting in value to £15. How many coins has he of each kind?

17. A sum of £4 6*s*. is made up of sovereigns, three-times as many crowns and four times as many florins. Find the total number of coins.

18. A and B have together £14; if A gives B one-fourth of his money, B will have now what A had at first. Find what money each has at first.

19. A has Rs. 4 more than B. If Rs. 10 from A's money were given away to B, B would have then $\frac{3}{2}$ times what A would have. Find B's money.

20. A began to play a game with B with twice as much money as B had. A gained Rs. 4 and found that he had 3 times as much as B then had. Find with what amount each began the game.

21. A room is 3 ft. longer than it is broad. If the length and breadth are each diminished by 2 ft. the area of the room will be diminished by 50 sq. ft. Find the length and breadth of the room.

22. The floor of a room is a perfect square; if the length is increased by 5 ft. and the breadth diminished by 4 ft. the area remains unaltered. Find the area of the floor of the room.

23. The length of a room is twice its breadth; if the length is diminished by 5 ft. and the breadth increased by 5 ft. the area exceeds the original area by 25 sq. ft. Find the area of the room.

24. The length of a field is twice its breadth; another field which is 50 yards longer and 10 yards broader, contains 6,800 square yards more than the former. Find the dimensions of each.

25. The length of a room exceeds its breadth by 3 ft.; if the length had been increased by 3 ft. and the breadth diminished by 2 ft. the area would not have been altered. Find the dimensions.

26. *A* does a piece of work in 3 days and *B* in 4 days. Find in what time they can together do it.

27. A person bought a certain number of tables and twice as many chairs with Rs. 115. If the cost of each chair is Rs. 5-8-0 and that of each table is Rs. 12, find the number of chairs bought.

28. A man bought 20 books consisting of copies of Algebra and Geometry with a total sum of Rs. 26. If each copy of Algebra costs Re 1-4-0 and each copy of Geometry Rs. 1-8-0, find the number of copies of Geometry bought.

29. A carriage, horse and harness are together worth £120; the carriage is worth three-fourths of the horse's value and the harness one-seventh of the value of the carriage and horse together. Find the value of each.

30. A person bought a certain number of eggs at 15 for a shilling and sold them at 14*d.* per dozen gaining thereby 22*d.* Find the number of eggs.

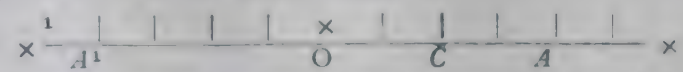
31. A boy says to his friend "I am 18 years older than you and in two years I shall be twice as old as you." What are their ages?

32. In 10 years a father will be $2\frac{1}{2}$ times as old as his son whose present age is one-fourth of his father's age. Find their ages.

CHAPTER IX.

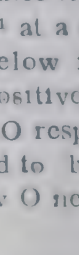
CO-ORDINATES.

69. Take a straight line $x^1 O x$ and a point O on it. We can mark a point on it at any given distance

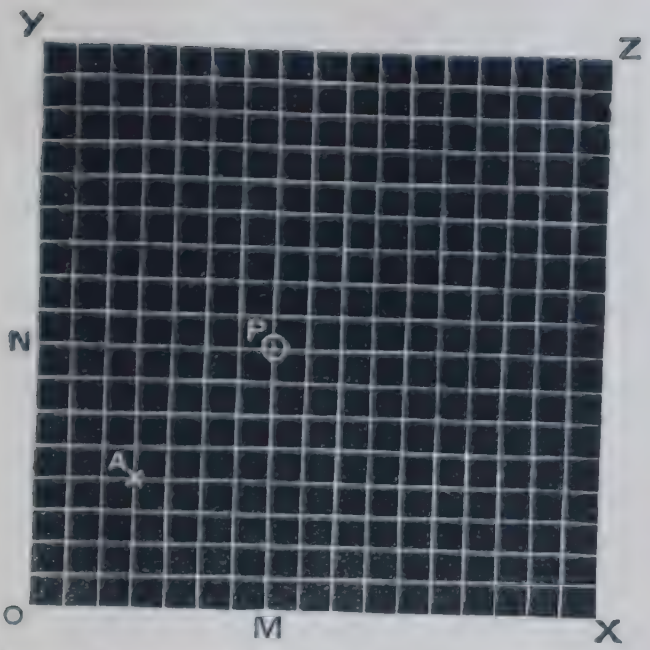


from it. Suppose we wish to mark a point on the straight line at a distance of 4 steps from O. It can be noticed that we can do so in two ways, as we can measure 4 steps either to the right or left of O. Thus we get two points A, A^1 as shown in the figure. But suppose we wish to mark a point 4 steps from O to its right then we get only one point, *viz*, A and if it is to the left, then also we get one point. Therefore if it is our object that only one point should be marked, we should state not only its distance from a fixed point, but also its direction. It was already mentioned in chapter III that the distances to the right of O are considered to be positive and those to the left negative. Therefore instead of saying that A is 4 steps or units from O to its right, we can use + sign before 4 and omit "to its right". C is therefore + 2 steps from O. Similarly A^1 may be said to be − 4 steps from O using − sign before 4 and omitting "to its left".

70. Take a vertical straight line YOY^1. As above we can mark a point on this straight line if its distance from O is given, and we get here also two points B and B^1 at a distance of 5 steps from O one above it and the other below it. It is convenient here as in the previous case to use positive and negative signs to denote distances above and below O respectively. Generally distances above O are considered to be positive and are denoted by the sign + and those below O negative and denoted by the sign −.



Thus B is + 5 steps and B^1 is − 5 steps from O. So we find that points can be located on any straight line, if their distances from any point on that line are known with the direction. The point from which distances are measured is called the starting point or the *origin*



The distance of the point from this origin is called the co-ordinate of that point.

71. We shall see how we can locate a point on a plane.

Suppose you are required to fix the position of a point on the page of your note book. This can be done if its distances from two adjacent edges are known. We find that it is not enough if the distance of the point from only one edge is known, as we can mark a very large number of points which are at a given distance say 5'' from the lower edge.

Let the rectangle $Y\ O\ X\ Z$ represent a page of the note book. Let OX & OY represent the edges from which distances are measured to fix the position of any point. Let each division of the lines represent an inch.

The points that are at a distance of 5'' from $O\ Y$ are on a straight line parallel to $O\ Y$ drawn at a distance of 3'' and all points that are at a distance of 4'' from $O\ X$ lie on the parallel to $O\ X$ at a distance 4'' from $O\ X$. These two parallels meet at A. This point is at a distance of 4'' or 4 units from $O\ X$ and 3'' or 3 units from $O\ Y$. Similarly any

point may be marked on the page if such two distances are given and conversely these distances may be found out when the position of a point is given. Unless these two distances are known, the position of A cannot be fixed. Unlike in the case of fixing a point on a line, we require two measurements or two co-ordinates to fix the position of a point on a plane. We can also view that A is reached by going + 4 units from O along $O\,X$ and then + 3 units vertically. However two co-ordinate distances are quite necessary to fix the position of a point.

72. Instead of taking the edges of the page, we can take two straight lines $X'\,O\,X$ and $Y\,O\,Y'$ at right angles to each other in the plane of the given page and fix the position of a point, its distances from these lines with proper signs being given.

These two straight lines that are at right angles to each other are called the lines or axes of reference or axes to which the measurements are referred and the horizontal which is generally denoted by $X'\,O\,X$ is called the axis of x or x axis and the vertical line $Y\,O\,Y'$ the axis of y or y axis. As was said previously, distances measured along $O\,X$ to the right of O are positive and to the left negative and those measured parallel to $O\,Y$ in the upward direction from O are positive and in the downward direction negative. The two distances are called the co-ordinates. The measurement in the direction parallel to the x axis is called the x co-ordinate and the parallel to the

y axis y co-ordinate. In the above figure we know that the point A is 3 units in a direction parallel to the x axis and 4 in the direction parallel to the y axis.

3 is called the x co-drdinate and 4 is called the y co-ordinate of A. Generally we omit to state that 3 is the x co-ordinate and 4 is the y co-ordinate of A, but only write 3, 4 to denote it, always taking care to write the x co-ordinate first and the y co-ordinate next. The order of writing the co-ordinates is very important as the point (4, 3) means a different point altogether. The x co-ordinate is also known as *abscissa* and the y co-ordinate, the *ordinate* of the point.

Considering the point P on the plane, we find
$O\,M$ = the co-ordinate on the x axis.
$O\,N$ = " " y axis.

If $O\,M$ and $O\,N$ are denoted by x and y respectively the point P is spoken of as the point (x, y).

It may be viewed that $P\,N$ and $P\,M$ to be the co-ordinates of P since $O\,M = P\,N$ and $O\,N = P\,M$, and therefore the x co-ordinate of a point may be considered as its distance from the y axis and y co-ordinate its distance from the x axis.

The four compartments into which the page is divided by the axes of references, are called the quadrants.

Y O X is called the first quadrant,
$Y\,O\,X^1$ " second "
$X^1\,O\,Y^1$ " third "
$Y^1\,O\,X$ " fourth "

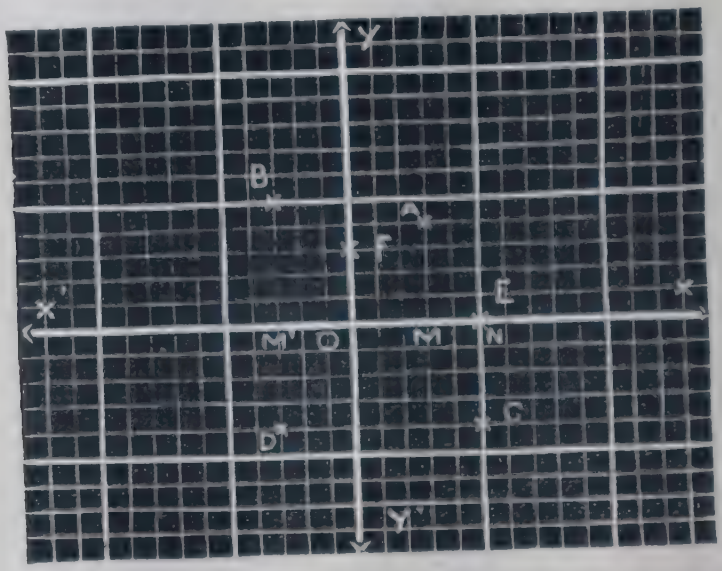
In the first quadrant x co-ordinate is $+e$ & y co-ordinate is $+e$
" second " " $-e$ " $+e$
" third " " $-e$ " $-e$
" fourth " " $+e$ " $-e$

73. To determine the position of a point whose co-ordinates are 6, 3, we have to draw two parallels one to the Y axis at a distance of 6 units (inch or half inch) and the other parallel to the x axis at distance of 3 units from it. Thhe point of intersection is the required position of the point. Or we can go a distance of 6 units from O along the x axis and from there go vertically a distance of 3 units to

reach the required point. To determine more points similar procedure is to be adopted. So for this purpose we have to draw a number of straight lines parallel to the axes of reference. We find it very convenient to use paper ruled into squares at very small distances for this purpose. Such paper is called the squared paper, which gives us parallel lines ready for our use.

74 The fixing of the position of a point when the co-ordinates are given is called the plotting of the point.

Ex. 1. Plot the points (3, 4), (−3, 5), (5, −4) (−3, −4), (5, 0), (0, 3).



(1) Let the unit be one small division. Measure O M = 3 units on the x axis and along the parallel through M to y axis, measure M A = 4 units. A is the point (3, 4).

(2) Measure O M = 3 units on O X' since there is negative sign before 3. Measure M' B = 5 units on the parallel through M' B is the point (−3, 5).

(3) Measure O N = 5 units on the x axis. N C = 4 units on the parallel through N to the y axis in the downward direction as there is a negative sign to the co-ordinate. C is the required point.

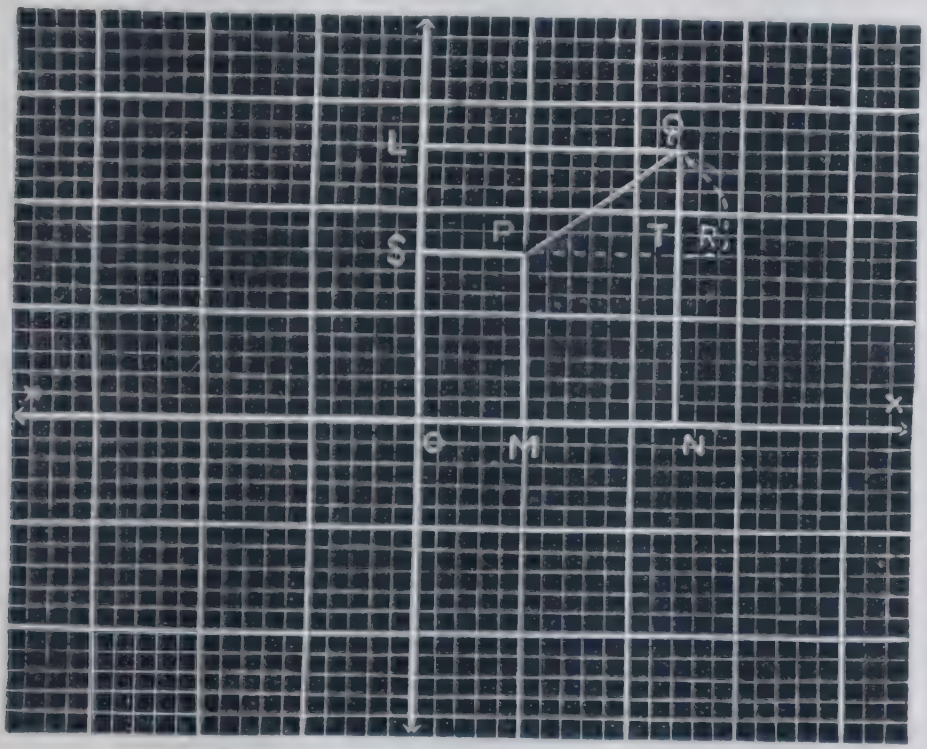
(4) Measure O M' = 3 units on O X' as the X co-ordinate is −3 and M' D = 4 units on the vertical through M' in a downward direction. Then D is the point (−3, −4).

(5) Measure O E = 5 units on the x axis and zero units on the vertical through E. ∴ E is the required point (5, 0)

(6) Measure 3 units on the x axis from O. Through this point, *i.e.*, O measure O F, 3 units on the vertical upwards, *i.e.*, on the y axis.

75 The following points should be noticed :

(1) **The x co-ordinate of all points on the y axis is 0.**
(2) **The y co-ordinate of all points on the x axis is 0.**
(3) **The x co-ordinate and y co-ordinate of the origin are (0. 0.)**



Ex. 2. Plot the points (5, 8) and (12, 13) find the length of the line joining them by calculation and by observation

Take two straight lines X'OX and YOY at right-angles to each other as the axes of reference. Plot the points (5, 8) and (12, 13) as shown above taking each small division or ·1 of an inch as the unit and call them P & Q.

Join P Q with the centre P and radius P Q describe an arc cutting P R a parallel drawn through P to the x axis ; then count the number of units in P. R.

∴ P R = (8·6) (approx).

Or thus

Draw the perpendiculars from P & Q to OX; PM & Q N are the ordinates of these points.

These are to be by the question 8 & 13.

$\therefore$ Q T = 13 − 8 = 5

P T = Q L − P S

= The abscissa of Q—the abscissa of P

= 12 − 5

= 7.

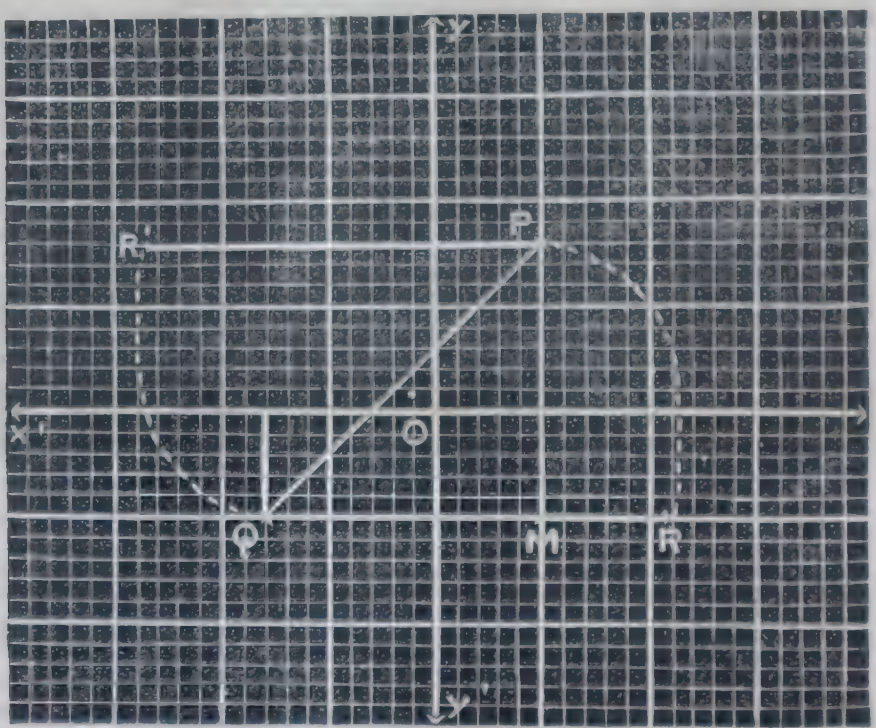
and $PQ^2 = PT^2 + RT^2$

$= 5^2 + 7^2$

$= 25 + 49$

$= 74$

$PQ = \sqrt{74} = 8{\cdot}6$ (approx).



Ex 3. Find the length of the straight line joining the points (5, 8) and (−8, −5).

Take X OX and YOY' two straight lines as the axes of reference and plot the points as in the above case and denote them by P & Q. Join P Q.

With P or Q as centre and P Q as radius describe an arc of a circle cutting P R' or Q R at R' or R the parallel from P or Q to the x axis. Measure the numbers of units on Q R or P R' without taking the direction into consideration.

The length of P Q = 18·5 (approx).

Note — The arcs may be cut on the parallel to y axis also.

or thus,

Obtain the right-angled triangle P M Q of which P Q is the hypotenuse as shown in the figure.

P M = the ordinate of P + the ordinate of Q

= 8 + 5 (the sign need not be taken into consideration).

= 13.

Q M = the abscissa of P + the abscissa of Q

= 5 + 8 (the signs before the co-ordinates need not be taken into consideration)

= 13.

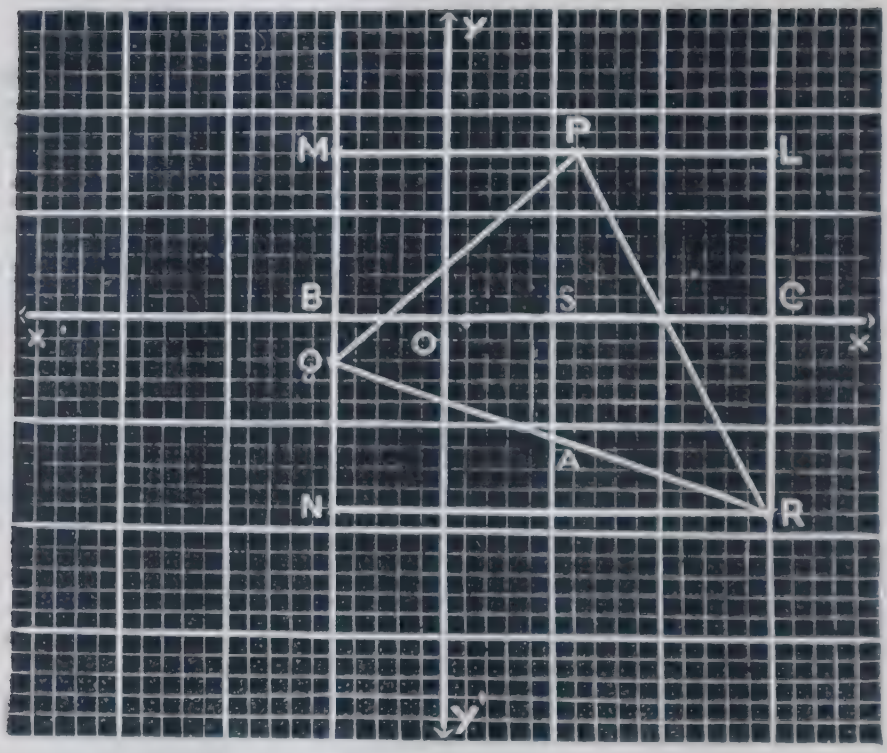
$PQ^2 = PM^2 + QM^2$

$= 13^2 + 13^2$

$= 169 + 169$

$= 338$

$\therefore PQ = \sqrt{338} = 18{\cdot}5$ (approx.)



Ex. 4. Find the area of the triangle whose vertices are (6, 8), (- 5, − 2), (15, − 9).

Taking the axes of reference as before plot the three points with ·1 of an inch as unit and denote them by P, Q, R as shown in the figure. Join them and obtain the triangle P Q R. Count up the number of squares, neglecting fractions of them if they are less than half and counting as complete squares if they are half or above half.

Thus the number of squares whose side is the unit taken, enclosed by the triangle is 139 square units nearly.

If the unit is an inch, the area is so many square inches, if it is a cm, it is so many sq. cm., etc.

Second Method:—The area may also be found by drawing the perpendicular from one of the points to the opposite side and finding the length of the perpendicular and the base.

The area of the $\triangle = \frac{1}{2}$ P M & Q R
$= 138\frac{1}{2}$ sq. units.

Third Method:—Obtain the trapezium Q R L M as shown in the figure.

The area of the trapezium Q R L M $= \frac{1}{2}$ L M (MQ + LR)
$= \frac{1}{2}$ NR {(ordinate of P + ordinate of Q) + ordinate of P + ordinate of R)}
$= \frac{1}{2} \times 20\ (8+2+8+9)$
$= 10 \times 27 = 270$ sq. units.

The area of the right-angled triangle PMQ $= \frac{1}{2}$ P M × Q M
$= \frac{1}{2} \times 11 \times 10 = 55$ sq. units

.....$\triangle$ P L R
$= \frac{1}{2}$ P L × R L
$= \frac{1}{2}\ (15-6)\ (8+9)$
$= \frac{1}{2} \times 9 \times 17$
$= \frac{153}{2} = 76{\cdot}5.$

$\therefore$ The area of the $\triangle$ P Q R = the area of the trapezium QRLM
− the area of the QPM$\triangle$ − the area of the $\triangle$ R L P
$= 270 - 55 - 76\frac{1}{2}$
$= 138\frac{1}{2}$ sq. units.

Fourth Method:—The area of the rectangle N R L M may be calculated and from it the area of the right-angled triangles R L P, P M Q & N Q R may be subtracted.

Ex. 5. Find the co-ordinates of the middle point A of Q R in the figure of the previous question.

Draw A S $\perp$ to XX'. Then AS is the ordinate and SO the abscissa of A.

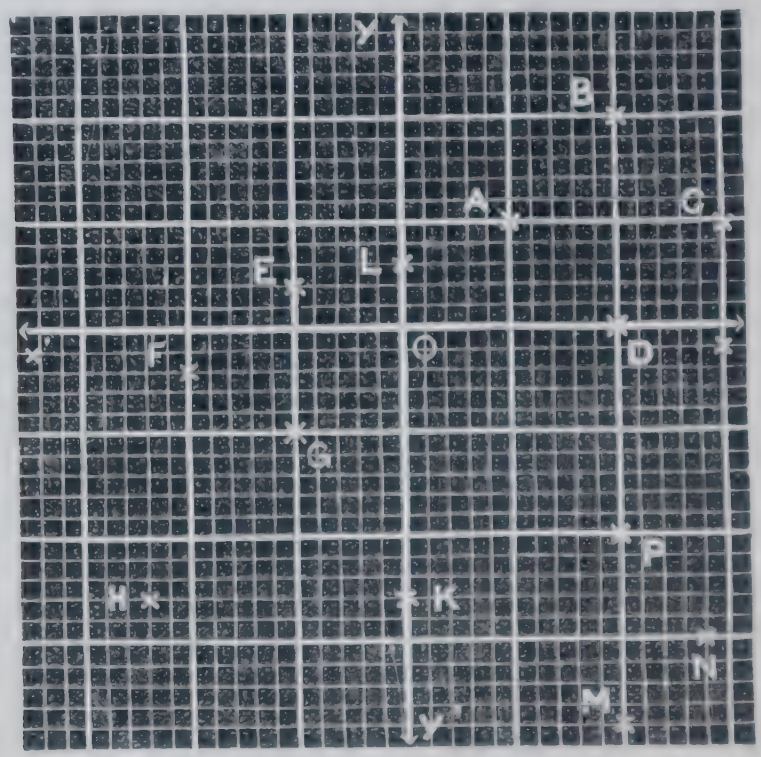
SO = S B − OB
$= \frac{1}{2}$ BC − OB
$= \frac{1}{2}$ (BO + OC) − OB
$= \frac{1}{2}$ (OC − OB)
$= \frac{1}{2}\ (15-5)$
$= 5$

$\because$ RQBC is a trapezium.

AS $= \frac{1}{2}$ (BQ + CR)
$= \frac{1}{2}\ (2+9)$
$= 5\frac{1}{2}$
$\therefore$ A is $(5, -5\frac{1}{2})$

EXERCISE 37.

1. Taking 0·1 of an inch as unit write down the co-ordinates of A, B, C.....P in the adjoining diagram.



2. Taking ·5 of an inch as unit write down the co-ordinates of A, B, C, D, G, H, N in the same figure.

3. Taking 1" as unit write down the co-ordinates of A, B, C, D, L, E, K, H, N, P in the same figure.

4. Plot the following points :

(2, 2) (5, − 8), (10, − 1), (− 9, 3), (− 15 − 15),
(0, 8), (8, 0), (− 20, 0), (0, − 15), (− 7, − 7),
(− 12, 6), (− 7, − 10), (0 − 12), (7 − 2),

5. Write down the co-ordinates of the points of inter-section of the straight lines joining the following pairs of points :—

(i) $(-8, 12)$ and $(7, -14)$ and $(10, 3)$ and $(-5, -9)$

(ii) $(6, 6)$, $(-8, 3)$ and $(-10, -8)$, $(-7, 7)$

(iii) $(-9, 5)$, $(5, 3)$ and $(12, 12)$, $(0, 0)$.

6. Find the co-ordinates of the middle point of the lines obtained by joining each of the following points with the origin.

$(12, 15)$, $(-18, 10)$, $(-15, -15)$.

7. Find the co-ordinates of the middle point of the lines joining each of the following pairs of points :—

(i) $(7, 8)$ and $(20, 22)$

(ii) $(5, 5)$ and $(-10, -10)$.

(iii) $(10, 10)$ and $(-10, 5)$.

8. Plot the points whose co-ordinates are given in the following table:—

$x =$	0	1	2	-4	-6	3	-6	-9
$y =$	5	7	-3	$+8$	$+12$	-4	-7	0

9. Taking 1 inch as unit, plot the following points :—

(i) $(4, 1)$, $(1, 0)$, $(0, 1)$, $(2, 2)$ $(-2, 0)$, $(3, -1)$, $(-1, 3)$

(ii) $(1{\cdot}5, {\cdot}5)$, $(2{\cdot}7, -1{\cdot}2)$, $(-2{\cdot}5, 1{\cdot}7)$, $(-{\cdot}5, -2)$, $(2{\cdot}3, -1)$

(iii) $(1\frac{1}{3}, 2\frac{1}{2})$, $(-2\frac{1}{4}, -3)$, $(\frac{1}{2}, -\frac{1}{2})$, $(\frac{3}{5}, \frac{1}{5})$, $(-\frac{3}{5}, -\frac{2}{5})$

(iv) $(1{\cdot}6, -2{\cdot}8)$, $(-2{\cdot}1, {\cdot}8)$, $(-3{\cdot}2, 1{\cdot}5)$, $(1{\cdot}3, -2{\cdot}5)$.

10. Taking 0·5 of an inch as unit plot the points :

(i) $(3, 4)$, $(-1, 2)$, $(-5, -5)$, $(3, -6)$, $(-2, -2)$.

(ii) $(-1{\cdot}2, 0)$, $(-{\cdot}6, -8)$, $(-2{\cdot}4, 3{\cdot}2)$, $(0, 2{\cdot}6)$ $(3{\cdot}3, 1{\cdot}5)$.

(iii) $(\frac{4}{5}, 1\frac{3}{5})$, $(0, -2\frac{1}{5})$, $(2\frac{3}{5}, 0)$, $(-2\frac{1}{2}, -1\frac{2}{5})$, $(3, -2\frac{1}{2})$.

11. Plot the following points on the same diagram: $(4, 2)$, $(4, 10)$, $(4, -5)$ $(4, 0)$, $(4, -8)$, $(4, -12)$.

What peculiarity do you observe about them ?

12. Plot the following points on the same diagram and notice the peculiarity about them. $(0, -5)$, $(2, -5)$, $(3, -5)$, $(8, -5)$, $(10, -5)$ $(-15, -5)$. $(-8, -5)$, $(-12, -5)$.

13. Find the lengths of the sides of the triangle whose vertices are (8, 6), (10, – 5), and (– 7, 10).

14. Show that the following points are the corners of a square (8, 12), (8, 8), (– 18, 12), (– 12, 8).

15. Find the area of the triangle whose vertices are :

(i) (12, 0), (0, 5), (0, 0).

(ii) (15, 0), (10, 8), (10, 0).

(iii) (– 8, 0), (– 8, – 6), (5, 0).

16. Find the areas of the triangles whose vertices are :

(9, 7), (– 4, – 3), (13, – 11).

17. Find the area of the trapezium whose vertices are :

(i) (6, 5), (18, 10), (6, 0), (18, 0).

(ii) (4, 10), (– 9, 6), (– 9, 0), (4, 0).

18. Find the area of the trapezium whose vertices are :

(4, 8), (– 6, 2), (– 6, 6), (4, – 6).

19. Find the perimeter of the triangle whose vertices are the points (1, 4), (5, 12), (13, 3).

20. A man walks 4 miles due west and then 3 miles due south. How far will he have walked if he has to reach a point 5 miles due east of the starting point?

21. Plot the points (13, 0), (12, 5), (5, 12), (0, 13) and show that they lie on the circumference of a circle whose centre is the origin. Find its radius.

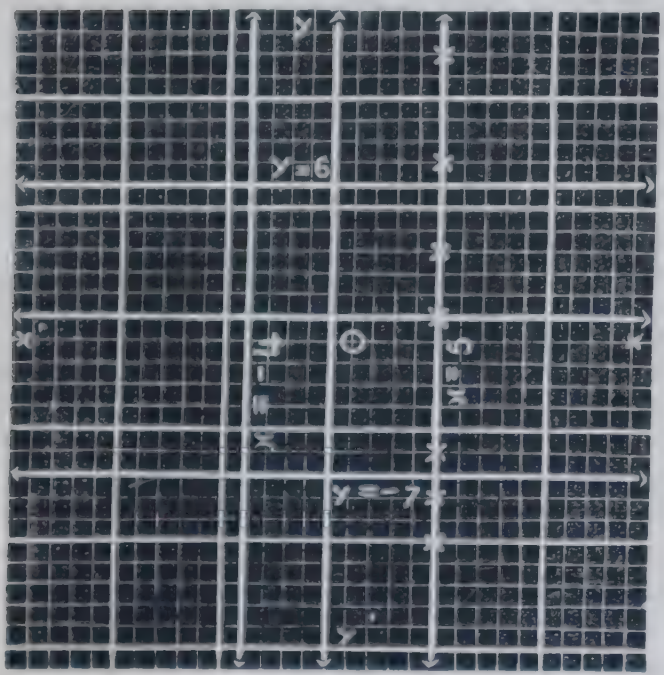
22. Find the length of the perpendicular drawn from (3, 4) on the line joining the points (6, 5) and (10, 13).

CHAPTER X.

THE STRAIGHT LINE GRAPH.

76. Plot the points whose x co-ordinates and the corresponding y co-ordinates are arranged in rows as in the following table taking 0·1 of an inch as unit (See figure below) :—

x	5	5	5	5	5	5	5
y	0	3	7	12	−10	−6	−8



and join them. We notice that all of them lie on a straight line parallel to the y axis at a distance of 5 units from it to its right.

The peculiarity noticed about these points is that the x co-ordinate of every point is the same, *viz.* 5. Taking any other point on the parallel we notice that its x co-ordinate is also 5. So every point on this straight line has its x co-ordinate equal to 5. If we take any point outside the parallel we see that its x co-ordinate is not 5 but some other quantity.

Therefore summing up what we have observed we find that every point on the parallel straight line has its x co-ordinate equal to 5, and no point outside it has its x co-ordinate $= 5$.

This straight line is called the *locus* of the points whose x co-ordinate $= 5$ and $x = 5$ is the equation of the straight line. The straight line is called the graph of the equation.

77. The *graph* of an equation is the line, such that every point on it satisfies the relation given by the equation regarding its co-ordinates.

Similarly if we take $x = -4$ into consideration we get a straight-line parallel to the y axis at a distance of 4 units to its left by plotting a number of points whose x co-ordinates $= -4$, and joining them.

78. Generally the equation $x = a$ represents therefore a straight line parallel to the y axis. It will be to its right or left according as a is positive or negative.

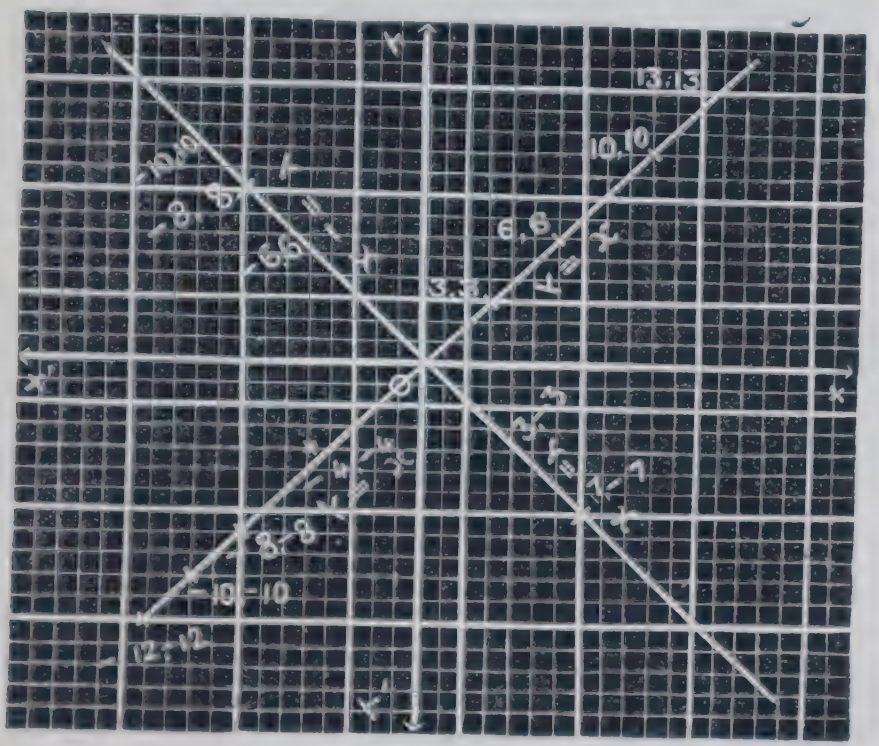
79. Again, the y co-ordinate of a point is said to be its distance from the x axis. If we plot a number of points whose y co-ordinate $=$ 6 and join them, we get a straight line parallel to the x axis at a distance of 6 units above it. We notice that every point on it and no other point has its $y = 6$. Hence this straight line represents the equation $y = 6$ or is the *graph* of the equation $y = 6$.

Similarly $y = -7$ represents a straight line parallel to the x axis at a distance of 7 units below it. We see, therefore, that $y = b$ represents a straight line parallel to the x axis ; it is above or below it according as b is a positive or a negative quantity.

80. Every point on the x axis and no other has $y = 0$ and every point on the y axis but no other has $x = 0$; therefore $y = 0$ represents the x axis and $x = 0$, the y axis.

81 Plot the points whose co-ordinates are given in the following table (see the figure given below) and join them; evidently all of them lie on a straight line :—

x	3	6	10	13	−4	−8	−10	−12
y	3	6	10	13	−4	−8	−10	−12



It may be readily seen that the y co-ordinate of every one of these points = the x co-ordinate. As in the previous case it may also be observed that the y co-ordinate of any other point on it taken at random = x co-ordinate of the same and no other point outside it has its $y = x$.

Since the condition satisfied by every point on the line and by no other point is $y=x$, $y=x$ is the equation of the line. Conversely the line thus drawn is such that every point on it satisfies the relation given by $y=x$, and is called the *graph* of the equation $y=x$.

82. Again plot the points given by the following table :—

x	3	7	−6	−8	+15	−12	−10
y	−3	−7	6	8	−15	12	10

We find that these points also lie on a straight line; we also notice that the co-ordinates of every point are numerically equal but are of opposite signs. It is therefore evident that the x and y co-ordinates of the points on this line satisfy the equation $y=-x$ but not of any other point. Therefore $y=-x$ is the equation of the line, and the line is the graph of the equation $y=-x$.

Let us next find the graph of $y=2x$. As shown in the previous case make a table of values of x and the corresponding values of y thus:

when

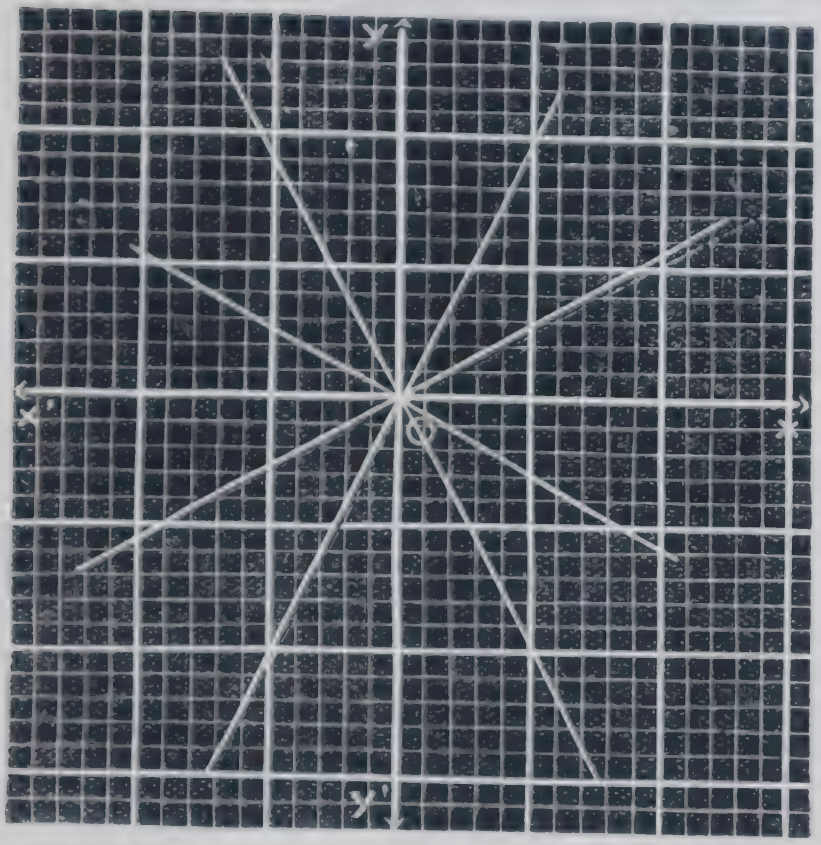
$x=$	0	1	3	6	−4	−3	−5
$y=$	0	2	6	12	−8	−6	−10

Plot the points given by the table, *viz.*, (0,0), (1, 2), (3, 6) . . . . . and join them.

Evidently a straight line is got and the student may satisfy himself that the co-ordinates of any point taken on it at random and of no other point satisfy the equation. Therefore the line thus obtained is the required graph of the equation $y=2x$.

83. In a similar manner we can find the graph of the following equations :—

(i) (*a*) $y+2x=0$ (*b*) $y-2x=0$.
(ii) $2y-x=0$
(iii) $2y+x=0$
(iv) $3y+4x=0$



(i) (*a*) $y+2x=0$ (*b*) $y-2x=0$

$y=-2x$ $y=2x$.

when

$x=$	0	1	2	−3	−2	4
$y=$	0	−2	−4	6	+4	−8

when

$x=$	0	1	3	−2
$y=$	0	2	6	−4

(ii) $2y = x$

$y = \frac{1}{2}x$

when

$x =$	0	1	2	3	4	. . . . .
$y =$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	. . . . .

(iii) $2y + x = 0$

$2y = -x$

$y = -\frac{1}{2}x$

when

$x =$	0	1	2	-4	. . . .
$y =$	0	$-\frac{1}{2}$	-1	2	. . . .

(iv) $3y + 4x = 0$

$3y = -4x$

$y = -\frac{4}{3}x.$

when

$x =$	0	3	6	-3	-6	. . . .
$y =$	0	-4	-8	$+4$	$+4$	. . . .

It is advisable to give such values to x as would give us integral values of y, when the equation involves fractions.

In each of the above cases we may notice that all the points given by each of the above tables all lie on a straight line. The unit taken in each case is 0·1 of an inch. We may take a larger unit, if the values in the table are small.

84. It is to be noticed that every one of the equations treated above involves only terms containing x and y without involving one independent of x or y. These equations may be put in the general form, $y = mx$ where m may be fractional or integral and positive or negative. It may also be further noticed that in each equation, when $x = 0$, y is equal to o. Therefore the values of the co-ordinates, o, o, satisfy the equation. The point whose co-ordinates are o, o, *i.e.*, the origin is a point on the line.

Therefore it may be said from what we have observed that $y = mx$ represents a straight line and also that it passes through the origin.

EXERCISE 32.

1. Draw the graph of $x = a$ for values of a, $3, -2, 4, 10, -5, -10$ with reference to the same axes and using the same unit.

2. Draw the graphs of $y = a$ for values a, $4, -1, 5, -3, 6, -6$ with reference to the same axes and with same unit.

3. Draw in the same diagram and using the same unit the graphs of

(i) $y = 3x$, (ii) $y + 3x = 0$ and state whether the point (10, 20) is on (i) and (−6, 18) is a point on (ii).

4. Draw on the same diagram using the same unit the graphs of

(i) $2y + 5x = 0$. (ii) $2y - 5x = 0$.

(iii) $5y + 2x = 0$. (iv) $5y - 2x = 0$.

and state on which of the lines represented by the above, the points (10, 25), (−5, −2), (−10, 25) lie.

85. *Variables*:—Certain quantities will be changing in value without being constant, *e.g.*, the population of a town, the temperature of atmosphere, etc.

One quantity is sometimes related to another in such a way that a change in value of the one produces a corresponding change in the other. Such quantities are said to be directly proportional to each

other ; for example, suppose the cost of a certain number of books is something, then if the number of books is doubled, the cost is doubled, if the number is trebled the cost is trebled and so on The value of the expression $x+2$ depends on the value of x which can take a number of values. If $x+2$ is denoted by y, then for every value of x, there will be a corresponding value of y; and these may be exhibited in the following table for the sake of clearness:—

when	$x=$	0	1	2	-1	-2	. . . .
	$y=$	2	3	4	$+1$	0	. . . .

All the above quantities, *viz.*, the population of a town, the temperature, the cost of books, etc., are called *variable* quantities or *variables* as their values are changing. Therefore x also is a variable and hence y, whose values depend on the value of x which is a variable, is also a variable. The expression $x+2$, is an expression in the variable x, and the second term 2 is called the *constant*. The expression in the variable x is also called a *function of* x

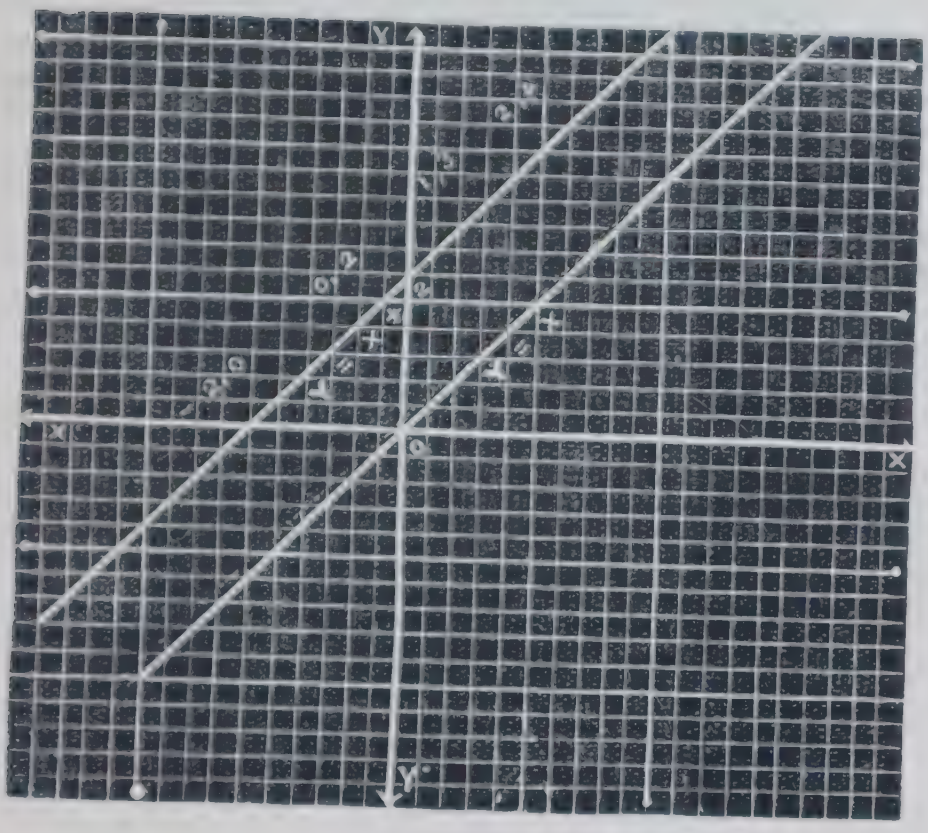
86. To draw the graph of the equation $y=x+2$.

Give a series of values to x and obtain the corresponding values of y from the given equation, as shown in the table:—

when	$x=$	0	1	2	-2	-3	. . .
	$y=$	2	3	4	0	-1	. . . .

Exhibit these changes by means of points. That is, plot the points (0, 2), (1, 3), (2, 4,) (-2, 0,) (-3, -1).....and join them.

[The unit chosen is ·3 of an inch]



A straight line is obtained as shown in the figure. The correctness of the graph can be verified by taking any point at random on it and seeing whether the co-ordinates of that point are related to each other by the equation (or satisfy the equation) or not.

In the same diagram the graph of $y = x$ is also drawn with the same unit. The y in this question, $= x$ while the y in the question $y = x + 2$ is as much as that in the first *plus* 2. That is the ordinate of every point on the second graph is 2 units more than the ordinate of every point on the first graph. It can therefore be

observed that the two straight lines are parallel to each other and the second graph, *i.e.*, $y = x + 2$ can be got by producing the ordinate of any point on the first line to 2 units more and drawing a parallel to it through the extremity. Or more easily by taking 2 units on the y axis from the origin (since the origin also is a point on the graph of $y = x$ and the y co-ordinate of the origin is O) and by drawing parallel through that point. The constant term of the equation $y = x + 2$, is equal to the length of the intercept on the y axis, which can be got by putting $x = 0$, and the intercept on the x axis can be found by putting $y = 0$ in the equation. As two points are quite enough to draw a straight line, these intercepts can be found and the points joined. But to ensure greater accuracy of the graph more points will have to be found, through which the straight line passes.

87. It may also be observed that it will not be easy to plot these points of intersection of the graph on the axes of reference particularly when these intercepts are so large that the size of the squared paper used does not permit marking them. Therefore the method of obtaining a series of values of x and y as done in this case is to be adopted preferably.

2. To draw the graph of the equation—

$$2y - 4x + 3 = 0$$

Write the equation as

$$2y = 4x - 3$$

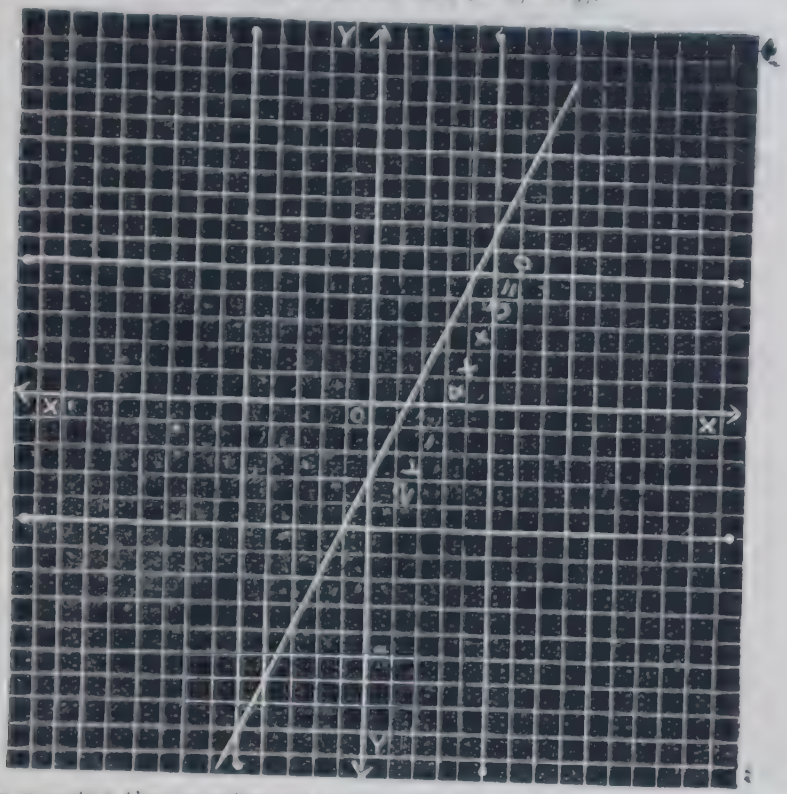
$$y = 2x - \tfrac{3}{2}.$$

By this arrangement the values of y corresponding to the various values of x can be found more readily than otherwise.

when

$x =$	0	1	2	-3	...
$y =$	$-\frac{3}{2}$	$\frac{1}{2}$	$2\frac{1}{2}$	$-7\frac{1}{2}$	...

Plot the points $(0, -\frac{3}{4})$, $(1, \frac{1}{2})$, $(2, 2\frac{1}{2})$, $(-3, -7\frac{1}{4})$.



The greater the number of points that are plotted, the more accurate will the graph be.

83. To draw the graph of the function $\frac{3x+2}{4}$ the value of the function depends on the value of x.

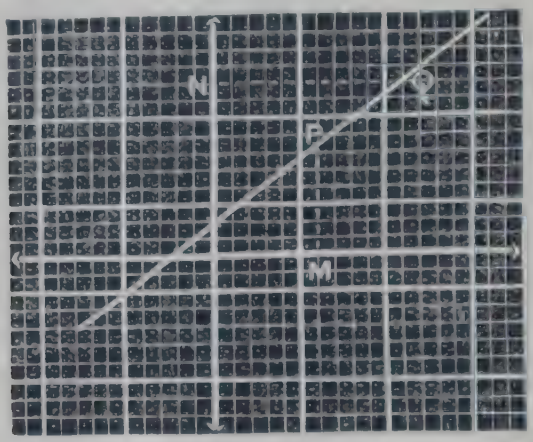
Let $y = \frac{3x+2}{4}$.

Give a series of values to x and find the corresponding values of y or the function as shown in the table :—

when $x =$	0	1	2	4	-2	-4	. . .
$y =$	$\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{14}{4}$	-1	$-\frac{10}{4}$	. . .

Here we find that the values of x involve fractions, so we will have to accurately know one-half, one-fourth of a unit, etc.

Therefore "1" may be chosen as the unit. Plot the points given by the table, we find that they all lie on a straight line which is the graph of the function $\frac{3x+2}{4}$.



This graph may be used to find out the values of y or the functions for a given value of x. Since these two values of x and y are to be related to each other as shown by the equation $y = \frac{3x+2}{4}$, they must be the co-ordinates of a point on the graph. Therefore when the value of x is given, it means the x co-ordinate of the point is given and its y co-ordinate can be found out. For example, suppose $x = 1\frac{1}{2}$, find the values of y or the function. Mark a point M at $1\frac{1}{2}$ units on the x axis from the origin. This length is the x co-ordinate of a point on the line. To find the position of that point erect at the point a perpendicular M P to the x axis to meet the graph. The point at which the perpendicular cuts the graph is the required point, the length of this perpendicular is its ordinate or the y co-ordinate and this represents the value of y or the function when $x = 1\frac{1}{2}$.

M P $= 1\frac{1}{2}$ (approx).

This can be verified by calculation. Similarly when the value of x is given, the corresponding value of y can be found.

Suppose $y = 2\frac{1}{2}$.

Then on the y axis mark a point N at a distance of $2\frac{1}{2}$ units from the origin. At N erect a perpendicular to the y axis N Q to meet the graph at Q. Then measure the number of units in N Q. This gives the value of x corresponding to $y = 2\frac{1}{2}$.

$\therefore$ N Q = 3 (approx). or $2\frac{2}{3}$.

89 To find graphically the values of x and y which will satisfy both the equations.

(i) $2y - x = 3$.

(ii) $3x + 5y = 13$.

Write the equations thus :

$$y = \frac{x+3}{2} \quad \ldots \ldots \quad (1)$$

$$y = \frac{-3x+13}{5} \quad \ldots \ldots \quad (2)$$

Prepare tables of values of x and y from each of the equations thus :

From (1)

when

$x =$	0	1	2	-1
$y =$	$1\frac{1}{2}$	2	$2\frac{1}{2}$	1

From (2)

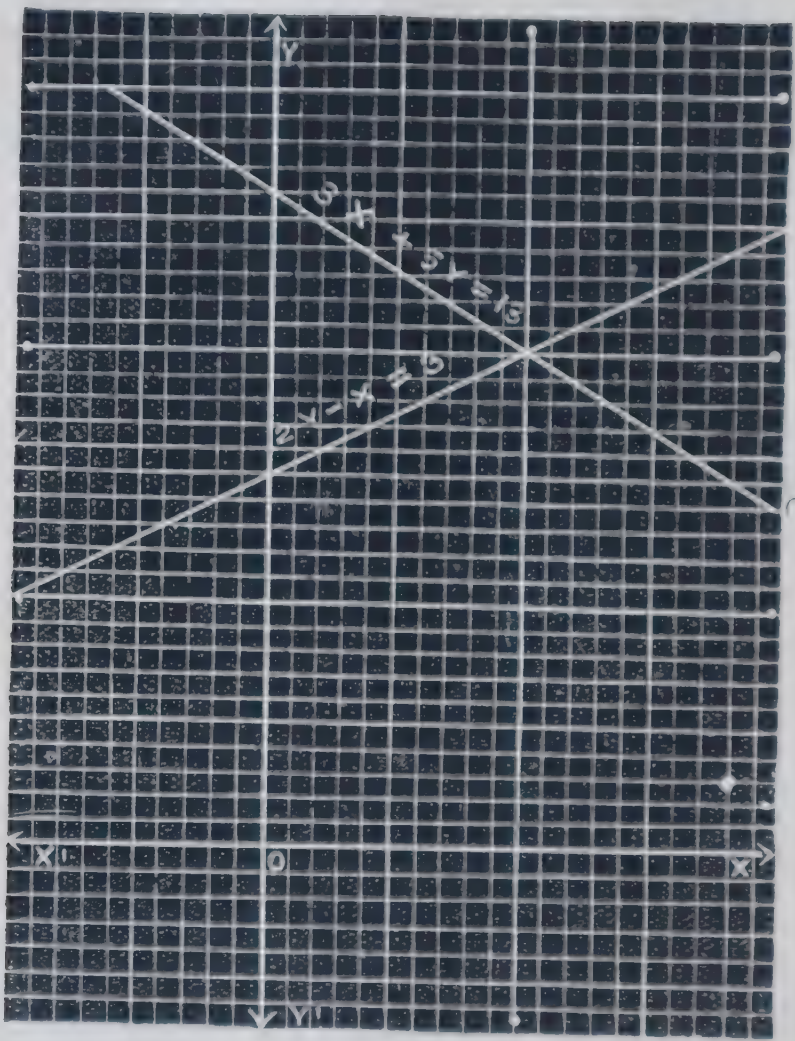
when

$x =$	0	1	2	-1
$y =$	$\frac{13}{5}$	2	$\frac{7}{5}$	$\frac{16}{5}$

Let us take one inch as our unit and plot the graphs as shown in the figure below. The two straight lines intersect at P. Since P is on the graph of the equation (1) its co-ordinates satisfy the equation (1) and as it is also on the graph of equation (2) also its co-ordinates satisfy equation (2).

Therefore the co-ordinates of P satisfy the two equations simultaneously. The co-ordinates of the point are 1, 2.

Therefore the values of x and y that satisfy both the equations are 1, 2.



It must be remembered that the unit for both the graphs should be the same ; and the unit chosen should be as large as possible.

Hence to find the values of x, y which satisfy two equations of the first degree simultaneously we should draw the graphs of the equations and read off the co-ordinates of the point of intersection. These co-ordinates will be the required values of x and y.

90. If it is required to solve graphically a simple equation in one unknown, *e.g.*

$\frac{2x+3}{2} = \frac{4x+2}{3}$ we can proceed thus.

Put $y = \frac{2x+3}{2}$ . . . . (1).

$\therefore \quad y = \frac{4x+2}{3}$ . . . . (2).

As in the previous case, the graph of each equation should be drawn using the same unit and the point of intersection of the two lines should be marked. As the value of x alone should be found, the x co-ordinate of the point of intersection should be read off. It is this value of x that satisfies the given equation. The value obtained graphically may be verified by algebraical solution.

91. The procedure for drawing the graph of an equation may, for the sake of clearness, be summarised as follows :—

(1) Put the equation in the form $y = mx + c$ (keep y on the left hand side and remove all other terms to the other side. Particularly see that the co-efficient of y is unity).

(2) Give a series of values to x and find the corresponding values of y from the equation Arrange them in the form of a table separately.

(3) Plot the points whose co-ordinates are given by the table.

(4) Join them by means of a foot-rule.

92. The various points that have been observed during the course of the foregoing chapters are given below to enable the students to have a full grasp of all the points.

(1) $x = a$ represents a straight line parallel to the y axis at a distance of a units from it.

$x = 0$ represents the y axis as the x co-ordinate of every point on it is zero.

If a is positive the line will be in the first and fourth quadrants and if negative, in the second and third quadrants.

(2) $y=b$ represents a straight line parallel to the x axis at a distance of b units from it.

$y=0$ represents the x axis, as the y co-ordinate of every point on it is zero

If b is positive, the line lies in the first and the second quadrants and if negative, in the third and fourth quadrants.

(3) $y=mx$ represents a straight line passing through the origin.

(Notice the equation does not contain a term free from x and y and therefore (0, 0) the co-ordinates of the origin satisfy the equation.)

If m is positive, the line lies in the first and the third quadrants and if negative, in the second and fourth quadrants.

m is called the gradient or slope of the line.

(3) $y=mx+c$ represents a straight line which does not pass through the origin.

If the co-ordinates of the origin O, O be substituted for x and y in the equation it is not satisfied. Therefore the origin is not a point on the graph.

As m denotes the slope of the graph, it is also observed that $y=mx+c$ is parallel to $y=mx$, making an intercept of c units on the y axis.

93. To get the intercept on y axis cut by any graph of an equation put $x=0$ and find the value of y; and the intercept on the x axis is found by putting $y=0$ and finding the value of x. The graph may be obtained by finding the points of intersection of the graph with the axes by putting x and y equal to zero in order, and joining them, as two points are quite enough to determine a straight line. But this is not always an easy method to adopt.

94. As it has been observed that every equation of the first degree in the two variables x and y can always be put in the form $y=mx+c$ and that the equation always represents a straight line, it is called a *linear equation* and expression $mx+c$ a *linear function*.

EXERCISE 33.

In the following exercises, in cases where the unit is not stated, the student should choose a suitable unit and state it clearly :

A small scale is not at all advisable.

1. In the same diagram and with the same unit draw the graphs of :

(i) $y=2x+3$. (ii) $y=-2x+3$. (iii) $y=2x-3$.
(iv) $y=-2x-3$.

2. In the same diagram and with the same unit draw the graphs of :

(i) $6x+3y+15=0$ (ii) $6x-3y-15=0$.
(iii) $y+2x+5=0$ (iv) $2x-y-5=0$.

3. In the same diagram draw the graphs of :

(i) $2y=3x$ (ii) $2y=3x+4$
(iii) $2y=3x-4$.

4. In the same diagram draw the graphs of.

(i) $y+2x=0$. (ii) $y+2x=4$. (iii) $y+2x+4=0$.

Draw the graphs of the following functions by finding intercepts on the axes :

5. $x+y=3$ 6. $3x+2y=6$
7. $4x-3y=12$ 8. $6x-y+3=0$
9. $4y-x=4$ 10. $3y-2x+9=0$.

(In each case a sufficiently large unit should be chosen).

What is the equation to the path of a point which moves as described in examples 11 – 13. Draw in each case the graph.

11. The abscissa of the point is always -10

12. The ordinate of the point is always 9.

13. The abscissa of the point is thrice the ordinate.

14. A point moves such that the sum of its distances from two straight lines at right angles to each other is always 8 units of length. Find the equation to its path and also draw the graph.

Draw the graphs of the following :—

15. $2y+5x=0$
16. $5y-2x=0$.
17. $10y+8x=7$
18. $3x-4y=15$.
19. $2x=9y-3$.
20. $12x-5y=10$.
21. $3x+4y-24=0$
22. $6y+3x-15=0$.
23. $6x=3-2y$.
24. $x=6+3y$.
25. $\frac{y}{2}=\frac{x}{3}+1$.
26. $\frac{x}{3}+\frac{y}{4}=2$.
27. $\frac{x-y}{3}=\frac{1}{4}$.
28. $\frac{x+y}{2}=3$.

Draw the graphs of the following functions :

29. $2x+1$
30. $-3x+1$
31. $\frac{x-1}{2}$
32. $\frac{3x-4}{2}$.
33. $\frac{3}{4}x+5$.
34. $\frac{2}{5}(x+10)$.

35. Draw the graph of the function $\frac{4x-3}{5}$ and from it read off the value of the function when $x=2{\cdot}5$. (A suitable unit should be chosen).

36. Draw the graph of the function $\frac{5-2x}{4}$ and from it read off the value of the function when $x=6$; also find the value of x which makes the function equal to (i) 1 (ii) 0.

Solve graphically the following pairs of equations :—

37. $2x+y=6$; $x-y=0$ . . . . . (1 inch as unit).
38. $3x-y=7$; $x+y+1=0$ . . . . (one inch .).
39. $2x+3y=23$; $3x+2y=22$ . . . . (0·4" . . .).
40. $x-y=1$; $2x+4y=11$ . . . . (0·5" . . .).
41. $\frac{2x+y}{4}-2=0$; $\frac{3x+2y}{2}=3$ (·1" : : :)
42. $2x \quad -\frac{y-2}{5}=4$

$3=9-y \quad \frac{x-2}{3}$.

43. $x+y=8$; $x-y=2$.

44. Draw the graph of each of the functions $2x-3$ and $\frac{2x-1}{3}$; and find the value of x for which the two functions are equal

Solve graphically the following equations :

45. $\frac{3x}{4}+3=\frac{5x}{6}+2$

46. $7x-6=6x-4$

47. $7x-3=5x+4$.

48. Draw with the same axes and unit the graphs of :

(1) $2x=y+1$ (2) $x+y=5$ (3) $x-2y+4=0$.

Show that the three straight lines pass through the same point.

49. Show graphically that the three straight lines represented by the following equations are concurrent:

(1) $y=4$, (2) $x+3=y$, (3) $3x-2y+5=0$.

50. Draw the graphs of the following equations with reference to the same axes using the same unit.

(1) $x=y$. (2) $2x-y=4$; (3) $x-3y+3=0$.

Also find the co-ordinates of the points of intersection of every two of the above three straight lines.

51. Find the co-ordinates of the vertices of the triangle whose sides are represented by :—

(1) $y=2$; (2) $x+y=1$; (3) $3x-4y=10$.

CHAPTER XI.

FORMULAE AND THEIR APPLICATION.

95. A Formula is the most general expression for any theorem respecting algebraical or numerical quantities.

96. Formula $(a+b)^2=a^2+b^2+2ab$.

The square of the sum of two quantities is equal to the sum of their squares increased by twice the product of the quantities.

[Art. 47, Ex. 2.]

Ex. 1. Find the square of $3x+4y$

$$(3x+4y)^2=(3x)^2+(4y)^2+2(3x)(4y)$$
$$=9x^2+24xy+16y^2.$$

Ex. 2. Find the square of $2a^3+5b^2$.

$$(2a^3+5b^2)^2=(2a^3)^2+(5b^2)^2+2(2a^2)(5b^2)$$
$$=4a^6+20a^3b^2+25b^4.$$

97. Formula $(a-b)^2=a^2+b^2-2ab$.

The square of the difference of two quantities is equal to the sum of their squares diminished by twice the product of the two quantities.

[Art. 47, Ex. 2.]

Ex. 1. Find the square of $2a-3b$

$$(2a-3b)^2=(2a)^2+(3b)^2-2(2a)(3b)$$
$$=4a^2-12ab+9b^2.$$

Ex. 2. Find the square of $3a^4-4b^5$.

$$(3a^4-4b^5)^2=(3a^4)^2+(4b^5)^2-2(3a^4)(4b^5)$$
$$=9a^8-24a^4b^5+16b^{10}.$$

98. To find the square of $a+b+c$.

$$(a+b+c)^2=\{a+(b+c)\}^2 \text{ ...considering } b+c \text{ as one term}$$
$$=a^2+2a(b+c)+(b+c)^2$$
$$=a^2+2ab+2ac+b^2+2bc+c^2$$
$$=a^2+b^2+c^2+2ab+2ac+2bc$$
$$=a^2+b^2+c^2+2a(b+c)+2b(c) \quad \ldots\ldots\ldots\ldots (1)$$

To find the square of $a+b+c+d$.

$(a+b+c+d)^2$

$= (a+b)+(c+d)\}^2$ ..... (considering $a+b$ as one term and $c+d$ as another.)

$=(a+b)^2+2(a+b)(c+d)+(c+d)^2$

$=a^2+2ab+b^2+2ac+2ad+2bc+2bd+c^2+2cd+d^2$

$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$

$=a^2+b^2+c^2+d^2+2a(b+c+d)+2b(c+d)+2c(d)$ .....(11)

From (1) and (11), it is plain that *the square of the sum of any number of quantities is equal to the sum of the squares of all the quantities together with twice the product of each term and the sum of all the terms which follow it.*

Formula $(a+b+c+d+e \ldots)^2$

$=a^2+b^2+c^2+d^2+e^2+\ldots+2a(b+c+d+e\ldots)$.

$+2b(c+d+e\ldots)+2c(d+e\ldots)+2d(e+\ldots)$ *etc.*

Ex. 1. Find the square of $x-y-z$.

$(x-y-z)^2=\{x+(-y)+(-z)\}^2$

$=x^2+(-y)^2+(-z)^2+2x(-y-z)+2(-y)(-z)$

$=x^2+y^2+z^2-2xy-2xz+2yz$.

Ex. 2. Find the square of $2x+3y+4z$

$(2x+3y+4z)^2=(2x)^2+(3y)^2+(4z)^2+2(2x)(3y+4z)+2(3y)(4z)$

$=4x^2+9y^2+16z^2+12xy+16xz+24yz$

Note. The rules stated in Arts. 58 and 59 may be applied to deduce the squares of all multinomial expressions.

Ex. 3. Find the square of $x-y-z$.

$(x-y-z)^2=(x-y)-z\}^2$ .....(considering $x-y$ as one term)

$=(x-y)^2-2(x-y)z+z^2$

$=x^2-2xy+y^2-2xz+2yz+z^2$

$=x^2+y^2+z^2-2xy-2xz+2yz$.

Ex. 4. Find the square of $a-b-c-d$.

$(a-b-c-d)^2$.

$=(a-b)-(c+d)\}^2$ ..... (considering $a-b$ as one term and $c+d$ as another.)

$=(a-b)^2-2(a-b)(c+d)+(c+d)^2$

$=a^2-2ab+b^2-2(ac-bc+ad-bd)+c^2+2cd+d^2$

$=a^2+b^2+c^2+d^2-2ab-2ac+2bc-2ad+2bd+2cd$.

Note.—These two results may be deduced from the expansions of $(x+y+z)^2$ and $(a+b+c+d)^2$ respectively by putting—y for y,—z for z in the first and—b for b,—c for c, and—d for d, in the second.

EXERCISE 34.

Find the square of each of the following expressions :—

1. $4x+3$.
2. $11x+8$.
3. $3a+8b$.
4. $4m+6n$.
5. $ax+2by$.
6. $3abc+c^2$.
7. $ax+bcy$.
8. $mp+qn$.
9. $\frac{3}{5}x+\frac{5}{7}y$.
10. $3a^2+7b^2$.
11. $3p^3+q^2$.
12. $100+2$.
13. $100{\cdot}3$.
14. $600{\cdot}4$.
15. $5x-3$.
16. $9x-7$.
17. $ab-ac$.
18. $-5a-4b$.
19. $4p-8q$.
20. $-ab-cd$.
21. $\frac{3}{8}a-\frac{3}{4}b$.
22. b^2c-a^2d.
23. p^3-2qr.
24. $-acb-de$.
25. $4a^3-5c^3$.
26. $1000-2$.
27. $999{\cdot}8$.
28. $7{\cdot}996$.
29. $a-3b-4c$.
30. $3x-4y-5z$.
31. $4p-3m-n$.
32. $\frac{1}{2}a-\frac{2}{3}b-\frac{3}{4}c$.
33. $ab-bc+ca$.
34. a^2-a-1.
35. $a^2-2b^2-3c^2$.
36. $a-b+x+y$.
37. $2a-3b-4x+5y$.
38. $\frac{1}{2}a-\frac{1}{3}b-\frac{1}{4}c-\frac{1}{5}d$.
39. $a^2+2b^2-2c^2-d^2$.
40. $ab+ac+bc$.
41. $a+3b+4c$.
42. $2m+3n+4p$.
43. $\frac{1}{2}a+\frac{1}{3}b+\frac{1}{4}c$.
44. $x^2+y^2+c^2$.
45. $a^2+b^3+c^4$.
46. $a+b+2x+3y$.
47. $4a+3b+2c+d$.
48. $x+2y+3z+4n$.
49. $4m+3n+2p+3q$.

Simplify :—

50. $(a+b-c)^2+(b+c-a)^2+(c+a-b)^2$.
51. $(a+2b-3c)^2+(b+2c-3a)^2+(c+2a-3b)^2$
52. $(a+b+c-d)^2+(b+c+d-a)^2+(c+d+a-b)^2+(d+a+b-c)^2$.
53. If $x+y=a$, $xy=b$, express x^4+y^4 in terms of a and b.
54. If $m^2-n^2=x$, $m^2n^2=y$, express m^6+n^6 in terms of x and y.
55. $(x+y+z)^2+(x+z-y)^2+2(x+y+z)(x+z-y)$.
56. $(a+b-c-d)^2+(b+c-d-a)^2+(c+d-a-b)^2+(d+a-b-c)^2$.
57. $(l+m+n)^2+(m+n-l)^2-2(l+m+n)(m+n-l)$.

58. $(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2+2(a+b-2c)(b+c-2a)$
$+2(b+c-2a)(c+a-2b)+2(a+b-2c)(c+a-2b)$.

99. Formula $(a+b)^2+(a-b)^2=2(a^2+b^2)$.

The sum of the squares of the sum and difference of any two quantities is equal to twice the sum of the squares of the two quantities.

[Arts. 96 and 97.]

100. Formula $(a+b)^2-(a-b)^2=4ab$.

The difference of the squares of the sum and difference of any two quantities is equal to four times the product of the two quantities.

[Arts. 96 and 97.]

EXERCISE 35.

Simplify :—

1. $(3a+7)^2+(3a-7)^2$.
2. $(\frac{1}{2}a+\frac{2}{3}b)^2+(\frac{1}{2}a-\frac{2}{3}b)^2$.
3. $(a^2+b^2)^2+(a^2-b^2)^2$.
4. $(a^3+b^3)^2+(a^3-b^2)^2$.
5. $(3a+8)^2-(3a-8)^2$.
6. $(\frac{2}{5}x+\frac{5}{8}y)^2-(\frac{2}{5}x-\frac{5}{8}y)^2$.
7. $(p^2+q^2)^2-(p^2-q^2)^2$.
8. $(x^3+y^3)^2-(x^3-y^3)^2$.
9. $(a^2+b^2+ab)^2+(a^2+b^2-ab)^2$.
10. $(x^2-y^2+2xy)^2+(x^2-y^2-2xy)^2$.
11. $(a-b+c)^2+(a-c+b)^2$.
12. $(x^2+y^2+xy)^2-(x^2+y^2-xy)^2$.
13. $(a^2-b^2+2ab)^2-(a^2-b^2-2ab)^2$.
14. $(x-y+z)^2-(x-z+y)^2$.

101. Formula $(a+b)(a-b)=a^2-b^2$.

The product of the sum and difference of two quantities is equal to the difference of their squares. [Art. 47, Ex. 2.]

Ex. 1. Multiply $4a+3b$ by $4a-3b$.

$(4a+3b)(4a-3b)=(4a)^2-(3b)^2=16a^2-9b^2$.

Ex. 2. Multiply a^3+b^3 by a^3-b^3.

$(a^3+b^3)(a^3-b^3)=(a^3)^2-(b^3)^2=a^6-b^6$.

Ex. 3. Multiply $a+b-c$ by $b+c-a$.

$(a+b-c)(b+c-a)$.

$$= b+(a-c)\}\{b-(a-c)$$
$$= b^2-(a-c)^2$$
$$= b^2-(a^2-2ac+c^2)$$
$$= b^2-a^2+2ac-c^2.$$

Ex. 4. Multiply a^2-ab+b^2 by a^2+ab+b^2.

$(a^2+ab+b^2)(a^2-ab+b^2)$

$$= \{(a^2+b^2)+ab\}\{(a^2+b^2)-ab\}$$
$$= (a^2+b^2)^2-(ab)^2$$
$$= a^4+2a^2b^2+b^4-a^2b^2$$
$$= a^4+a^2b^2+b^4.$$

The product of any two numbers or quantities may be expressed as the difference of two squares.

Ex. 5. Express 760×431 as the difference of two squares.

Now 760×431 is to be put in the form $(a+b)\times(a-b)$

so that $a+b=760$; and $a-b=431$.

$\therefore$ by addition $2a=1191$

$$\therefore \quad a=595\tfrac{1}{2}$$
$$\therefore \quad b=760-595\tfrac{1}{2}=164\tfrac{1}{2}$$

By formula $(a+b)(a-b)=a^2-b^2$

where a and b are any two numbers.

$$\therefore (595\tfrac{1}{2}+164\tfrac{1}{2})(595\tfrac{1}{2}-164\tfrac{1}{2}) = (595\tfrac{1}{2})^2-(164\tfrac{1}{2})^2$$

i.e. $760\times431=(595\tfrac{1}{2})^2-164\tfrac{1}{2})^2$.

EXERCISE 36.

Multiply together :—

1. $ab+cd$ by $ab-cd$.
2. $5a+7$ by $5a-7$.
3. $ax+y^2$ by $ax-y^2$.
4. $xy+yz$ by $xy-yz$.
5. $4x+5b$ by $4x-5b$.
6. $3x^2-7b^2$ by $3x^2+7b^2$.
7. $4xy-5mn$ by $4xy+5mn$.
8. p^2-3qr by p^2+3qr.
9. a^2+b^2, $a+b$ and $a-b$.
10. $p+1$, $p-1$ and p^2+1.
11. $x+y-z$ by $z-y+x$.
12. $x+y+z$ by $x-y+z$.

13. p^2+pq+q^2 by p^2-pq+q^2.
14. a^4+a^2+1 by a^4-a^2+1.
15. $x^2+2xy+y^2$ by $x^2-2xy+y^2$.
16. $(2a-3b+c)(2a+3b-c)$.
17. $(a+b-c+d)(a+b+c-d)$.
18. $(a^5+a^3+a^2+a+1)(a^5-a^3+a^2-a+1)$.
19. $a+\sqrt{ab}+b$ by $a-\sqrt{ab}+b$.
20. $(x+\sqrt{x-1})(x+\sqrt{x+1})$.
21. $x+\sqrt{3x}+3$, x^2-3x+9 and $x-\sqrt{3x}+3$.
22. a^2-2a+1, a^2+2a+1 and a^4+2a^2+1.
23. $(x^2-3x+9)(x^2+3x+9)(x^4-9x^2+81)$.
24. $(a^2-4a+8)(a^2+4a+8)(a^2-6a+18)(a^2+6a+18)$.
25. 203×197.
26. $1{\cdot}72\times1{\cdot}68$.
27. $20{\cdot}004\times19{\cdot}996$.

102. Formula $(a+b)^3=a^3+b^3+3ab(a+b)$.

The cube of the sum of any two quantities is equal to the sum of the cubes of the two quantities, increased by thrice their product multiplied by their sum.

[Art. 48, Ex. 2.]

Ex. 1. Find the cube of $2x+3y$,

$$\begin{aligned}(2x+3y)^3&=(2x)^3+(3y)^3+3(2x)(3y)(2x+3y)\\&=8x^3+27y^3+18xy(2x+3y)\\&=8x^3+27y^3+36x^2y+54xy^2\\&=8x^3+36x^2y+54xy^2+27y^3.\end{aligned}$$

Ex. 2. If $p+q=5$ and $pq=6$, find p^3+q^3.

$$\begin{aligned}&(p+q)^3=p^3+q^3+3pq(p+q)\\&\therefore\quad(5)^3\quad=p^3+q^3+3\times6\times5\\&\therefore\quad125\quad=p^3+q^3+90\\&\therefore p^3+q^2=125-90=35.\end{aligned}$$

Ex. 3. If $x+\frac{1}{x}=2$, find $x^3+\frac{1}{x^3}$.

$$\begin{aligned}\left(x+\frac{1}{x}\right)^3&=x^3+\frac{1}{x^3}+3x\times\frac{1}{x}\left(x+\frac{1}{x}\right)\\&=x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)\end{aligned}$$

$\therefore\ 2^3 = x^3 + \dfrac{1}{x^3} + 3 \times 2$

$\therefore 8 = x^3 + \dfrac{1}{x^3} + 6$

$\therefore x^3 + \dfrac{1}{x^3} = 8 - 6 = 2.$

Similarly $x^9 + \dfrac{1}{x^9} = 2$; $x^{27} + \dfrac{1}{x^{27}} = 2$, &c.

The cube of the sum of three or more quantities can be deduced from the formula of Art. 64.

$(a+b+c)^3$

$= \{(a+b)+c\}^3$...considering $a+b$ as one term)

$= \{(a+b)^3 + c^3 + 3(a+b)(c)(a+b+c)$

$= a^3 + b^3 + 3ab(a+b) + c^3 + 3c(a+b)^2 + 3c^2(a+b)$

$= a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3ca^2 + 3cb^2 + 6abc + 3c^2a + 3c^2b$

$= a^3 + b^3 + c^3 + 3ab(a+b) + 3bc(b+c) + 3ac(a+c) + 6abc$...(i)

$= a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(a+c) + 3c^2(a+b) + 6abc$...(ii)

$= a^3 + b^3 + c^3 + 3a(b^2 + c^2) + 3b(a^2 + c^2 + 3c(a^2 + b^2) + 6\ abc$ (iii)

$= \mathbf{a^3 + b^3 + c^3 + 3(a+b+c)(ab+ac+bc) - 3abc}$... .... (iv)

$= \mathbf{a^3 + b^3 + c^3 + 3(a+b)(b+c)\ c+a)}$ . .....:.....(v)

103. Formula $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$.

The cube of the difference of any two quantities is equal to the difference of the cubes of the two quantities, diminished by thrice their product multiplied by their difference. [Art. 48, Ex 2.]

Ex. 1. Find the cube of $2x - 3y$.

$(2x-3y)^3 = (2x)^3 - (3y)^3 - 3(2x)(3y)(2x-3y)$

$= 8x^3 - 27y^3 - 18xy(2x-3y)$

$= 8x^3 - 27y^3 - 36x^2y + 54xy^2$

$= 8x^3 - 36x^2y + 54xy^2 - 27y^3.$

Ex. 2. Find the cube of $a - b - c$.

$(a-b-c)^3$

$= \{(a-b)-c\}^3$...(considering $a-b$ as one term)

$= (a-b)^3 - c^3 - 3(a-b((c)(a-b-c)$

$= a^3 - b^3 - 3ab(a-b) - c^3 - 3c(a-b)^2 + 3c^2(a-b)$

$= a^3 - b^3 - c^3 - 3a^2b + 3ab^2 - 3ca^2 - 3cb^2 + 6abc + 3c^2a - 3c^2b$

$= a^3 - b^3 - c^3 - 3ab(a-b) + 3bc(-b-c) - 3ac(a-c) + 6abc.$

Note.—The same may be deduced from the expansion of $(a+b+c)^3$ by putting—b for b, and—c for c.

EXERCISE 37.

Find the cube of :—

1. $4a+3$.
2. $3x+2y$.
3. $\frac{2}{3}a+\frac{1}{3}b$.
4. $4b-5a$.
5. $ab-cd$.
6. $2p-5q$.
7. $\frac{1}{2}x-\frac{1}{2}y$.
8. x^2+2y.
9. $max+xam$.
10. $2x^a+3y^b$.
11. $xy+yz+zx$.
12. $ab-bc+ac$.
13. $a^2-b^2-c^2$.
14. $a+2b-c$.
15. $p^2+q^2-r^2$.
16. $a+2b+3c$.
17. $\frac{1}{2}a-\frac{1}{3}b-\frac{1}{4}c$.
18. 102.
19. 996.
20. 5·03.
21. 99·2.

Find the value of :—

22. x^3+y^3 when $x+y=7$ and $xy=6$.
23. x^3-y^3 when $x-y=6$ and $xy=16$.
24. $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=3$.
25. $a^3-\frac{1}{a^3}$ when $a-\frac{1}{a}=5$.
26. x^3+y^3+15xy when $x+y=5$.
27. p^3+q^3+6pq when $p+q=2$.
28. x^3-y^3-9xy when $x-y=3$.
29. $8x^3-27y^3-90xy$ when $2x-3y=5$.

Simplify :—

30. $(a+b+c)^3-(a^3+b^3+c^3)$.
31. $(x-y)^3+(y-z)^3+(z-x)^3$.
32. $(2a-3b)^3-(2b-3a)^3-15(2a-3b)(2b-3a)(a-b)$.
33. $(x-1)^3-(y-1)^3-3(x-1)(y-1)(x-y)$.
34. $(a+b+c)^3+(c-a-b)^3+6c\{c^2-(a+b)^2\}$.
35. $(x+2y)^3-(x+2y)^2(x-2y)+(x+2y)(x-2y)^2-(x-2y)^3$.

104. Formulæ $(a+b)(a^2-ab+b^2)=a^3+b^3$ and $(a-b)(a^2+ab+b^2)=a^3-b^3$. [Art. 48, Ex. 1.]

Ex. 1. Multiply a^4-a^2+1 by a^2+1.

Putting p for a^2 and q for 1, we have

$$a^4-a^2+1=(a^2)^2-a^2\times1+1^2=p^2-pq+q^2$$

$$\therefore(a^4-a^2+1)(a^2+1)=p^2-pq+q^2)(p+q)$$

$$=p^3+q^3$$

$$=(a^2)^3+1^3$$

$$=a^6+1.$$

Ex. 2. Multiply $9a^2+12a+16$ by $3a-4$.

$9a^2+12a+16=(3a)^2+3a\times 4+4^2$

$=(x^2+xy+y^2)$...(putting x for $3a$ & y for 4)

$\therefore (9a^2+12a+16)(3a-4)=(x^2+xy+y^2)(x-y)$

$=x^3-y^3$

$=(3a)^3-4^3$

$=27a^3-64.$

EXERCISE 38.

Multiply :—

1. x^2-2x+4 by $x+2$.
2. x^2-3x+9 by $x+3$.
3. $4a^2-2a+1$ by $2a+1$.
4. $a^2b^2-abc+c^2$ by $ab+c$.
5. $25x^2-10xy+4y^2$ by $5x+2y$.
6. $16a^2-28ab+49b^2$ by $4a+7b$.
7. $1+3x+9x^2$ by $1-3x$.
8. $a^2+2abc+4b^2c^2$ by $a-2bc$.
9. x^2+2x+4 by $x-2$.
10. $m^2+4mnp+16p^2n^2$ by $m-4pn$.
11. $4a^2b^4+2ab^2+1$ by $2ab^2-1$.
12. $a^8+a^4b^4+b^8$ by a^4-b^4.
13. $a^2+\frac{1}{2}ab+\frac{1}{4}b^2$ by $a-\frac{1}{2}b$.
14. $\frac{4}{9}a^2-ab+\frac{3}{4}b^2$ by $\frac{2}{3}a+\frac{3}{2}b$.
15. $a+\sqrt{ab}+b$ by $\sqrt{a}-\sqrt{b}$.
16. $9x-12\sqrt{xy}+16y$ by $3\sqrt{x}-4\sqrt{y}$.
17. $\frac{1}{4}m^2+\frac{1}{2}n^2-\frac{1}{6}mn$, $\frac{1}{2}m+\frac{1}{3}n$, and $\frac{1}{8}m^3-\frac{1}{27}n^3$.
18. $a^2x^2+b^2y^2+abxy$, $ax-by$, $ax+by$ and $a^2x^2+b^2y^2-abxy$.
19. $9x^2+4y^2-6xy$, $9x^2+6xy+4y^2$, $2y+3x$, and $3x-2y$.

105. When two or more binomial factors have one term common to them all, their product can be readily written down.

1. **Formula** $(x+a)(x+b)=x^2+(a+b)x+ab$. [Art. 50, Ex. 2.]

The following may be deduced from the above formula :—

i. $(x-a)(x-b)=x^2-(a+b)x+ab$.

ii. $(x+a)(x-b)=x^2+(a-b)x-ab$.

iii. $(x-a)(x+b)=x^2+(b-a)x-ab$.

Ex. 1. Multiply $x+7$ by $x+11$.

$$(x+7)(x+11)$$
$$=x^2+(7+11)x+7\times 11$$
$$=x^2+18x+77.$$

Ex. 2. Multiply $x-7$ by $x+10$.

$$(x-7)(x+10)=x^2+(-7+10)x+(-7)(10)$$
$$=x^2+3x-70$$

II. Formula $(\mathbf{x+a})(\mathbf{x+b})(+\mathbf{xc})$
$$=\mathbf{x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.}$$

[Art. 50, Ex. 2.]

Similarly $(x+a)(x+b)(x+c)(x+d)$
$$=x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$$
$$+(abc+abd+acd+bcd)x+abcd.$$

We observe the following laws in Formulæ I and II:—

1. The number of terms in the product is one more than the number of binomial factors.

2. The index of the highest power of x is the same as the number of factors, and the indices of x decrease by unity in each succeeding term.

3. The co-efficient of the highest power of x is unity, and the co-efficients of the succeeding powers of x are the sum, the sum of the products in pairs, in threes, &c., respectively, of the second terms of the factors, and the last term is the product of all the second terms of the factors.

Find the continued product of :—

Ex. 1 $(x+1)(x+2)(x+3)$.

$$(x+1)(x+2)(x+3)$$
$$=x^3+(1+2+3)x^2+(1\cdot 2+1\cdot 3+2\cdot 3)x+1.2.3.$$
$$=x^3+6x^2+11x+6.$$

Ex. 2. $(x-2)(x-3)(x-4)$.

$$(x-2)(x-3)(x-4)$$
$$=x^3+(-2-3-4)x^2$$
$$+\{(-2)\times(-3)+(-3)\times(-4)+(-2)\times(-4)\}x$$
$$+(-2)\times(-3)\times(-4)$$
$$=x^3-9x^2+26x-24$$

Ex. 3. $(x+1)(x-2)(x-4)$.

$$(x+1)(x-2)(x-4)$$
$$= x^3 + (1-2-4)x^2 + \{1\times(-2)+1\times(-4)+(-2)\times(-4)\}x + 1\times(-2)\times(-4)$$
$$= x^3 - 5x^2 + 2x + 8.$$

Ex. 4. $(a+b)(b+c)(c+a)$.

$$(a+b)(b+c)(c+a)$$
$$= \{(a+b+c)-c\}\{(a+b+c)-a\}\{(a+b+c)-b\}$$
$$= (a+b+c)^3 - (a+b+c)^2(a+b+c) + \{(-c)(-a) + (-c)(-b) + (-a)(-b)\}(a+b+c) + (-c)(-a)(-b)$$
$$= (a+b+c)^3 - (a+b+c)^3 + (ac+bc+ab)(a+b+c) - abc$$
$$= \mathbf{(ab+ac+bc)(a+b+c) - abc.}$$

EXERCISE 39.

Write down the products of :—

1. $(x+7)(x+5)$.
2. $(x+2)(x-3)$.
3. $(x^2-3)(x^2-5)$.
4. $(x-3y)(x+4y)$.
5. $(2x-3)(2x+5)$.
6. $(ax+12)(ax-13)$.
7. $(2x-\frac{1}{3}y)(2x+\frac{1}{4}y)$.
8. $(2x+3y)(3x+4y)$.
9. $(x+2)(x+3)(x+4)$.
10. $(x+2)(x-4)(x+6)$.
11. $(x+5)(x+6)(x-7)$.
12. $(2x-1)(2x-3)(2x+4)$.
13. $(x-1)(x-3)(x-5)$.
14. $(x-1)(y-1)(z-1)$.
15. $(x+\frac{1}{2})(x+\frac{1}{3})(x+\frac{1}{4})$.
16. $(x+ab)(x+bc)(x+ca)$.
17. $\left(1+\frac{1}{m}\right)\left(1+\frac{m}{n}\right)\left(1+\frac{n}{l}\right)$.
18. $\left(1-\frac{q}{p}\right)\left(1-\frac{q}{r}\right)\left(1-\frac{r}{p}\right)$.
19. $\left(x-\frac{a}{b}\right)\left(x-\frac{b}{c}\right)\left(x-\frac{c}{a}\right)$.
20. $(x+2)(x+3)^2$.
21. $(x-4)^2(x+3)$.
22. $2(1+2x)(1+x)(3+2x)$.
23. $3(3x-a)(x-a)(3x-2a)$.
24. $(a-1)(2a-3)(3a-5)$.
25. $(2x+1)(3x+2)(4x+3)$.
26. $(a^2-1)(a^2-2)(a^2-3)$.
27. $(x-1)(x+1)(x+3)(x-3)$.
28. $(a+1)(a+2)(a+3)(a-1)(a-2)(a-3)$.

29. $(x+a+b)(x+b+c)(x+c+a)$.

30. 101.102.103.

31. 97.98.99.

32. 95.93.97.99.

105-A. Formula $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$
$=a^3+b^3+c^3-3abc$.
[Art. 49, Ex. 2.]

Ex. 1. Multiply $x^2+y^2+z^2-xy+xz+yz$ by $x+y-z$.

Putting p for x, q for y and r for $-z$, we have

$(x+y-z)(x^2+y^2+z^2-xy+xz+yz)$.

$=(p+q+r)(p^2+q^2+r^2-pq-pr-qr)$

$=p^3+q^3+r^3-3pqr$

$=x^3+y^3+(-z)^3-3xy(-z)$

$=x^3+y^3-z^3+3xyz$.

Ex. 2. Multiply $a^2+4b^2+9c^2-2ab-3ac-6bc$ by $a+2b+3c$.

Putting p for a, q for $2b$ and r for $3c$, we have

$(a+2b+3c)(a^2+4b^2+9c^2-2ab-3ac-6bc)$

$=(p+q+r)(p^2+q^2+r^2-pq-pr-qr)$

$=p^3+q^3+r^3-3pqr$

$=a^3+(2b)^3+(3c)^3-3a(2b)(3c)$

$=a^3+8b^3+27c^3-18abc$.

EXERCISE 40.

Multiply :–

1. $x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2$ by $x^2+y^2+z^2$.
2. $a^2+b^2+c^2+ab+ac-bc$ by $a-b-c$.
3. $4x^2+9y^2+z^2+6xy+2xz-3yz$ by $2x-3y-z$.
4. $9p^2+25q^2+16+15pq+12p-20q$ by $3p-5q-4$.
5. $m^2+n^2+1+mn-m+n$ by $m-n+1$.
6. $a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)$ by $ab+bc+ca$.
7. $x^3+\frac{1}{x^3}-x-\frac{1}{x}$ by $x+\frac{1}{x}+1$.
8. $2+\frac{l^2}{m^2}+\frac{m^2}{l^2}+\frac{l}{m}-\frac{m}{l}$ by $1-\frac{l}{m}+\frac{m}{l}$.
9. $x^4+\frac{1}{x^4}+3+2x^2+\frac{2}{x^2}$ by $\left(x-\frac{1}{x}\right)^2$.

10. $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{c}-\frac{c}{b}-\frac{b}{a}$ by $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.

11. $\frac{x^2}{y^2z^2}+\frac{y^2}{z^2x^2}+\frac{z^2}{x^2y^2}+\frac{1}{z^2}+\frac{1}{y^2}-\frac{1}{x^2}$ by $\frac{x}{yz}-\frac{y}{zx}-\frac{z}{xy}$.

12. $(a-b)^2+(b-c)^2+(c-a)^2$ by $a+b+c$.

13. $(a+b)^2+(b+c)^2+(c+a)^2-(a+b)\ (b+c)$
$-(b+c)\ (c+a)-(c+a)\ (a+b)$ by $2\ (a+b+c)$.

14. $(2x-y)^2+(2y-z)^2+(2z-x)^2-(2x-y)\ (2y-z)$
$-(2y-z)\ (2z-x)-(2z-x)\ (2x-y)$ by $x+y+z$.

106. By actual multiplication we have

Formula $(a+b)\ (b+c)\ (c+a)$

$=a\ (b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc$ ... (i)

$=a\ (b+c)^2+b(c+a)^2+c(a+b)^2-4abc$ ... (ii)

$=a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc$ ... (iii)

$=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$ ... (iv)

$=ab(a+b)+bc(b+c)+ca(c+a)+2abc$ .. (v)

$=(a+b+c)(ab+ac+bc)-abc$ ... (vi)

Ex. Multiply $(a+2b+c)\ (b+2c+a)\ (c+2a+b)$.

Putting p for $a+b$, q for $b+c$ and r for $c+a$, we have

$(a+2b+c)\ (b+2c+a)\ (c+2a+b)$

$=(p+q)\ (q+r)\ (r+p)$

$=p\ (q^2+r^2)+q(r^2+p^2)+r(p^2+q^2)+2pqr$

$=(a+b)\{(b+c)^2+(c+a)^2\}+(b+c)\{(c+a)^2+(a+b)^2\}$
$+(c+a)\{(a+b)^2+(b+c)^2\}+2(a+b)(b+c)(c+a)$.

This can be expanded by Formulæ.

107. By actual multiplication we have

Formula $(a-b)(b-c)(c-a)$

$=a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$ ... (i)

$=-\{a^2(b-c)+b^2(c-a)+c^2(a-b)\}$ ... (ii)

$=-\{ab(a-b)+bc(b-c)+ca(c-a)\}$ ... (iii)

Ex. Multiply $(x^2-y^2)(y^2-z^2)(z^2-x^2)$.

The given expression $=(a-b)(b-c)(c-a)$, putting a for x^2, b for y^2 and c for z^2.

$=\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}$

$=\{x^2(y^4-z^4)+y^2(z^4-x^4)+z^2(x^4-y^4)\}$

$=x^2y^4-x^2z^4+y^2z^4-y^2x^4+x^4z^2-y^4z^2$,

108. By actual multiplication we have

Formula $(a+b+c)(a^2+b^2+c^2)$
$=a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b)$.

$(a+b+c)(a^2+b^2+c^2)$
$=(a+b+c)\{a^2+b^2+c^2-ab-ac-bc\}+(a+b+c)(ab+ac+bc)$
$=a^3+b^3+c^3-3abc+a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$
$=a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b)$.

Ex. Multiply $(x+y+2)$ by (x^2+y^2+4).

The product $=x^3+y^3+2^3+x^2(y+2)+y^2(x+2)+2^2(x+y)$.
$=x^3+y^3+8+x^2y+y^2x+2x^2+2y^2+4x+4y$.

EXERCISE 41.

Simplify by formulæ :—

1. $(1+a+b)(a+b+ab)$.
2. $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}\right)$.
3. $2\{x+y+z)(x+y)(x+z)+(x+z)(z+y)+(z+y)(x+y)\}$.
4. $2(x-z)\{x-y)(y-z)-(y-z)(z-x)-(z-x)(x-y)\}$.
5. $(x-2y+z)(y-2x+z)(x-2z+y)$.
6. $(a+b)^2(b-a)+(b+c)^2(c-b)+(c+a)^2(a-c)$
$+(b-a)(c-a)(a-c)$.
7. $(a+2b+3c)^2(a-2b+c)+(b+2c+3a)^2(b-2c+a)$
$+(c+2a+3b)^2(c-2a+b)+(a-2b+c)(b-2c+a)(c-2a+b)$.
8. $(1+x)(x+y)(y+a)$.
9. $(2b+c)(b+c)(2c+b)$.
10. $2(2a-3b)(3b-4c)(2c-a)$.
11. $\frac{1}{4}(\frac{1}{2}x+\frac{1}{3}y)(\frac{1}{3}y+\frac{1}{2}z)(\frac{1}{2}z+x)$.
12. $(2a+2b+c)(b+c)(2c+2a+b)$.
13. $(x^2+4y^2+9c^2)(x-2y-3c)$.
14. $\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{a}+\frac{a}{b}\right)$.
15. $(a-b)(b-c)(a-2b+c)+(b-c)(c-a)(b-2c+a)$
$+(c-a)(a-b)(c-2a+b)$.

CHAPTER XII.

RESOLUTION INTO FACTORS.

109. From Chapter VI it is seen, that a product is *composed* of factors, one of which is multiplicand and the other, the multiplier. A product may, therefore, be called a compound of *elements* which are its component factors. When these factors have to be found out, we have to *resolve* or *decompose* the product. The process by which this is effected, is called the Resolution of Expressions into their component factors.

Expressions in which each of the terms contains a common factor.

Ex. 1. $a^2+ab+ac=a(a+b+c)$.

Ex. 2. $a(a-b)+(a-b)^2=(a-b)(a+a-b)=(a-b)(2a-b)$.

Ex. 3. $4x^5-3x^4+7x^2=x^2(4x^3-3x^2+7)$.

Ex. 4. $10x^3y^2-6x^4y+4xy^3=2xy(5x^2y-3x^3+2y^2)$.

EXERCISE 42.

Resolve into factors :—

1. $b^2+ba+bc$.
2. $xy+xz+x^2$.
3. $a^2-ab-ac$.
4. $abc-bcd+b^2c$.
5. $a^2bc+ab^2c-abc^2$.
6. $(a+b)^2-2(a+b)$.
7. $x-y-4(x-y)^2$.
8. $(c-a)^2+(c-a)(a-b)$.
9. $x^5y-4x^3y^2-3x^2y^3$.
10. $a^{n+1}-a^n$.
11. $4x+4^2x$.
12. 2^n+4^n.
13. $3x+1-6x+2$.
14. $a^2(b-c)+bc(b-c)^2+(b-c)^3$.
15. $(x-y)^3-2(y-x)^2+4(y+x)$.

73. Expressions of the form a^2+b^2+2ab or a^2+b^2-2ab.

We know that

$(a+b)^2=(a+b)(a+b)=a^2+b^2+2ab$.

$(a-b)^2=(a-b)(a-b)=a^2+b^2-2ab$ [Arts. 96 and 97.]

Hence conversely

$a^2+b^2+2ab=(a+b)(a+b)=(a+b)^2$.

$a^2+b^2-2ab=(a-b)(a-b)=(a-b)^2$.

Ex. 1. $a^4+b^4+2a^2b^2=(a^2)^2+(b^2)^2+2a^2b^2=(a^2+b^2)^2$.

Ex. 2 $16x^2+25y^2-40xy=(4x)^2+(5y)^2-2(4x)(5y)=(4x-5y)^2$.

Ex. 3. Simplify $(m+n)^2-2(m+n)(m-n)+(m-n)^2$.

Let $m+n=a$; and $m-n=b$; then

the expression $=a^2-2ab+b^2$

$=(a-b)^2$

$=\{(m+n)-(m-n)\}^2=(2n)^2=4n^2$.

Ex. 4. Find the value of $25x^2+60xy+36y^2$ when $x=3$, and $y=-2$.

$25x^2+36y^2+60xy=(5x)^2+(6y)^2+2(5x)(6y)$.

$=(5x+6y)^2$

$=(15-12)^2$

$=9$.

EXERCISE 43.

Resolve into factors :—

1. a^2+4b^2+4ab.
2. $4x^2+9y2-12xy$.
3. $9m^2+16n^2+24mn$.
4. $16p^2-40pq+25q^2$.
5. $25a^2c^2+10abcd+b^2d^2$.
6. $72x^2+84xy+98y^2$.
7. $9x^4+42x^3+49x^2$.
8. $27a^2-6ab+\frac{1}{3}b^2$.
9. $\frac{4}{9}x^2-xy+\frac{9}{16}y^2$.
10. $\frac{4}{3}a^5b-4a^3b+3ab^3$.
11. $16l^2-8(lm-nl)+n^2-2mn+m^2$.
12. $a^{2n}+2a^nb^n+b^{2n}$.
13. $x^{2m}-2x^m+^n+x^{2n}$.
14. $2^{2x}+2^{x+y+1}+2^{2y}$.
15. $3^{4n}-2\cdot18^n+4^n$.

Simplify :—

16. $(a+b)2\pm2(a+b)(a-b)+(a-b)^2$.
17. $(a-b+c)^2\pm2(a-b+c)(b+c-a)+(b+c-a)^2$.
18. $(4a-3b)^2\pm2(4a-3b)(3a-4b)+(3a-4a)^2$.
19. $(ab+ac-bc)^2\pm2(ab+ac-bc)(bc+ab-ac)+(bc+ab-ac)^2$.
20. $(2a^2-b^2-c^2)^2\pm2(2a^2-b^2-c^2)(2b^2-a^2-c^2)+(2b^2-a^2-c^2)^2$.

Find the values of :—

21. $4a^2+28a+49$ when $a=-4$.
22. $b^2c^2-12abc+36a^2$ when $a=4$, $b=13$, and $c=2$.
23. $16p^2-24pq+9q^2$ when $p=3$ and $q=4$.
24. $25(x+y)^2+20(x+y)z+4z^2$ when $x=-2$, $y=-3$, and $z=12$.
25. $9(a+b)^2-30(a+b)c+25c^2$ when $a=4$, $b=3$, and $c=6$.

110. Expressions of the form $a^2 - b^2$ or those which can be put into that form.

We know that $(a+b)(a-b) = a^2 - b^2$. [Art. 101.]

Hence conversely $a^2 - b^2 = (a+b)(a-b)$.

Ex. 1. $a^8 - b^8 = (a^4)^2 = (a^4+b^4)(a^4-b^4)$.

Again $a^4 - b^4 = (a^2)^2 - (b^2)^2$

$= (a^2+b^2)(a^2-b^2)$

$= (a^2+b^2)(a+b)(a-b)$

$\therefore a^8 - b^8 = (a^4+b^4)(a^2+b^2)(a+b)(a-b)$.

Ex. 2. $x^4+4 = (x^4+4x^2+4) - 4x^2$.

$= (x^2+2)^2 - (2x)^2$

$= (x^2+2+2x)(x^2+2-2x)$

$= (x^2+2x+2)(x^2-2x+2)$

Note.—The given expression is made a perfect square by adding a perfect square to it.

Ex. 3. $a^2 - b^2 - 2bc - c^2$

$= a^2 - (b^2+2bc+c^2)$

$= a^2 - (b+c)^2$

$= \{a+(b+c)\}\{a-(b+c)\}$.

$= (a+b+c)(a-b-c)$.

Ex. 4. $x^4 - 18x^2y^2 + y^4$

$= x^4 - 2x^2y^2 + y^4 - 16x^2y^2$

$= (x^2-y^2)^2 - (4xy)^2$

$= (x^2-y^2+4xy)(x^2-y^2-4xy)$.

Ex. 5. $4a^2b^2 - (a^2+b^2-c^2)^2$

$= (2ab)^2 - (a^2+b^2-c^2)^2$

$= \{2ab+(a^2+b^2-c^2)\}\{2ab-(a^2+b^2-c^2)\}$

$= \{(a^2+b^2+2ab)-c^2\}\{c^2-(a^2+b^2-2ab)\}$

$= \{(a+b)^2-c^2\}\{c^2-(a-b)^2\}$

$= \{(a+b)+c\}\{(a+b)-c\}\{c+(a-b)\}\{c-(a+b)\}$

$= (a+b+c)(a+b-c)(c+a-b)(c-a+b)$.

Ex. 6. Find the value of $(1000)^2 - (999)^2$.

The expression $= (1000+999)(1000-999)$

$= 1999 \times 1$

$= 1999$.

Ex. 7. Find the value of :—

$p^2+q^2+2pq-r^2$ when $p=2$, $q=3$ and $r=-5$.

The expression $=(p+q)^2-r^2$

$=(p+q+r)(p+q-r)$

$=(2+3-5)(2+3+5)$

$=0\times 10$

$=0$.

EXERCISE 44

Resolve into factors :—

1. a^8-81.
2. x^6-256.
3. $25a^2-9$.
4. a^4-81b^4.
5. $49c^2-64d^2$.
6. $4a^2x^2-9b^2$.
7. $a^2-(b-c)^2$.
8. $(a+b)^2-(c+d)^2$.
9. $(3x+4y)^2-49z^2$.
10. $(2a+7b)^2-(4a-5b)^2$.
11. $(4p-5q)^2-(3p+7q)^2$.
12. $(a+b-c)^2-(a-b+c)^2$.
13. $(a-2b+3c)^2-(3a-2b+c)^2$.
14. $(2x+3y-4z)^2-(4x-3y+2z)^2$.
15. $(3p-4q+r)^2-(4r-3q+p)^2$.
16. x^4+64.
17. $4a^4+81$.
18. a^4+4b^4.
19. x^4+x^2+1.
20. x^8+x^4+1.
21. $x^4+x^2y^2+y^4$.
22. x^4+2x^2+9.
23. x^4-7x^2+9.
24. $4a^4+3a^2+9$
25. x^4+8x^2+144.
26. $9x^4-33x^2+16$.
27. $9x^4-x^2+16$.
28. $9a^4-19a^2x^2+25x^4$.
29. $9x^4-25x^2+16$.
30. $x^2+4y^2-9z^2-4xy$.
31. $16a^2b^2-(a^2+4b^2-c^2)^2$.
32. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.
33. $a^2-2ab-c^2+2bc$.

34. $a^2-b^2-c^2+d^2-2(ad-bc)$.

35. $a^4-b^4+c^4-2a^2c^2+4ab^2c$.

36. $(a^2-b^2)^2+(c^2-d^2)^2-(a+b)^2(c-d)^2-(a-b)^2(c+d)^2$.

Find the value of :—

37. $x^2+y^2+2xy-z^2$ when $x=3$, $y=2$ and $z=5$.

38. $a^2+b^2-2ab-c^2$ when $a=1$, $b=3$ and $c=2$.

39. $(49346)^2-(49336)^2$.

40. $3434\times3434-3432\times3432$.

75. Expressions of the form $a^3+b^3+3ab(a+b)$ or $a^3-b^3-3ab(a-b)$.

We know that

$(a+b)^3=(a+b)(a+b)(a+b)=a^3+b^3+3ab(a+b)$

$(a-b)^3=(a-b)(a-b)(a-b)=a^3-b^3-3ab(a-b)$

[Arts. 102 and 103.]

Hence conversely

$a^3+b^3+3ab(a+b)=(a+b)(a+b)(a+b)=(a+b)^3$.

$a^3+b^3-3ab(a-b)=(a-b)(a-b)(a-b)=(a-b)^3$.

Ex. 1. $8a^3+36a^2b+54ab^2+27b^3$

$=8a^3+27b^3+36a^2b+54ab^2$

$=(2a)^3+(3b)^3+3(2a)(3b)(2a+3b)$

$=(2a+3b)^3$.

Ex. 2. $m^3x^3-3m^2nx^2+3mn^2x-n^3$

$=m^3x^3-n^3-3m^2nx^2+3mn^2x$

$=(mx)^3-n^3-3mnx(mx-n)$

$=(mx-n)^3$.

Ex. 3. Simplify :—

$(x+y)^3+(x-y)^3+3(x+y)^2(x-y)+3(x+y)(x-y)^2$.

Putting a for $x+y$ and b for $x-y$, we have the given expression

$=a^3+b^3+3a^2b+3ab^2$

$=a^3+b^3+3ab(a+b)$

$=(a+b)^3$

$=\{(x+y)+(x-y)\}^3$

$=(2x)^3$

$=8x^3$.

EXERCISE 45.

Resolve into factors :—

1. $a^3+9a^2+27a+27$.
2. $27x^3-27x^2+9x-1$.
3. $a^3b^3-3a^2b^2c+b^3c^2(3a-c)$.
4. $8p^3+4p^2q+\frac{2}{3}pq^2+\frac{1}{27}q^3$.
5. $\frac{1}{27}b^6-\frac{1}{3}b^4c^3+b^2c^6-c^9$.
6. $\frac{1}{8}m^3-\frac{1}{27}n^3-\frac{1}{2}mn(\frac{1}{2}m-\frac{1}{3}n)$.
7. $1+x+3(\sqrt{x^2}+\sqrt{x})$.
8. $l^3-3l^2(m-n)+3l(m^2-2mn+n^2)$
$-m^3+3m^2n-3mn^2+n^3$.
9. $3^{n+3}+3^{n+1}.4^n+3^{2n+1}.2^n+27^n$.

Simplify :—

10. $(x-2)^3+9(x-2)(x+1)+27$.
11. $x^6+3x^4y^2+3x^2y^4+y^6$ when $x=a+b$, $y=a-b$.
12. $(a-b)^3+3(a-c)(a-b)(b-c)+(b-c)^3$.
13. $(2a-3b)^3+3(a+b)(2a-3b)(3a-2b)-(3a-2b)^3$.
14. $(a+b+c)^3+(c-a-b)^3+6c\{c^2-(a+b)^2\}$.
15. $101^3-3.101^2+3.101-1$.

111. Expression of the form a^3+b^3 or a^3-b^3.

We know that $(a+b)(a^2-ab+b^2)=a^3+b^3$.

$(a-b)(a^2+ab+b^2)=a^3-b^3$.

Hence conversely [Art. 104.]

$$a^3+b^3=(a+b)(a^2-ab+b^2).$$
$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

Ex. 1. $a^6-b^6=(a^3)^2-(b^3)^2$
$=(a^3+b^3)(a^3-b^3)$
$=(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

Ex. 2. $8+x^3=2^3+x^3$
$=(2+x)(2^2-2x+x^2)$
$=(2+x)(4-2x+x^2)$

Ex. 3. $27-x^3=3^3-x^3$
$=(3-x)(3^2+3x+x^2)$
$=(3-x)(9+3x+x^2)$.

Ex. 4. $a^3+3a^2b+3ab^2+2b^3$
$=a^3+b^3+3ab(a+b)+b^3$
$=(a+b)^3+b^3$
$=(a+b+b)\{(a+b)^2-b(a+b)+b^2\}$
$=(a+2b)(a^2+ab+b^2)$.

EXERCISE 46.

Resolve into factors :—

1. x^3-8.
2. a^3+27.
3. $216b^3c^3-a^3$.
4. $64a^3-27b^3d^3$.
5. $a^3x^2y+27x^5y^4$.
6. $x^4-2xy^3+y^4$.
7. $x^7-2x^3y+3xy^3$.
8. $(a^2-bc)^3+8b^3c^3$.
9. x^6+x^3-2.
10. $\frac{8}{27}a^3-\frac{27}{64}b^3$.
11. $27x-64y$.
12. $81a^3-375$.
13. $24n+81n$.
14. a^3-3a^2+3a+7.
15. $(a+b)^3-(b+c)^3$.
16. $(2a-3b)^3+(2b-3a)^3$.
17. $8(m-n)^3-27(n-p)^3$.
18. $a^3+3a^2+3a-26$.
19. $2a^3+3a^2+3a+1$.
20. $(x+y+z)^3-x^3$.

42. Expression of the form x^2+px+q.

First Method—

We know that $(x+a)(x+b)=x^2+(a+b)x+ab$. [Art. 103]

Hence conversely $x^2+(a+b)x+ab=(x+a)(x+b)$.

And by changing the signs of a and b.

We have $(x-a)(x-b)=x^2-(a+b)x+ab$.

$(x+a)(x-b)=x^2+(a-b)x-ab$.

$(x-a)(x+b)=x^2-(a-b)x-ab$.

and also the respective converse results.

Examining the above results we find that *to resolve an expression of the form* x^2+px+q *into factors we have to find by trial two quantities like* a *and* b *such that* $a+b=p$ *and* $ab=q$.

Ex. 1. $x^2+7x+10$.

Here $a+b=7$ and $ab=10$, therefore $a=5$ and $b=2$.

$\therefore x^2+7x+10=x^2+(5+2)x+5\times2=(x+5)(x+2)$.

Ex. 2. $x^2-12x+35$.

Here $a+b=-12$ and $ab=35$, therefore $a=-7$ and $b=-5$.

$\therefore x^2-12x+35=x^2-(5+7)x+7\times5=(x-7)(x-5)$

Ex. 3. $x^2+3x-40$.

Here $a+b=8$ and $ab=-40$, therefore $a=8$ and $b=-5$.

$\therefore x^2+3x-40=x^2+(8-5)x-8\times5=(x+8)(x-5)$.

Ex. 4. $x^2-5x-36$.

Here $a+b=-5$ and $ab=-36$, therefore $a=-9$ and $b=4$.

$\therefore x^2-5x-36=x^2-(9-4)x-9\times4=(x-9)(x+4)$.

Ex. 5. $x^2+(c-a)x+(a-b)(b-c)$.

Here the sum of $-(a-b)$ and $-(b-c)=c-a$ and their product $=-(a-b)(b-c)$.

$\therefore x^2+(c-a)x+(a-b)(b-c)$

$=x^2-(a-b+b-c)x+(a-b)(b-c)$

$=\{x-(a-b)\}\{x-b(b-c)\}$

$=(x-a+b)(x-b+c)$.

Ex. 6. $(x^2-x)^2-(x^2-x)-6$. Putting y for x^2-x.

We have y^2-y-6

$=y^2-(3-2)y-3\times2$

$=(y-3)(y+2)$

$=(x^2-x-3)(x^2-x+2)$.

Second Method.—

The factors of the given expression can also be found by writing it as the difference of two squares.

Ex. 7. x^2+6x+8

$=x^2+6x+9-9+8$ [where $9=(\frac{6}{2})^2$.]

$=(x+3)^2-1$

$=(x+3+1)(x+3-1)$

$=(x+4)(x-2)$.

Ex. 8. $x^2-11x-60$

$=x^2-11x+1\frac{2}{4}\frac{1}{4}-1\frac{2}{4}\frac{1}{4}-60$ [where $1\frac{2}{4}\frac{1}{4}=(\frac{11}{2})^2$.

$=(x-\frac{11}{2})^2-(\frac{19}{2})^2$

$=(x-\frac{11}{2}+\frac{19}{2})(x-\frac{11}{2}-\frac{19}{2})$

$=(x+4)(x-15)$

EXERCISE 47.

Resolve into factors :—

1. x^2+x-6.
2. $x^2+25x+144$.
3. x^2-6x+5
4. $x^2-8xy+12y^2$.
5. $1+15x+14x^2$.
6. $x^2+(a+2b)x+2ab$.
7. $x^2-(2a+b)x+(a^2+ab)$.

8. $x^2-2bx+b^2-a^2$.

9. $x^2-2(a^2+b^2)x+(a^2-b^2)^2$.

10. $x^2-(a+b+c)x+ab+ac$.

11. $x^2+\left(\frac{a}{b}+\frac{b}{a}\right)x+1$. 12. $x^2+2ax+a^2-(b-c)^2$.

13. $x^2+c(a+b)x-ab(a-c)(b+c)$.

14. $x^2+(a+b-c)x-ac-bc$.

15. $(x^2+5x)^2-8(x^2+5x)-84$.

16. $(a^2-3a)^2-3(a^2-3a)-4$.

17. $(x^2+2x)^2-(x^2+2x)-2$.

18. $m^2+mn-30n^2$. 19. x^4+3x^2-28.

20. a^6-10a^3+16. 21. a^8-11a^4-80.

22. $x^2+(a+b)^2x+ab(a+b)^2$.

23. $x^2+(a-b)^2x-ab(a-b)^2$.

24. $x^2+(a+b)x+(a-2b)(b-2a)$.

25. $(x-y)^2+3(y-z)(x-y)-(2y-z)(2z-y)$.

113. Expressions of the form ax^2+bx+c.

When a is a perfect square, the factors may be found by the methods of the previous Article.

When a is not a perfect square, two methods may be used. *First Method.*

If we multiply and divide ax^2+bx+c by a,

we get $=\frac{1}{a}(a^2x^2+ab+ac)=\frac{1}{a}\{(ax)^2+b(ax)+ac$

$=\frac{1}{a}(y^2+by+ac)$, where $y=ax$

Now the expression $y^2+by+ac$ may be resolved by the methods of the previous Article.

Ex. 1. $4x^2+14x+12=(2x)^2+7(2x)+2$

$=(2x)^2+(4+3)(2x)+4\times3$

$=(2x+4)(2x+3)$

or thus: $4x^2+14x+12=4x^2+14x+\frac{49}{4}-\frac{49}{4}+12$

$=\left(2x+\frac{7}{2}\right)^2-\left(\frac{1}{2}\right)^2$

$=\left(2x+\frac{7}{2}+\frac{1}{2}\right)\left(2x+\frac{7}{2}-\frac{1}{2}\right)$

$=(2x+4)(2x+3)$.

Ex. 2 $2x^2+5x+2$. Multiplying and dividing by 2, we have $2x^2+5x+2=\frac{1}{2}\{2\times 2x^2+2\times 5x+2\times 2\}$.

$$\begin{aligned}&\tfrac{1}{2}\{(2x)^2+5(2x)+4\}\\&=\tfrac{1}{2}(y^2+5y+4), \text{ where } y=2x\\&=\tfrac{1}{2}(y+4)(y+1)\\&=\tfrac{1}{2}(2x+4)(2x+1)\\&=(x+2)(2x+1).\end{aligned}$$

Ex 3 $8x^2-26x+15$. Multiplying and dividing by 8, we have $8x^2-26x+15=\frac{1}{8}\{8\times 8x^2-8\times 26x+8\times 15\}$

$$\begin{aligned}&\tfrac{1}{8}\{(2x)^2+5(2x)+4\}\\&=\tfrac{1}{8}\{(8x)^2-26\times 8x+120\}\\&=\tfrac{1}{8}(y^2-26y+120), \text{ where } y=8x\\&=\tfrac{1}{8}(y-20)(y-6)\\&=\tfrac{1}{8}(8x-20)(8x-6)\\&=(2x-5)(4x-3).\end{aligned}$$

Second Method.—

By actual multiplication we have—

$$\begin{aligned}(px+q)(rx+s)&=prx^2+(qr+ps)x+qs.\\(px-q)(rx+s)&=prx^2-(qr-ps)x-qs.\\(px+q)(rx-s)&=prx^2+(qr-ps)x-qs.\\(px-q)(rx-s)&=prx^2-(qr+ps)x+qs.\end{aligned}$$

Examining the above results we find that *an expression of the form* ax^2+bx+c *can also be resolved by taking the product of the co-efficient of the first term and the last term and splitting it up into two factors such that their algebraical sum may be equal to the co-efficient of the middle term.*

Ex 4 $15x^2+38x-24$.

Here $15.24=3.5.3.2.2.2=18.20$.

And $38=18+20$.

$$\begin{aligned}\therefore 15x^2+38x+24&=15x^2+18x+20x+24\\&=3x(5x+6)+4(5x+6)\\&=(5x+6)(3x+4).\end{aligned}$$

Ex. 5 $12x^2+7xy-10y^2$.

Here $12.(-10y^2)=-3.2.2.2.5\,y.y=(+15y)\times(-8y)$.

And $7y=15y-8y$.

$$\begin{aligned}\therefore 12x^2+7xy-10y^2&=12x^2+15xy-8xy-10y^2.\\&=3x(4x+5y)-2y(4x+5y)\\&=(4x+5y)(3x-2y).\end{aligned}$$

EXERCISE 48.

1. $49x^2+49xy+6y^2$.
2. $9(x+1)^2-3(x+1)-6$.
3. $2v^2+v-1$.
4. $8x^2-6x-9$.
5. $2x^2+5xy-12y^2$.
6. $2x^2+x-15$.
7. $6a^2-a-15$.
8. $6x^2-13x+6$.
9. $8x^2-2x-3$.
10. $6x^2-19a^2x+10a^4$.
11. $20x^2+221x+600$.
12. $12m^2-mn-20n^2$.
13. $10a^2-41ab+21b^2$.
14. $18x^2-33x+14$.
15. $24a^2x^2-14avx-3v^2$.
16. $28m^2+23mn-15n^2$.
17. $33a^2+61ab+10b^2$.
18. $35x^2-2xy-48y^2$.
19. $42x^2+5xy-132y^2$.
20. $63x^2+19xy-104y^2$.
21. $168l^2-262lm+195m^2$.
22. $255a^2+542ab+288b^2$.
23. $8p^2(2p+3)^2-6(2p^2+3p)-9$.
24. $2(ab+ca)^2-11(abc+ac^2)+5c^2$.
25. $(x-y)(x^2-y^2)+(x-z)(x-y)-(y+z)$.

114. Expressions which contain a Single Power only of some Letter or Quantity.

Such expressions can be arranged in two groups, one group containing all the terms in which this letter occurs, and the other group containing the remaining terms.

Ex. 1. $ab+cd+ac+bd$. Arranging the expression in two groups, one containing the terms in which b occurs and the other containing the remaining terms,

we have $ab+cd+ac+bd=(ab+bd)+(cd+ac)$

$=b(a+d)+c(d+a)$

$=(a+d)(b+c)$.

Ex. 2. $a^2bx-a^2c-b^2cx+bc^2$.

Here x occurs in the first power only ; hence we have the expression $=(a^2bx-b^2cx)-(a^2c-bc^2)=bx(a^2-bc)-c(a^2-bc)$

$=(a^2-bc)(bx-c)$.

Ex. 3. $ax+bx-ay-by$.

The expression $=x(a-b)-y(a+b)=(a+b)(x-y)$.

EXERCISE 49.

Resolve into factors :—

1. $ac-ad+bc-bd$.
2. $abx^2+(b-a)xy-y^2$.
3. $2v^2+4ax+6bx+12ab$.
4. $2a^3-a^2b+4abx-2b^2x$.

5. $(2x^2-3a^2)x+(2x^2-3a^2)a$.
6. $ax^3+(a+b)x^2+(a+b)x+a$.
7. $1-x-y+xy$.
8. $ax+a+x+1$.
9. $a^2-ab-ac+bc$.
10. $p^2+pq+pr+rq$.
11. $4x-4y-4y+bx$.
12. $a^2+a+b+ab$.

115. Expressions which when arranged in groups of two or more terms, have a factor common to all the groups.

Ex. 1. $x^3-x^2y+xy^2-y^3=(x^3-y^3)-xy(x-y)$
$=(x-y)(x^2+xy+x^2-xy)$
$=(x-y)(x^2+y^2)$

or thus : the expression $=(x^3-x^2y)+(xy^2-y^3)$
$=x^2(x-y)+y^2(x-y)$
$=(x-y)(x^2+y^2)$.

Ex. 2 $ab(x^2+y^2)+xy(a^2+b^2)=abx^2+aby^2+xya^2+xyb^2$
$=(abx^2+xya^2)+(aby^2+xyb^2)$
$=ax(bx+ay)+by(ay+bx)$
$=(bx+ay)(ax+by)$.

EXERCISE 50.

Resolve into factors :—

1. $(3x^2-4b^2)a+(3x^2-4x^2)b$.
2. $x^4+x^2y^2-y^2z^2-z^4$.
3. $x^2-y^2-z^2+2yz+x+y-z$.
4. $x^2-2xy+y^2-x+y$.
5. $a(a+c)-b(b+c)$.
6. $x^4+2x^3a-xa^3-a^4$.
7. $x^4-2x^3y+2xy^3-y^4$.
8. $bx-ac-cx+ab$.
9. $a^2-b^2-c^2-2bc+a-b-c$.
10. $a^3+a^2b+ab^2$.
11. $a^3-2a^2b+ab^2-2b^3$.
12. $xy(1+z^2)+z(x^2+y^2)$.
13. $a(b^2+c^2-a^2)+b(c^2+a^2-b^2)$.
14. $x(xy+x+1)-y(xy+y+1)$.
15. $a(a+1)+2ab+b(b+1)$.
16. $b(a^2+c^2-b^2)+c(a^2+b^2-c^2)$.

116. Expressions of the form $a^3+b^3+c^3-3abc$.

We know that

$$a+b+c)(a^2+b^2+c^2-ab-ac-bc)=a^3+b^3+c^3-3abc.$$

Hence conversely [Art. 105A.]

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$

Or thus,

$$\begin{aligned}
&a^3+b^3+c^3-3abc\\
&=(a+b)^3-3ab(a+b)+c^3-3abc\\
&=(a+b)^3+c^3-3ab\,(a+b)-3abc\\
&=(a+b+c)\{(a+b)^2-c(a+b)+c^2\}-3ab(a+b+c)\\
&=(a+b+c)\{(a+b)^2-c(a+b)+c^2-3ab\}\\
&=(a+b+c)(a^2+2ab+b^2-ac-bc+c^2-3ab)\\
&=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).
\end{aligned}$$

417. Expressions of the form $(a+b+c)^3-a^3-b^3-c^3$.

$$\begin{aligned}
&(a+b+c)^3-a^3-b^3-c^3\\
&=(a+b+c)^3-a^3-(b^3+c^3)\\
&=(b+c)\{(a+b+c)^2+a(a+b+c)+a^2-(b^2-bc+c^2)\}\\
&=(b+c)\{a^2+b^2+c^2+2ab+2bc+2ca+a^2+ab+ac+a^2-b^2\\
&\qquad\qquad +bc-c^2\}\\
&=(b+c)(3a^2+3ab+3bc+3ac)\\
&=3(b+c)\{a(a+b)+c(a+b)\}\\
&=3(b+c)(a+b)(a+c)\\
&=3(a+b)(b+c)(c+a)
\end{aligned}$$

Ex. 1. $(x+2y+2z)^3-(x+y)^3-(y+z)^3-z^3$.

Putting a for $x+y$, b for $y+z$ and c for z, we have the expression $=(a+b+c)^3-a^3-b^3-c^3$

$$\begin{aligned}
&=3(a+b)(b+c)(c+a).\\
&=3(x+y+y+z)(y+z+z)(z+x+y)\\
&=(3x+2y+2z)(y+2z)(x+y+z).
\end{aligned}$$

Ex. 2. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.

Putting x for $b+c-a$, y for $c+a-b$ and z for $a+b-c$, we have the given expression

$$\begin{aligned}
&=(x+y+z)^3-x^3-y^3-z^3\\
&=3(x+y)(y+z)(z+x)\\
&=3(b+c-a+c+a-b)(c+a-b+a+b-c)(a+b-c+b+c-a)\\
&=3(2c)(2a)(2b)\\
&=24abc.
\end{aligned}$$

EXERCISE 51.

Resolve into factors :—

1. $b^3+c^3-a^3+3abc$.
2. $c^3+36c+37$.
3. $28b^3-9b^2c+c^3$.
4. $9a^3+6a^2b-b^3$.

5. $a^3-b^3-c^3-3abc$.

6. $x^6+y^6-z^6+3x^2y^2z^2$.

7. $\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}-3$.

8. $l^3-\frac{1}{8}m^3-\frac{1}{27}n^3-\frac{1}{2}lmn$.

9. $x^3+y^3+1-3xy$.

10. $a^3+8b^3+27c^3-18abc$.

11. $(a+b)^3+(b+c)^3+(c-a)^3-3(a+b)(b+c)(c+a)$.

12. $x^3+\frac{1}{x^3}-2$.

13. $p^3-\frac{1}{27p^3}-2$.

14. $8a^6-9a^3+1$.

15. $1+28a^6-27a^{12}$.

16. $(3+a+b+c)^3-(1+a)^3-(1+b)^3-(1+c)^3$.

17. $(3-x-y-z)^3-(1-x)^3-(1-y)^3-(1-z)^3$.

18. $(a^2+b^2+c^2-ab-ac-bc)^3-(a^2-bc)^3-(b^2-ac)^3-(c^2-ab)^3$.

19. $(x+y+z)^3-(2x-y)^3-(2y-z)^3-(2z-x)^3$.

20. $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$.

118. To prove that $a^3+b^3+c^3=3abc$, if $a+b+c=0$.

Since $a+b+c=0$, $a+b=-c$

$\therefore (a+b)^3=-c^3$

or $a^3+b^3+3ab(a+b)=-c^3$

$\therefore a^3+b^3+3ab(-c)=-c^3$, since $a+b=-c$

$\therefore a^3+b^3-3abc=-c^3$

$\therefore a^3+b^3+c^3=3abc$.

or thus: $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

Since $a+b+c=0$

$a^3+b^3+c^3-3abc=0$

$\therefore a^3+b^3+c^3=3abc$.

Hence, *when the sum of three quantities is zero, the sum of their cubes is equal to three times their product.*

Ex. Resolve into factors $(a-b)^3+(b-c)^3+(c-a)^3$.

Putting x for $a-b$, y for $b-c$ and z for $c-a$, we have the expression $=x^3+y^3+z^3$;

and $x+y+z=(a-b)+(b-c)+(c-a)=0$

$\therefore x^3+y^3+z^3=3xyz$,

i.e., $(a-b)^3+(b-c)^3+(c-a)^3$

$=3(a-b)(b-c)(c-a)$.

EXERCISE 52.

Find the factors of :—

1. $(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3$.
2. $(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^3$.

3. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.
4. $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$.
5. $(x+a)^3-(x-b)^3-(x+b)^3$.
6. $a^3\left(\frac{b}{c}-\frac{c}{b}\right)^3+b^3\left(\frac{c}{a}-\frac{a}{c}\right)^3+c^3\left(\frac{a}{b}-\frac{b}{a}\right)^3$.
7. $(x-2y)^3+(2y-1)^3+(1-x)^3$.
8. $(ax-by)^3+(by-cz)^3+(cz-ax)^3$.
9. $a^3(bz-cy)^3+b^3(cx-az)^3+c^3(ay-bx)^3$.
10. $(a-2b+c)^3+(b-2c+a)^3+(c-2a+b)^3$.
11. $(a+2b-3c)^3+(b+2c-3a)^3+(c+2a-3b)^3$.
12. $(s-a)^3+(s-b)^3-c^3$, where $2s=a+b+c$.
13. $(2a-b)^3+(2b-c)^3+(2c-a)^3$, if $a+b+c=0$.
14. $(3a-s)^3+(3b-s)^3+(3c-s)^3$, if $s=a+b+c$.
15. $a^9(b-c)^3+b^9(c-a)^3+c^9(a-b)^3$, if $a+b+c=0$.

CHAPTER XIII.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

119. Suppose we have an equation of the form $ax+by=c$ where x and y are both unknown quantities, and a, b, c known quantities, it is clear that we shall obtain from it $x=\frac{c-by}{a}$;

and since y is an unknown quantity, the value of x must still be unknown also.

For every value we choose to give to y, we get a corresponding value of x. Thus we get an infinite number of values of x and y which will satisfy the equation. In such a case the solution is said to be *indeterminate* and the equation an *indeterminate* equation.

But if we have a second equation $px+qy=r$ where x and y have the same values as in the first equation, we may reduce this second equation to the form $x=\frac{r-qy}{p}$; p, q and r being known quantities.

Since the value of x is the same in both equations, we have $\frac{c-by}{a}=\frac{r-qy}{p}$; an equation containing only one unknown quantity y, from which therefore the value of y may be found in terms of a, b, c, p, q and r which are known quantities, and x can be determined by substituting the value of y thus found in either of the equations $ax+by=c$ and $px+qy=r$.

As these equations are satisfied by the *same* values of the unknown quantities, they are called **Simultaneous Equations**.

Hence it follows that if we have two unknown quantities, we must have two independent equations.

120. There are *three* methods by which such equations can be solved.

Let the equations be $4x+3y=31$ and $3x+2y=22$.

First Method.—

Equalize the co-efficients of *one* of the unknown quantities.

$4x+3y=31$ ..... (1)

$3x+2y=22$ ..... (2)

The L. C. M of the co-efficients of x is 6, and of those of y is 12. so that it is easier to equalize the co-efficients of x than those of y.

Multiplying (1) by 2, we have $8x+6y=62$ ..... (3)

Multiplying (2) by 3, we have $9x+6y=66$ ..... (4)

Subtracting (3) from (4), we have $x=4$.

Substituting 4 for x in (1),

we get $4\times 4+3y=31$.

$\therefore 3y=15$.

$\therefore y=5$.

The required solution is $x=4$ and $y=5$.

Note.—*The method is the one generally employed.*

Second Method—

Find x in terms of y, or y in terms of x from both equations, and equate the values so obtained.

$4x+3y=31$ ..... (1)

$3x+2y=22$ ..... (2)

From (1), $4x=31-3y$. $\therefore x=\dfrac{31-3y}{4}$ ..... (3)

From (2), $3x=22-2y$. $\therefore x=\dfrac{22-2y}{3}$ ..... (4)

From (3) and (4) $\dfrac{31-3y}{4}=\dfrac{22-2y}{3}$.

$\therefore 93-9y=88-8y$.

$\therefore -y=-5$.

$\therefore y=5$.

From (3), $x=\dfrac{31-15}{4}=4$.

Third Method.—

Find x in terms of y, or y in terms of x from one of the equations, and substitute the value so obtained in the other.

$4x+3y=31$ ..... (1)

$3x+2y=22$ ..... (2)

From (1), $3y=31-4x$. $\therefore y=\dfrac{31-4x}{3}$ ..... (3)

Substituting this value of y in (2),

we get $3x+\dfrac{2(31-4x)}{3}=22$.

$$\therefore 9x+62-8x=66$$
$$\therefore x=4.$$

From (3), $y=\frac{31-16}{3}=5$.

121. Examples worked out.

Ex. 1. Solve the equations

$$5x-8y=12 \quad \ldots(1)$$
$$11x+12y=56 \quad \ldots(2)$$

Multiplying (1) by 3, we have $15x-24y=36 \quad \ldots(3)$

Multiplying (2) by 2, we have $22x+24y=112 \quad \ldots(4)$

Adding (3) and (4), $37x=148$.

$\therefore x=4$.

Substituting 4 for x in (1), we get—

$$20-8y=12.$$
$$\therefore -8y=-8.$$
$$\therefore y=1.$$

Ex. 2. Solve the equations—

$$ax+by=c \quad \ldots(1)$$
$$a'x+b'y=c' \quad \ldots(2)$$

Multiplying (1) by b', $ab'x+bb'y=cb' \quad \ldots(3)$

Multiplying (2) by b, $a'bx+bb'y=c'b \quad \ldots(4)$

Subtracting (4) from (3), $x(ab'-a'b)=cb'-c'b$.

$$\therefore x=\frac{cb'-c'b}{ab'-a'b}$$

Instead of substituting this value of x in either of the equations, it is easier to get the value of y by the same method.

Thus from (1), $a'ax+a'by=a'c$

from (2), $a'ax+ab'y=ac'$.

Subtracting, $y(a'b-ab')=a'c-ac'$.

$$\therefore y=\frac{a'c-ac'}{a'b-ab'}$$

EXERCISE 53.

Solve the following equations according to the first method.

1. $x+y=8$.
   $x-y=2$.
2. $4x+3y=19$.
   $11x-2y=1$.
3. $17x-10y=31$.
   $17x+9y=69$.
4. $7x+5y=17$.
   $8x-3y=2$.
5. $10x+15y=25$.
   $12x-3y=42$.
6. $5x-2y=26$.
   $3y+x=46$.

7. $2x+3y=2$
$6x-15y=-2.$

8. $4x=5y.$
$13x=2y+2\frac{17}{20}.$

9. $2x+3y=2.$
$8y-x=56.$

10. $10x-7y=51.$
$5x+y=12.$

11. $4x+15y=18.$
$3x+5y=1.$

12. $10x-3y=23.$
$5x+2y=8.$

13. $5x+13y=102.$
$x+y=14.$

14. $15x-9y=27.$
$3x+2y=13.$

15. $7x-2y=43.$
$14x+y=66.$

16. $8x-5y=10.$
$7x+y=41.$

Solve the following equations by either the second or the third method.

17. $x=3y-7.$
$5y=4x+7.$

18. $2x=5y-8.$
$3x=4x+2.$

19. $x+y=9.$
$10x-3y=38.$

20. $x-y=7.$
$2x+3y=-1.$

21. $2x-3y=7.$
$3x+2y=30.$

22. $4x+5y=6.$
$3x-y=-5.$

23. $y=1-x.$
$5x+3y=-3.$

24. $10x=15y+10.$
$x=11-y.$

25. $5x+10y=0.$
$2x-3y=21.$

26. $2x=y.$
$5x-3y+4=0.$

[In Examples 27–31 first from two equations by adding the two equations and by subtracting one from the other.]

27. $7x+3y=4.$
$3x+7y=16.$

28. $13x+17y=50.$
$17x+17y=70.$

29. $2x+3y=5.$
$3x+2y=25.$

30. $5x+7y=45.$
$7x+5y=39.$

31. $10x+11y=32.$
$11x+10y=31.$

32. If $4x+3y=2$ and $3x-y=8$, find the value of x and y and also of $7x-y$.

33. If $7x-5y=1$ and $3x+y=13$, find the value of $2x-y$ and $5x+11y$.

34. If $2x=3y$ and $14x-15y=2$ find the value of $10x+21y$ and $4x-9y$.

35 If $4x+10y=3$ and $20x-20y=1$, find the value of $4x-5y$ and $16x-5y$.

36 If the equation $ax+b=y$ is satisfied by $x=4$, $y=1$ and $x=7$, $y=10$, find the values of a and b.

37. If $y=mx+c$ is satisfied by $x=10$, $y=5$ and $x=1$, $y=2$, find the values of a and b.

38 The expression $ax+b$ takes the values of -3, 10 respectively when x takes the values $\frac{2}{3}$ and $-\frac{5}{3}$; find a and b.

39 The expression $mx+c$ will vanish when $x=\frac{1}{2}$ and will be equal to $-\frac{1}{2}$ when $x=\frac{1}{4}$. Find m and c.

40 If the equation $y=ax+b$ is satisfied by the values $(2,-3)$ and $(-3,2)$ of x and y respectively, find a and b.

Ex. 2 Solve the equations :—

$$\frac{x+y}{3}+\frac{x-y}{4}=11 \quad \text{.....A.}$$

$$\frac{x+y}{2}+\frac{x-y}{3}=8 \quad \text{.....B.}$$

Clearing of fractions and simplifying we get

$4x+4y+3x-3y=172$, or $7x+y=132$ – (i).

$3x+3y+2x-2y=48$, or $5x+y=48$ – (ii).

Subtracting (ii) from (i) we get $2x=84 \therefore x=42$.

Substituting 42 for x in (i) or (ii) we get $y=-162$

Ex. 3. Solve the equations :—

$$2x+\cdot 4y=1\cdot 2 \quad \text{.....(1)}$$

$$3\cdot 4x-\cdot 02y=\cdot 01 \quad \text{.....(2)}$$

Multiplying (1) by 3·4, we have $6\cdot 8x+1\cdot 36y=4\cdot 08$.....(3)

Multiplying (2) by 2 we have $6\cdot 8x-\cdot 04y=\cdot 02$ .....(4)

Subtracting, $1\cdot 4y=4\cdot 06$

$\therefore \quad y=2\cdot 9$

Substituting 2·9 for y in (1), we have

$2x+1\cdot 16=1\cdot 2$

$2x \quad =\cdot 04$

$x \quad =\cdot 02$

41. $3x-y=18$

$\frac{x}{2}+\frac{y}{3}=9$

42. $\frac{x}{8}+\frac{y}{3}=6.$

$2x-\frac{y}{4}=29.$

43. $\frac{x}{4}+\frac{y}{5}=2$

$\frac{x}{2}+y=7$

44. $\frac{2x}{5}+\frac{3y}{5}=17$

$\frac{x}{4}-\frac{y}{5}=8$

45. $\frac{2x}{5}-\frac{3y}{5}+5=0$

$3x-y=15$

46. $\frac{3x}{5}+\frac{y}{4}=13$

$\frac{x}{3}-\frac{y}{8}=3$

47. $\frac{x+y}{3}+2x=13.$

$\frac{x-y}{2}+y=\frac{9}{2}.$

48. $\frac{x+y}{3}-\frac{x-y}{5}=0$

$\frac{x+y}{7}-\frac{x-y}{35}+\frac{1}{7}=0$

49. $2x-\frac{y-3}{5}=4.$

$3y=9-\frac{x-2}{3}.$

50. $2x-\frac{y-3}{5}=\frac{5x-2}{2}.$

$2y-\frac{x-5}{3}=\frac{7y-7}{2}$

51. $(x+5)(y+7)=(x+1)(y+9)+1122-x+10=3y+1.$

52. $\cdot 3x+\cdot 125y=3x-y.$

$3x-\cdot 5y=2\cdot 25-3y.$

53. $4x+8y=2\cdot 4$

$10\cdot 2x+\cdot 06y=\cdot 03$

54. $7\cdot 2x+3\cdot 6y=54.$

$2\cdot 3x+6\cdot 9y=23.$

55. $\frac{1}{2}(3x-y)+\frac{1}{3}(x+y)=3$

$\frac{1}{5}(5x-y)-1=0$

56. $(x+1)(y-3)-(x-1)(y+3)+10=0.$

$5x-2y=7.$

57. If $4x+y=63$ and $8x-y=21$ prove that $5x=4y.$

58. If $5x-3y=26$, and $2x+3y=44$ prove that $4x=5y.$

59. Eliminate y between the two equations $y=2x+1$, $3y-4=a$. Also find the values of x and y when $a=11$.

60. Eliminate y between the equations $y=2x-7$, and $3x-2y=a$. Also find the values of x and y when $a=9$.

61. If $5x=4y$ and $x-y=z$,

(i) Find an equation connecting y and z.

(ii) Find an equation connecting x and z.

62. If $x+y=1$ and $2x-y=z$, find an equation connecting.

(i) y and z. (ii) x and z.

63. If the three expressions $4x$, $2x+y$, and $3x-y+15$ are equal, find the values of x and y and also the value of each expression.

64. If the three expressions $3y$, $3x-6$ and $x+y+6$ are equal, find the values of each expression.

CHAPTER XIV.

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

121. The *number* of unknown quantities which enter a problem and the *number* of independent conditions connecting them with known quantities must be the same.

The unknown quantities are usually denoted by x and y.

Ex. 1. Four-fifths of the sum of two numbers is 56, and if the greater is subtracted from three times the less, the remainder is equal to the greater number; find the numbers.

Let x be the greater number and y the less.

From the first condition, $\frac{4}{5}(x+y)=56$

or $4x+4y=280$.....(1)

From the second, $3y-x=x$

$3y=2x$....., ...(2)

Substituting $6y$ for $4x$ in (1) we have

$10y=280$

$y=28$

From (2) $2x=84$

$x=42$.

The two numbers are 42 and 28.

Ex. 2. There is a fraction, such that if its numerator be increased by 1, its value is $\frac{1}{3}$; and if its denominator be increased by 2, its value is $\frac{1}{2}$. What is the fraction?

Let the fraction be $\frac{x}{y}$

If we increase the numerator by 1, the fraction becomes $\frac{x+1}{y}$.

If we increase the denominator by 2 the fraction becomes $\frac{x}{y+2}$

By the conditions of the problem,

$$\frac{x+1}{y}=\frac{1}{3} \text{ and } \frac{x}{y+2}=\frac{1}{2}$$

$\therefore 4x+4=3y$.....(1)

$2x=y+2$.....(2)

$$\therefore 4x = 2y + 4$$
$$\therefore 2y + 4 + 4 = 3y$$
$$\therefore y = 8$$

From (2) $x = 5$

The required fraction is $\frac{5}{8}$.

Ex 3 There is a number consisting of two digits ; the number is equal to seven times the sum of its digits ; and if 27 be subtracted from the number, the digits interchange their places ; find the number.

Let x be the digit in the unit's place and

y be " " ten's "

Then $10y + x$ = the number,

and $10x + y$ = the number when the digits are interchanged

By the conditions of the problem, $10y + x = 7(x + y)$ .....(1)

and $10y + x - 27 = 10x + y$.....(2)

From (1) $3y - 6x = 0$

$\therefore 3y = 6x$

$\therefore y = 2x$

Substituting $2x$ for y in (2) we get $20x + x - 27 = 10x + 2x$

$\therefore 9x = 27$

$\therefore x = 3$

$y = 6$

Hence the required number is 63.

Ex 4. The dimensions of a rectangular court are such that if the length were increased by 3 feet and the breadth diminished by the same, its area would be diminished by 18 square feet; and if its length were increased by 1 foot and the breadth increased by the same, its area would be increased by 18 square feet. Find the dimensions.

Let x = the length in *feet*

and y = the breadth in *feet*

Then xy = the area in *sq. ft.*

By the question, $(x + 3)(y - 3) = xy - 18$.....(1)

and $(x + 1)(y + 1) = xy + 18$.....(2)

From (1), $3(y - x) - 9 = -18$,

$\therefore y - x = -3$

From (2) $x+y+1=18$

$\therefore x+y=17.$

$\therefore x=10$ and $y=7.$

$\therefore$ Length $=10$ft. and breadth $=7$ft.

EXERCISE 84.

1. The sum of two numbers is 25 and their difference is 7; find the numbers.

2. One-third of the sum of two numbers is 23, the sum of twice the greater number and three times the other is 162. Find the numbers.

3. The sum of two numbers is equal to three times the smaller increased by 10, and twice the greater is less than five times the smaller by 10. Find the numbers.

4. There are two numbers, the difference between twice the greater number and the less is 108; three-fourths of the smaller number exceeds half the greater number by 19. Find the numbers.

5. Find two numbers such that one-half of the first and one-third of the second is 14, and one-third of the first and one-half of the second is 11.

6. One-tenth of the sum of two angles is 30° and twice their difference is 20°. Find the angles.

7. One-fifth of the sum of two numbers is less than twice their difference by 1, and one-fifth of the greater is equal to one-fourth of the less. Find the numbers.

8. A said to B, "If you give me one-third of your money, I shall have Rs. 14." B replied, "If you give me only one-fifth of your money, I shall have Rs. 14." How much had each?

9. Eight years ago A was 5 times as old as B, and in 2 years he will be 3 times as old; find their present ages.

10. Twelve years ago A's age was $1\frac{1}{2}$ times that of B, and in 3 years eleven times A's age will be fifteen times that of B. Find their ages.

11. Ten years ago a father was twice as old as his son, and the father's present age is less than twice the son's age by 30 years. Find their present ages.

12. Find the fraction which becomes $\frac{1}{2}$ when 8 is added to the numerator and $\frac{1}{10}$ when 8 is subtracted from the denominator.

13. If the numerator is increased by 2, a fraction becomes $\frac{1}{2}$ and if the denominator is increased by 2, it becomes $\frac{1}{4}$ Find the fraction.

14. If the numerator of a fraction is diminished by 1 it becomes $\frac{1}{3}$, and if the denominator is increased by 2, it becomes $\frac{3}{4}$. Find the fraction.

15. If 1 be subtracted from the numerator and 1 be added to the denominator, a fraction becomes equal to $\frac{1}{2}$; and if the numerator is increased by 1 and the denominator diminished by 1, it becomes unity. Find the fraction.

16. A number consisting of two digits is equal to six times the sum of the digits, and four times the digit in the tens' place is equal to five times the digit in the unit's place. Find the number.

17. A number consists of two digits; if 27 be added to it, the number is reversed, and five times the sum of the digits exceeds the number by 8. Find the number.

18. A number of two digits is reversed if 18 is subtracted from it; and if 4 is subtracted from it, six times the sum of the digits is obtained. Find the number.

19. Two-thirds of a number of two digits is equal to four times the sum of the digits increased by 4, and twice the digit in the ten's place is three times the digit in the unit's place. Find the number.

20. A number of two digits when divided by the sum of the digits gives 4, and the difference between the digits is 3. Find the number.

21. If the length of a rectangular plot of ground is increased by 2 yds. and the breadth diminished by 1 yd, its area remains unaltered But if the length is diminished by 1 yd. and breadth increased by 1 yd. its area is increased by 1 sq. yd. Find the dimensions of the plot.

22. If the length of a rectangular room is increased by 2 ft and the breadth diminished by 2 ft. the area would be increased by 60 sq. ft; but if the length is diminished by 2 ft. and the breadth increased by 2 ft. the area would be increased only by 4 sq. ft. Find the dimensions of the room.

23. The length and breadth of a rectangular room are in the ratio of 5 : 3 ; and if the length be increased by 4 ft and the breadth by 2 ft. the area would be increased by 96 sq. ft. Find the dimensions of the room.

24. If the length and breadth of a retangular room be each diminished by 3 ft. the area would be diminished by 72 sq ft ; and four times the length is equal to five times the breadth Find the area of the room.

25. A person buys 8 lbs. of tea and 5 lbs. of sugar for 19s 11d ; and at another time he buys 5 lb. of tea and 8 lbs. sugar for 18s 8d, the prices being the same as before; find the price of each.

26. The cost of 3 cows and 4 horses is Rs. 635, and the cost of 2 cows and 5 horses is Rs 840. Find the cost of each.

27. A person wishing to relieve a certain number of beggars finds that if he gives them 2s. each, he will require 3s. more, but if he gives them 1s 6*d* each, he will have 4s 6*d* to spare. What money had he in his pocket and how many beggars did he relieve ?

28. A certain sum of money is to be divided among a certain number of men; if there were three men less, each man would have £150 more; but if there were 6 men more, each man would have £120 less ; find the sum of money and the number of men.

29. Two men A and B play at cards; if B wins Rs. 10, A has only five-sixths of what B then has; but if, instead, A wins Rs 20, he has Rs. 5 less than four times the money B then has. Find how much money each starts with.

30. Two men A and B play at cards; if B wins Rs. 5 he will have twice as much as A will then have ; but if, instead, A wins Rs. 5 he will have as much money as B had at first Find with how much money each starts.

EXAMINATION PAPERS.

SECOND SERIES ON CHAPTERS VI to XIV.

1.

1. Substitute $a-3$ for y in the expression $3y^3-2y^2+3y+1$ and arrange the result in descending powers of a.

2. Resolve :—(1) $6x^2+5x-6$

(2) $(3a-b)^2-(2a+b)^2$

3. Simplify $(a-b)^2+(b-c)^2+(c-a)^2+2(a-b)(b-c)+2(b-c)(c-a)+2(c-a)(a-b)$.

4. If $S=\frac{n}{2}\{2a+(n-1)d\}$, find n when $S=275$, $a=11$, $d=3$.

5. Draw the graph of the function $\frac{2x-5}{3}$ and find from it the value of the function when $x=4$.

6. A man bought a house for a rupees and sold it for b rupees; find his gain per cent.

2.

1. Find the value of $2x^2-3xy+2y^2$ in terms of a when $x=a+1$ and $y=a-1$.

2. Resolve into factors :—(1) $x^2-3x-70$.

(2) x^3+x+2.

3. Simplify: $\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)-\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)$.

Hence find the value of

$xy+\frac{1}{xy}$, when $x+\frac{1}{x}=2a$, $y+\frac{1}{y}=2b$, $x-\frac{1}{x}=2m$, $y-\frac{1}{y}=2n$.

4. A man buys $(x+y)$ maunds of sugar at $(a+b)$ rupees a maund and $(x-y)$ maunds at $(a-b)$ rupees per maund; he sells the whole at c rupees per maund. What is his gain ?

5. Solve : $\frac{1}{2}(x-5)+\frac{1}{3}(9x+6)=5x-9$.

6. Plot the graph of $21y=2x+3$ and find the co-ordinates of the point of its intersection with the axis of x.

3.

1. Resolve into factors the expression a^8+a^4+1.

2. Find the co-efficients of x and x^2 in $(ax^3+2x+b)(bx^2-x+c)$.

3. A father is three times as old as his son; in 5 years his age will be $2\frac{1}{2}$ times that of his son. Find their ages.

4. The volume of a right cone is given by the Formula $V=\frac{1}{3}\pi r^2 h$ where V is the volume, r the radius and h the height.

Find r when $V=154$ c. in, $h=3{\cdot}5$ in and $\pi=3\frac{1}{7}$.

5. Solve: $13x+11y=68$.
$11x+13y=76$.

6. Find by drawing the graphs of $x+y=5$ and $3x-y=3$, the point of intersection of the straight lines represented by the equations.

4.

1. What value of x will make the expression $\frac{3x-2}{2}+\frac{1}{3}$ equal to $\frac{4x+1}{3}+\frac{1}{2}$.

2. Solve: $5(x-1)+3(y-1)=21$.
$5(y+1)-3(x+1)=5$,

3. A number consists of two digits whose sum is 9; if 9 is added to the number, it is reversed. Find the number.

4. Use the formula $(a^2-b^2)=(a+b)(a-b)$ to find the difference between the squares of 694057 and 305943.

5. What must be added to
$y(xy+y+1)$ to make it equal to
$x(xy+x+1)$. Express the result in factors.

6. Express $(x^2+5x)^2-8(x^2+5x)-84$ as the product 4 factors.

5.

1. Draw the line traced out by a point which moves so that its co-ordinates are connected by the equation $y=3x-2$. Find also the co-ordinates of the point of intersection of this line with the axis of x.

2. Factorise (1) $4a^2-9b^2+12bc-4c^2$
(2) $x^5+x^3-x^2-1$

3. If the expressions $12x-5y$, $2x+3y$ and $3x+4y-18$ have equal values, find x and y.

4. If $y=a-2$, find the value of the expression y^3-2y^2+3y+5 in terms of a.

5. Find the continued product of $x^2+\sqrt{3}x+3$, $x^2-\sqrt{3}x+3$, x^4-3x+9.

6. The cost of 2 cows and 3 horses is Rs. 620, and the cost of 3 cows and 2 horses is Rs. 505. Find the cost of a cow and a horse.

6

1. Use the identity $(x-y)^2=x^2-2xy+y^2$ to find the square of 9995

2. If $x=a^3+a^2$, $y=a^2+a$, $z=a+1$, find in terms of a the value of (i) xyz (ii) x^2-yz.

3. Solve :—(1) $\frac{x}{8}-\frac{2(x-1)}{5}=\frac{3x-4}{15}+\frac{x}{12}$.

(2) $2x+5y-9=2y=3x-y+9$.

4. Find the continued product of $a+b+c$, $b+c-a$, $c+a-b$ $a+b-c$.

5. If the numerator of a fraction is increased by 2, the fraction becomes $\frac{1}{2}$; if the denominator is increased by 2, it becomes $\frac{1}{4}$. Find the fraction.

6. If $b+c-3a=2x$, $c+a-3b=2y$, and $a+b-3c=2a$, express the value of $(x+2a)(x+2b)(x+2c)$ in terms of a.

7

1. Find a pair of numbers that satisfy $\frac{x}{6}-y=\frac{3y}{4}-17$ and $x-\frac{4y}{5}=\frac{2x}{3}$ simultaneously.

2. Factorise :—(1) $4abx^2-2(a^2+b^2)xy+aby^2$, (2) $abx^2-(a^2-b^2)x-ab$.

3. Verify the following statement for $a=1$, $b=2$, $c=-1$: $a^2(b-c)+b^2(c-a)+c^2(a-b)=(a-b)(b-c)(a-c)$.

4. If in the formula $S=2\pi r(h+r)$, $S=569\frac{1}{4}$, $r=5\frac{1}{4}$, $\pi=\frac{22}{7}$, find the value of h.

5. A and B play at cards ; if B wins Rs. 5, he will have seven-eighths of what A will then have, but if A wins Rs. 5, he will have twice as much as B. Find with how much money they start.

6. The expression $ax+b$ is equal to 16 when $x=3$ and is equal to -19 when $x=-2$; find a and b.

7. Show that the sum of the squares of the greatest and the least of any three consecutive numbers exceeds twice the square of the middle number by 2.

8

1. In a certain book the following equation is required to be solved: $(2x-1)(x+\ \ )=2x(x+2)$; a number is omitted by oversight. The answer is given to be $x=3$. Find the number omitted.

2. If $3x=4y$ and $x+y=z$; find an equation connecting (i) x and z (ii) y and z.

3. Eliminate y between the equations $y=2x-7$ and $3x-2y=a$. Also find the values of x and y when $a=9$.

4. If the length of a rectangular hall is diminished by 5 ft. and the breadth increased by 2 ft. its area would be diminished by 30 ft. but if only the breadth be increased by 6 ft. the area would be increased by 300 sq. ft. Find the dimensions of the hall.

5. Factorise :—(1) $x^4-2x^3y+2xy^3-y^4$.

(2) $a(a+1)+2ab+b(b+1)$.

6. Show that $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$. Use this to find the value of $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.

PART II.

CHAPTER XV

MULTIPLICATION DIVISION AND RESOLUTION INTO FACTORS.—*(Continued.)*

122. To multiply any two expressions by the Method of Detached Co-efficients.

In multiplying together two algebraical expressions, a good deal of labour may be saved by merely writing the co-efficients, and multiplying them together in the ordinary way inserting the powers of x in the product.

Ex. 1. Multiply $2x^3+x^2-3x+1$ by x^3-2x^2-x+3.

Write only the co-efficients and multiply thus :—

$$\begin{array}{r} 2+1-3+1 \\ 1-2-1+3 \\ \hline 2+1-3+1 \\ -4-2+6-2 \\ -2-1+3-1 \\ +6+3-9+3 \\ \hline 2-3-7+12+4-10+3 \end{array}$$

∴ The product is $2x^6-3x^5-7x^4+12x^3+4x^2-10x+3$.

We can see whether the process of multiplication is correct by giving particular values to x.

Thus in the above examples if $x=2$

the multiplier $=2^3-2.2^2-2+3=1$

the multiplicand $=2.2^3+2^2-3.2+1=15$

the product $=2.2^6-3.2^5-7.2^4+12.2^3+4.2^2-10.2+3.$

$=243-228=15.$

The process of multiplication is correct because $15\times1=15$.

Generally the process is verified by putting $x=1$, which is the same thing as taking into consideration only the co-efficients of the expressions.

Thus $(2+1-3+1)(1-2-1+3)=2-3-7+12+4-10+3$

$1\times1 \qquad =21-29=1.$

Note.—We may use the same method even in cases where all the powers of x are not present or where the co-efficients are literal provided we insert the missing powers of x with zero co-efficients and enclose within brackets the sum of the co-efficients of similar powers.

Ex. 2. Multiply $3x^4 - 2x^3 - ax + 4$ by $4x^2 - 3$.

Write only the co-efficients, supplying 0 for the co-efficients of the missing terms and multiply.

$$\begin{array}{r} 3 - 2 + 0 - a + 4 \\ 4 + 0 - 3 \\ \hline 12 - 8 + 0 - 4a + 16 \\ \quad - 9 + 6 - 0 + 3a - 12 \\ \hline 12 - 8 - 9 - 2(2a - 3) + 16 + 3a - 12 \end{array}$$

$\therefore$ The product is

$$12x - 8x^5 - 9x^4 - 2(2a - 3)x^3 + 16x^2 + 3ax - 12$$

Verification :

$(3 - 2 - a + 4)(4 - 3) = 12 - 8 - 9 - 2(2a - 3) + 16 + 3a - 12$. $(5 - a) \times 1 = 5 - a$.

EXERCISE 55.

Multiply by the *Method of Detached Co-efficients* and verify the results :—

1. $x^2 - 2x + 3$ by $x - 2$.
2. $x^3 + 3x^2 - 4x - 5$ by $2x + 1$.
3. $x - mx^2 + 3 - 5x^3$ by $1 - 3x$.
4. $6x^3 - x + 1$ by $6x^3 + x - 1$.
5. $x^4 - 2x^2 + 1$ by $x^4 + 2x^2 + 1$.
6. $x^3 - 2x^2y + 2xy^2 - y^3$ by $x^2 - xy + y^2$.
7. $x^3 + x^2 + x + 1$ by $x^3 + x^2 + x + 1$.
8. $x^3 + x^2y + xy^2 + y^3$ by $x^2 + 2xy + y^2$.
9. $1 + 7x + 10x^4$ by $1 + 11x + 30x^4$.
10. $\frac{1}{8}x^3 + \frac{1}{4}x^2 + \frac{1}{2}x + 1$ by $\frac{1}{4}x^2 - \frac{1}{2}x + 1$.
11. $2a^4 - 4a^2 + 5a - 6$ by $3a^2 - 2$.
12. $a^5 + 2a^3 - pa + 1$ by $a^2 - a - 1$.
13. $(a^2 + a + 1)(a^2 - a + 1)(a^4 - a^2 + 1)(a^8 - a^4 + 1)$.
14. $(x^3 + 3x^2 + 3x + 1)(x^3 - 3x^2 + 3x - 1)$.
15. $(a^4 + a^3 + a^2 + a + 1)(a^4 - a^3 + a^2 - a + 1) \times (a^8 - a^6 + a^4 - a^2 + 1)$.
16. $(x^4 - ax^3 + bx^2 - cx + d)(x^4 + ax^3 - bx^2 + cx - d)$.

128. To multiply any two expressions involving brackets without removing the same.

Ex. 1. Multiply without removing the brackets :—

$x^2(a-b)+x(a+b)+1$ by $x(a+b)+(a-b)$.

$x^2(a-b)+x(a+b)+1$

$x(a+b)+(a-b)$

$x^3(a^2-b^2)+x^2(a+b)^2+x(a+b)$

$+x^2(a-b)^2+x(a^2-b^2)+(a-b)$

$x^3(a^2-b^2)+2x^2(a^2+b^2)+x(a^2-b^2+a+b)+(a-b)$

Ex. 2. Multiply without removing the brackets :—

$(a^2-a+1)x^2+(a-1)x-2$ by $(a^2+a+1)x+(a+1)$.

$(a^2-a+1)x^2+(a-1)x-2$

$(a^2+a+1)x+(a+1)$

$(a^4+a^2+1)x^3+(a^3-1)x^2-2(a^2+a+1)x$

$+(a^3+1)x^2+(a^2-1)x-2(a+1)$

$(a^4+a^2+1)x^3+2a^3x^2-(a^2+2a+3)x-2(a+1)$

EXERCISE 56.

Multiply without removing the brackets:—

1. $a+b(x-y)$ by $b-a(x+y)$.
2. $x(a+b)-y$ by $x(a-b)+y$.
3. $(m+n)x-y(a-b)$ by $(m-n)x+y(a+b)$.
4. $x^2+(a+b)x+c$ by $(a-b)x-c$.
5. $(x-y)^2-2(x+y)+1$ by $(x-y)-1$.
6. $l^2(m+n)+l(m-n)+1$ by $l(m-n)+1$.
7. $a^2+(x-y)a+(x+y)$ by $a(x+y)-(x^2-xy+y^2)$.
8. $(a^2+ab+b^2)x^2+(a+b)x+1$ by $(a-b)x-1$.
9. $(x^2-xy+y^2)a^2-(x-y)a-2$ by $(x+y)a+2$.
10. $(a+b)x^2-(b+c)xy+(c+a)y^2$ by $(a-b)x+(b-c)y$.
11. $x^2(p+q-r)+x(q+r-p)+(r+p-q)$ by $(p+q+r)x+(p-q-r)$.
12. $(x-1)m^2+(x-2)m+(x-3)$ by $(x+1)+(x+2)m+(x+3)m^2$.
13. $x(a+b)+y(b+c)+z(c+a)$ by $x(a-b)+y(b-c)+z(c-a)$.
14. $(a-1)(m^2+m+1)p+(m+1)(a^2-a+1)q$
    by $(m-1)(a^2+a+1)-(a+1)(m^2-m+1)$.
15. $(p^2+pq+q^2)m^2-(p+q)m+(p-q)$
    by $(p^2-pq+q^2)m^2+(p-q)m-(p+q)$

134. To divide one expression by another by the Method of Detached Co-efficients

Ex 1. Divide :—

$3x^6 - 10x^5 + 4x^4 + 7x^3 - 3x^2 + 3x + 2$ by $x^2 - 2x - 1$.

Write only the co-efficients and divide thus :—

$$3 - 10 + 4 + 7 - 3 + 3 + 2 \,\big|\, \begin{matrix} 1 - 2 - 1 \\ \hline 3 - 4 - 1 + 1 - 2 \end{matrix}$$

$3 - 6 - 3$

$-4 + 7 + 7$

$-4 + 8 + 4$

$-1 + 3 - 3$

$-1 + 2 + 1$

$1 - 4 + 3$

$1 - 2 - 1$

$-2 + 4 + 2$

$-2 + 4 + 2$

∴ the quotient is $3x^4 - 4x^3 - x^2 + x + 2$

Verification :

$(3 - 10 + 4 + 7 - 3 + 3 + 2) + (1 - 2 - 1) = 3 - 4 - 1 + 1 - 2$

$6 + -2 = -3.$

Note.—We may use the same method even in cases where all the powers of x are not present or where the co-efficients are literal provided we insert the missing powers of x with zero co-efficients and enclose within brackets the sum of co-efficients of similar powers.

Ex. 2. Divide :—

$6x^5 - 8x^4 - 15x^3 + 2(m+10)x^2 - 5m$ by $2x^2 - 5$.

$$6 - 8 - 15 + 2\,(m+10) + 0 - 5m \,\big|\, \begin{matrix} 2 + 0 - 5 \\ \hline 3 - 4 + 0 + m \end{matrix}$$

$6 + 0 - 15$

$-8 + 0 + 2(m+10)$

$-8 - 0 + 20$

$2m \quad +0 - 5m$

$2m \quad +0 - 5m$

∴ The quotient is $3x^3 - 4x^2 + m$

Verification :

$\{6 - 8 - 15 + 2(m+10) - 5m\} + (2 - 5) = 3 - 4 + m$

$(3 - 3m) + -3 = -1 + m.$

EXERCISE 57.

Divide by the *Method of Detached Co-efficients* :—

1. $3x^4-7x^3+13x^2-7x+6$ by x^2-2x+3.
2. $5x^5-18x^4-8x^3+20x-5$ by x^3+2x^2-3.
3. $7x^5+19x^4-6x^3$ by $x+3$.
4. $11x^4-2x^3+14x-539$ by x^2-7.
5. $p^6-8p^5+17p^4-12p^3+10$ by p^2-5.
6. $x^7+x^6+x^5+2x^4-x^3-x^2-2x-1$ by x^4+x^3+x+1
7. $a^{12}+a^{10}+a^8+a^6+a^4+a^2+1$ by $a^6-a^5+a^4-a^3+a^2-a+1$.
8. $a^6+a^5+a^4+a^3+1$ by $a^4-a^3+a^2-a+1$.
9. $a^3-b^3+a^2b+ab^2$ by a^2+b^2.
10. a^4-81 by $a+3$.
11. $\frac{1}{7}a^3-\frac{1}{2}a^2b+\frac{1}{8}ab^2-\frac{1}{27}b^3$ by $\frac{1}{2}a-\frac{1}{3}b$.
12. $m^6n-5m^5n^2+15m^4n^3-21m^3n^4+18m^2n^5-5mn^6$ by m^2n-n^2m.
13. $acx^2-x(bc+ad)+bd$ by $cx-d$.
14. $2a+(3+2m)a^2+(2+m)a^3+(2+m)a^4+(4+5m)a+5$ by a^2+am+1.
15. $abx^4+x^3(b^2-ac)+a^2x^2+a(ab-c^2)+ca$ by bx^2-cx+a.

125 To divide one expression by another, either or both of them involving brackets, without removing the same.

Ex. 1. Divide without removing the brackets :—

$x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ by $x^2-(b+c)x+bc$.

$x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ | $x-a$ | $x^2-(b+c)x+bc$

$x^3-(b+c)x^2+bcx$

$-ax^2+(ab+ac)x-abc$

$-ax^2+a(b+c)x-abc$.

Ex. 2. Divide without removing the brackets :—

$(a^3+1)x^3+ax^2(a-1)+ax(a+3)+(a+1)$ by $x(a+1)+1$.

$(a^3+1)x^3+ax^2(2a-1)+ax(a+3)+(a+1)$ | $x(a+1)+1$ | $x^2(a^2-a+1)+(a-1)+(a+1)$

$x^2(a^2-1)+ax(a+3)+(a+1)$

$x^2(a^2-1)+x(a-1)$

$x(a+1)^2+(a+1)$

$x(a+1)^2+(a+1)$

EXERCISE 58.

Divide without removing the brackets :—

1. $(x^3+y^3)a^3+a^2y(x-2y)+2ay-1$ by $(x+y)a-1$.
2. $(m^3-1)x^3-(m^2+2m+3)x^2-(m+3)x-2$ by $(m-1)x-2$.
3. $(a+1)^4+2(a+1)^3-2(a+1)^2+2(a+1)-3$ by $(a+1)^2+1$.
4. $(p+q)^3-3(p+q)(q+r)^2+2(q+r)^3$ by $(p+q)+2(q+r)$.
5. $l^3(m^2-1)-2l^2(m^2+1)+l(m^2+3m+2)-3(m-1)$ by $l(m+1)-(m-1)$,
6. $a^3(x^2+5x+6)+a^2(4x+7)+4a(x+1)+(x+1)^2$ by $a(x+2)+(a+1)$.
7. $m^3(a^3-1)-m^2(a^3+a^2-2)+m(4a^2+3a+2)-3(a+1)$ by $m(a^2+a+1)-(a+1)$
8. $(a^2-a-20)x^2-2xy(7a+1)-(a^2+a-6)y^2$ by $(a-5)x-(a+3)y$.
9. $(p^2+3pq+2q^2)m^2+(5q^2-2p^2)m+(p^2-3pq+2q^2)$ by $(p+q)m+2q-p$.
10. $(a-b)^4-(a-b)^3-7(a-b)-3$ by $(a-b)^2+(a-b)+3$.
11. $(x-1)a^3+(x^3+2x-2)a^2+(3x^2-x^3)a-x^4$ by $a^2x+2a-x^2$.
12. $(x^3-y^3)a^2-(x^3-x^2+y^2-y^3)a^2-(4x^2+3xy+2y^2)a-3(x+y)$ by $(x-y)(a^2-a)-3$.
13. $3(c-b)a^2-(c^2-b^2)a+c^2(c-3b)+b^2(3c-b)$ by $3a^2-(b+c)a+(b-c)^2$.
14. $x^4(x-1)^2-x^2y^2(x^2+y^2-1)+y^4(y-1)^2$ by $x^2(x-1)+xy(x+y-1)+y^2(y-1)$,
15. $(b^2-c^2)x^4+2c(b-a)x^3+(c-a)(c+a-2b)x^2+2a(c-b)x+a^2-b^2$ by $(b-c)x^2+(c-a)x+a-b$.

Ex. Divide :—

$$x-a\;\Big)\;x^3+px^2+qx+r\;\Big(\;x^2+(p+a)x+q+a(p+a)$$

$$x^3-ax^2$$

$$(p+a)x^2+qx$$

$$(p+a)x^2-a(p+a)x$$

$$x\{q+a(p+a)\}+r$$

$$x\{q+a(p+a)\}-a\{q+a(p+a)\}$$

$$a\{q+a(p+a)\}+r$$

The co-efficients of the various powers of x in the quotient are 1, $p+a$, $q+a(p+a)$. These may be got without actual division by the following arrangement :—

Write down the co-efficients in the dividend thus :—

$$\begin{array}{cccc} 1+p & +q & + & r \\ +a & +a(p+a) & + & a\ \{a(p+a)+q\} \\ \hline 1+(p+a)+\{a(p+a)+q\}+[r+a\{a(p+a)+q\}] & & & \end{array}$$

The first term in the second line is got by multiplying 1 by a. The product a is placed below p and then added to it so as to give the term $p+a$ in the third line. This term $(p+a)$ is multiplied by a and written below q, and added to it so as to give the third term in the third line. Similarly a $\{a(p+a)+q\}$ added to r gives the remainder.

The first three terms in the third line give the quotient $x^2+x(p+a)+q+a(p+a)$, and the last term the remainder a^3+pa^2+qa+r.

This method is called synthetic division, and, when used with numerical co-efficients, this process looks very short.

Ex. 1. Divide :—

x^3-8x^2+9x-6 by $x-2$.

Detach the co-efficients thus :—

$$\left.\begin{array}{r} 1-8+9-6 \\ +2-12-6 \\ \hline 1-6-3-12 \end{array}\right\} 2 \text{ is the operator.}$$

$$\frac{1}{1}+\frac{-8}{1\times 2 \text{ or } 2}-\frac{+9}{6\times 2 \text{ or } 12}-\frac{-6}{3\times 2-6}$$

$\therefore$ x^2-6x-3 is the quotient and -12 is the remainder.

This may be verified by actual division.

Ex. 2. $x^4-6x^3+8x^2+9x+1$ by $x+3$.

$$\left.\begin{array}{r} 1-6+8+9+1 \\ -3+27-105+288 \\ \hline 1-9+35-96+289 \end{array}\right\} \text{Here } -3 \text{ is the operator, as } x+3=x-(-3)$$

$\therefore$ $x^3-9x^2+35x-96$ is the quotient ; $+289$ is the remainder.

Ex. 3. Divide

$6x^2+8x-9$ by $2x-1$.

By actual division.

$$2x-1\overline{)6x^2+8x-9}\left(3x+\frac{11}{2}\right.$$

$$\begin{array}{r} 6x^2-3x \\ \hline 11x-9 \\ 11x-\frac{11}{2} \\ \hline -\frac{7}{2} \end{array}$$

$\therefore$ $3x+\frac{11}{2}$ is the quotient and $-\frac{7}{2}$ is the remainder.

Q is the quotient and R the remainder.

$\therefore$ $6x^2+8x-9=Q(2x-1)+R$.

Divide throughout by 2

$\therefore$ $3x^2+4x-\frac{9}{2}=Q\left(x-\frac{1}{2}\right)+\frac{R}{2}$.

Hence divide $3x^2+4x-\frac{3}{2}$ by
$x-\frac{1}{2}$ the quotient will be Q
but the remainder will be only $\frac{R}{2}$.

$3+4-\frac{3}{2}$ operator $\frac{1}{2}$.
$0+\frac{3}{2}+\frac{11}{4}$.
$3+\frac{11}{2}-\frac{7}{4}$.

$\therefore$ $3x+\frac{11}{2}$ is the quotient.
$-\frac{7}{4}$ is $\frac{R}{2}$ $\therefore$ R $=-\frac{7}{2}$ or thus :
$6x^2+8x-9=Q\,(2x-1)+R$.
$=2Q\,(x-\frac{1}{2})+R$.

i.e. If we divide $6x^2+8x-9$ by $x-\frac{1}{2}$
the quotient will be $2\,Q$ and the remainder R.

$6+8-9$
$0+3+\frac{11}{2}$ operator $\frac{1}{2}$.
$6+11-\frac{7}{2}$

$\therefore$ $6x+11$ is $2\,Q$ $\therefore Q=3x+\frac{11}{2}$ and $R=-\frac{7}{2}=-3\frac{1}{2}$.

Thus when an expression in x is divided by $ax+b$, the remainder is the same as when it is divided by $x+\frac{b}{a}$ but the quotient is $\frac{1}{a}$ of the quotient got by dividing it by $x+\frac{b}{a}$, or every term in the expression may be divided by a and the resulting expression divided by $x+\frac{b}{a}$; the quotient is the same as when the original expression is divided by $ax+b$ but the remainder will have to be multiplied by a, to get the same remainder as when the original expression is divided by $ax+b$.

EXERCISE 59.

Divide without actual division and find the quotient and the remainder :—

(1) $8x^3+3x^2-5x+3$ by $x+1, x-1$
$x+2, x-2$, by $x+3$, $x-3$

(2) $7x^4-6x^2+8x+3$ by $x-2$, $x+8$, $x-7$.

(3) $6x^3-8x^2+3x-1$ by $3x+2$
$3x-2$, $2x+3$, $2x-3$, $2x+1$
$2x-1$, $3x-1$, $3x+1$ (Retain the dividend as it is).

(4) Work out all the questions in 3 by modifying the dividend suitably.

Functional Notation.

In the expression ax^3+bx^2+cx+d, x is said to be a variable because x may take any value; whereas a, b, c and d are said to be constants because a, b, c, and d take only a particular set of values, *e.g.*, the value of $6x^3-8x^2+9x-3$ depends on the value we assign to x and not on the co-efficients of the several powers of x which retain a fixed value. The expression which contains the variable x in its several powers with constant co-efficients is said to be a function of x usually represented by the symbol $f(x)$, [Art. 85.]

$f(x)\equiv 6x^3-8x^2+9x-3.$

when $x=1$, $6x^3-8x^2+9x-3=6-8+9-3=4.$

$x=2$, the expression $=6\cdot 8-8\cdot 4+9\cdot 2-3=31.$

$x=3$, the expression $=6\cdot 27-8\cdot 9+9\cdot 3-3=114.$

The value of the expression in x or the function of x when the variable x is given the value 1 is denoted for the sake of brevity $f(1)$; similarly $f(2)$ means the value of the expression or the function when the variable x is given the value 2 and so on. Thus: 8,

$f(x)=6x^3-8x^2+9x-3.$

$f(1)=6\cdot 1^3-8\cdot 1^2+9\cdot 1-3=4.$

$f(2)=6\cdot 2^3-8\cdot 2^2+9\cdot 2-3=31.$

$f(3)=6\cdot 3^3-8\cdot 3^2+9\cdot 3-3=114.$

Similarly

$f(a)=6a^3-8a^2+9a-3.$

$f(y)=6y^3-8y^2+9y-3.$

If there is another expression, *viz.*, $7x^2+9x+1$, or function of x and both have to be represented in the functional notation, we use

$F(x)$ for one and $\phi(x)$ for the other.

$F(x)\equiv 6x^3-8x^2+9x-3$

$\phi(x)\equiv 7x^2+9x+1.$

Examples.

(1) If $F(x)=6x^3+7x-1$,

find $F(x+1)-F(x-1)$

$F(x+1)=6(x+1)^3+7(x+1)-1$

$F(x-1)=6(x-1)^3+7(x-1)-1$

$$\therefore F(x+1)-F(x-1)=6\{(x+1)^2-(x-1)^2\}+7\{(x+1)-(x-1)\}-1+1$$

$$\Rightarrow 6\cdot 4x+7\cdot 2$$

$$=24x+14.$$

(2) If $\phi(x)=10x^2+11x+9$,

find $\phi(2)+\phi(-2)$

$$\phi(2)=10\cdot 4+11\cdot 2+9$$

$$\phi(-2)=10\cdot 4-11\cdot 2+9$$

$$\therefore \phi(2)+\phi(-2)=2\cdot 10\cdot 4+2\cdot 9=98$$

(3) $F(x)=6x^2+5x+3$

$\phi(x)=6x^2-5x+1$

Find $F(x+2)-\phi(x-2)$

$$F(x+2)=6(x+2)^2+5(x+2)+3$$

$$\phi(x-2)=6(x-2)^2-5(x-2)+1$$

$$\therefore F(x+2)-\phi(x-2)$$

$$=6\{(x+2)^2-(x-2)^2\}+5\{(x+2)+(x-2)\}+3-1$$

$$=6\cdot 8x+5\cdot 2x+2$$

$$=58x+2.$$

(4) If $F(x, y)=1+x+y+xy$,

find $F(1,2), F(1,0)$

$$F(1,2)=1+1+2+1\cdot 2=6$$

$$F(1,0)=1+1+0+1\cdot 0=2$$

(5) If $F(x, y, z)=x^3+y^3+z^3-3xyz$, find $F(1, 2, 3)$

$$F(1, 2, 3)=1^3+2^3+3^3-3\cdot 1, 2, 3=36-18=18$$

127. **Remainder Theorem**. *When an algebraical expression in x is divided by $x-a$, the remainder is obtained by substituting a for x in the expression.*

Dividing px^2+qx+r by $x-a$, we have

$$\begin{array}{l} x-a)px^2+qx+r(px+ap+q \\ \quad px^2-apx \\ \hline \quad x(ap+q)+r \\ \quad x(ap+q)-a(ap+q) \\ \hline \quad pa^2+qa+r \end{array}$$

The remainder is the same as the given expression with a written instead of x.

If we divide $x^4+px^3+qx^2+rx+s$ by $x-a$, the remainder will be $a^4+pa^3+qa^2+ra+s$.

Since $x+a=x-(-a)$,

when an expression in x is divided by $x+a$, the remainder is obtained by substituting $-a$ for x in the expression.

If $f(x)$ is divided by $x-a$, let Q be the quotient, and R the remainder; the degree of Q is less than that of $f(x)$ and R should be free from x (because the degree of remainder is always less than that of the divisor which is of the first degree.)

Then $f(x)\equiv Q(x-a)+R$.

This is true for all values of x and hence true when $x=+a$,

$\therefore f(a)=Q(a-a)+R$ (R is free from x)

$\therefore f(a)=0+R=R$.

which proves the theorem.

$\therefore$ if px^2+qx+r be divided by $x+a$, the remainder will be pa^2-qa+r.

If we divide $px^5+qx^4+rx^3+sx^2+tx+u$ by $x+a$, the remainder will be $-pa^5+qa^4-ra^3+sa^2-ta+u$.

If $f(x)$ is divided by $x-a$ the remainder is $f(a)$.

The condition that $f(x)$

is divisible by $x-a$ is that $f(a)=0$

128. In order that one expression may be exactly divisible by another, the remainder arising from the division must be equal to zero.

Hence, the rule for finding the condition of perfect divisibility is "*divide as far as possible in the ordinary way, and then put the remainder* $=0$, *and deduce the necessary condition from this.*"

Ex What value of x will make $x^3-5x^2+4x-16$ exactly divisible by x^2-3x-7?

Dividing in the usual way we get for the quotient $x-2$ and for the remainder $5x-30$.

For perfect divisibility, this remainder $5x-30$ must be equal to 0

$\therefore 5x=30$, or $x=6$.

129. Find the condition when x^3+ax^2+bx+c is divisible by x^2+px+q. Divide the first by the second in the usual way.

$$x^2+px+q \;\Big)\; x^3+ax^2+bx+c \;\Big(\; x+(a-p)$$

$$x^3+px^2+qx$$

$$x^2(a-p)+(b-q)x+c$$

$$x^2(a-p)+px(a-p)+q(a-p)$$

$$x\{(b-q)-p(a-p)\}+c-q(a-p)$$

Now this remainder should be equal to zero if x^2+px+q is a factor of the other expression.

$$\therefore x\{(b-q)-p(a-p)\}=q(a-p)-c$$

$$\text{or } x=\frac{q(a-p)-c}{b-q-p(a-p)}$$

Hence when x has this value, the first expression is divisible by the second.

But if $b-q-p(a-p)=0$, and also $c-q(a-p)=0$, the remainder is equal to zero irrespective of the value of x.

So, for all values of x, x^2+px+q is a factor of x^3+ax^2+bx+c, provided that $b-q-p(a-p)=0$ and $c-q(a-p)=0$.

130. An expression such as $px^4+qx^3+rx^2+sx+t$ will be exactly divisible by $x-a$ if the remainder arising from the division, viz., $f(a)$ or $pa^4+qa^3+ra^2+sa+t=0$; and by $x+a$ if the remainder, viz. $f(-a)$ or $pa^4-qa^3+ra^2-sa+t=0$.

In other words, an expression in x is exactly divisible by $x-a$, if it vanishes when a is substituted for x; and by $x+a$ if it vanishes when $-a$ is substituted for x.

Ex. 1. x^2-a^2 vanishes when a is put for x and also when $(-a)$ is put for x.

Therefore x^2-a^2 is divisible by $x-a$ and $x+a$.

Ex. 2. x^3-a^3 vanishes when a is put for x.

Therefore it is divisible by $x-a$.

When $(-a)$ is put for x, it does not vanish.

Therefore it is *not* divisible by $x+a$.

Ex. 3. To prove that x^n-a^n is divisible by $x-a$.

Since x^n-a^n vanishes when a is put for x.

x^n-a^n is divisible by $x-a$.

Ex. 4. To prove that x^n+a^n is divisible by $x+a$ when n is an odd integer.

$$x^n+a^n=(-a)^n+a^n=-a^n+a^n=0.$$

For $(-a)^n=-a^n$ as n is odd.

Hence x^n+a^n vanishes when we write $(-a)$ for x.

$\therefore$ it is divisible by $x+a$.

Ex. 5. To prove that x^n-a^n is divisible by $x+a$ when n is an over integer.

$$x^n-an=(-a)^n-an=an-an=0.$$

For $(-a)^n = +a^n$ when n is even.

Hence $x^n - a^n$ vanishes when we write $(-a)$ for x.

$\therefore$ it is divisible by $x+a$.

Ex. 6. To prove that $x^n - a^n$ is *not* divisible by $x+a$ when n is odd.

$$x^n - a^n = -a^n - a^n = -2a^n.$$

Hence the expression $x^n - a^n$ does not vanish when we write $(-a)$ for x.

$\therefore$ it is not divisible by $x+a$.

In the same way it can be shown that $x^n + a^n$ is *not* divisible by $x+a$ or by $x-a$, when n is even.

The results arrived at in Examples 3, 4, 5 and 6 may be stated thus :—

$x^n - a^n$ is divisible by $x-a$, when n is odd or even.
$x^n - a^n$ is divisible by $x+a$, when n is even.
$x^n - a^n$ is not divisible by $x+a$, whon n is odd.
$x^n + a^n$ is divisible by $x+a$, when n is odd.
$x^n + a^n$ is not divisible br $x+a$, or $x-a$, when n is even.

131. I. By actual division, $\dfrac{x^2 - a^2}{x-a} = x + a$;

$$\frac{x^3 - a^3}{x-a} = x^2 + xa + a^2 ;$$

$$\frac{x^4 - a^4}{x-a} = x^3 + x^2a + xa^2 + a^3 ; \text{ \&c.}$$

We observe that the number of terms in the quotient is the same as the index of the power of x or a in the dividend ; that the terms are *all positive*; and that while the indices of the powers of x decrease by unity, those of the powers of a increase by unity in each succeeding term.

$$\therefore \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1}.$$

II. By actual division,

$$\frac{x^3 + a^3}{x+a} = x^2 - xa + a^2 ;$$

$$\frac{x^5 + a^5}{x+a} = x^4 - x^3a + x^2a^2 - xa^3 + a^4 ; \text{ \&c.}$$

We observe that the terms in the quotient are alternately positive and negative; and that in other respects they obey the same laws as those of I.

$$\therefore \frac{x^n+a^n}{x+a} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 + \ldots + x^2a^{n-3} - xa^{n-2} + a^{n-1}.$$

III. By actual division,

$$\frac{x^2-a^2}{x+a} = x - a;$$

$$\frac{x^4-a^4}{x+a} = x^3 - x^2a + xa^2 - a^3;$$

$$\frac{x^6-a^6}{x+a} = x^5 - x^4a + x^3a^2 - x^3a^3 - x^2a^3 + xa^4 - a^5; \text{ \&c.}$$

Here also we observe the same laws as in II.

$$\therefore \frac{x^n-a^n}{x+a} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \ldots - x^2a^{n-3} + xa^{n-2} - a^{n-1}.$$

Ex. 1. If $a^{2n+1}+b^{2n+1}$ be divisible by a^3+b^3, shew that $n-1$ is a multiple of 3.

$$a^{2n+1}+b^{2n+1} = (a^3)^{\frac{2n+1}{3}} + (b^3)^{\frac{2n+1}{3}}.$$

Since this is divisible by a^3+b^3, $\frac{2n+1}{3}$ is an odd number.

$$\therefore \frac{2n+1}{3} = 2p+1 \text{ (general form of an odd number) where}$$

p has values, 0, 1, 2, 3, &c.

$$\therefore 2n+1 = 6p+3.$$

$$\therefore 2n-2 = 6p.$$

$$\therefore n-1 = 3p.$$

Ex. 2. $5^n - 4n - 1$ is divisible by 4^2 or 16.

$$5^n - 4n - 1$$

$$= (5^n-1) - 4n = (5-1)(5^{n-1}+5^{n-2}+\ldots+5+1) - 4n$$

$$= 4(5^{n-1}+5^{n-2}+5^{n-3}+\ldots+5+1-n)$$

$$= 4\{(5^{n-1}-1)+(5^{n-2}-1)+(5^{n-3}-1)+(5-1)+(1-1)\}.$$

[*Splitting n into n ones*].

$$= 4(5-1)\{(5^{n-2}+\ldots+1)+(5^{n-3}+\ldots+1)+\text{\&c.}\ldots+1\}.$$

$$= 4^2 \text{ or } 16\{(5^{n-2}+\ldots+1)+(5^{n-3}+\ldots+1)+\text{\&c.}\ldots+1\}.$$

Since 4^2 or 16 is a factor of the given expression, it is divisible by 4^2 or 16.

Ex. 132. If two algebraical expressions are identically equal for all values of the letters x, y, etc., involved in them, the coefficients of the like powers of x and y, etc., in each expression are equal.

If $px+qy \equiv ax+by$ (i.e., true for all value of x and y)

then $p=a$ and $q=b$.

Since $px+qy \equiv ax+by$ it must be true when $x=0$.

$\therefore qy=by \quad \therefore q=b$ and $\therefore p=a$

If $A+Bx+Cx^2+Dx^3+\text{etc.} \equiv a+bx+cx^2+dx^3+\text{etc.}$, then $A=a$, $B=b$, $C=c$, $D=d$, etc.

Since it is true for all values of x, it must be true when $x=0$.

$\therefore A=a$

$\therefore Bx+Cx^2+Dx^3+\text{etc.} \equiv bx+cx^2+dx^3+\text{etc.}$

Dividing both sides by x, we get

$B+Cx+Dx^2+\text{etc.} \equiv b+cx+dx^2+\text{etc.}$ Again by putting $x=0$, it may be shown that $B=b$; similarly $C=c$, and so on.

This is known as the **Principle of Indeterminate Co-efficients.**

Applications.

Ex. 1. If $4(x^2-2x) \equiv A(x^2+1)+Bx(x-1)+C$, find A, B, and C.

The right-hand expression, when arranged in descending powers of $x \equiv x^2(A+B)-Bx+A+C$.

$\therefore 4x^2-8x \equiv x^2(A+B)-Bx+A+C$.

This being an identity, equating the co-efficients of like powers of x on both sides, we get

$$A+B=4 \quad \text{(i)}$$
$$-B=-8 \quad \text{(ii)}$$
$$A+C=0 \quad \text{(iii)}$$

From (ii) $B=8$. Substituting the value of B in (i) we get $A=-4$

Putting $A=-4$ in (iii) we get $C=4$.

$\therefore A=-4$, $B=8$, and $C=4$.

Ex. 2. Express $8x^3+6x^2-3x+1$ as a function of $x+1$.

i.e., $A+B(x+1)+C(x+1)^2+D(x+1)^3$. It is evident that $(x+1)^4$ and higher powers cannot occur because the given expression is of the third degree.

$\therefore 8x^3+6x^2-3x+1 \equiv A+B(x+1)+C(x+1)^2+D(x+1)^3$ when A, B, C, D are to be determined. Divide both sides by $x+1$.

$$\therefore \frac{8x^3+6x^2-3x+1}{x+1} = \frac{A}{x+1}+B+C(x+1)+D(x+1)^2.$$

$$\left[\begin{array}{l|l} 8+6-3+1 & \\ 0-8+2+1 & -1 \text{ operator} \\ \hline 8-2-1+2 & \end{array}\right]$$

$$\therefore 8x^2-2x-1+\frac{2}{x+1} = \frac{A}{x+1}+B+C(x+1)+D(x+1)^2.$$

The integral expressions must be equal to one another and the fractions must be equal on both sides.

$$\therefore \frac{2}{x+1} = \frac{A}{x+1} \qquad \therefore A=2, \textit{ i.e.}, \text{ the first remainder.}$$

$$\therefore 8x^2-2x-1 = B+C(x+1)+D(x+1)^2$$

Again divide both sides by $x+1$.

$$\therefore \frac{8x^2-2x-1}{x+1} = \frac{B}{x+1}+C+D(x+1).$$

$$\left[\begin{array}{l} 8-2-1 \\ 0-8+10 \\ \hline 8-10+9 \end{array}\right] -1 \text{ is the operator.}$$

$$\therefore 8x-10+\frac{9}{x+1} = \frac{B}{x+1}+c+D(x+1)$$

$\therefore$ fractions, and integers on both sides must be equal

$$\therefore \frac{9}{x+1,} = \frac{B}{x+1} \quad \therefore B=9 \text{ the remainder in the 2nd division.}$$

$$\therefore 8x+10 = C+D(x+1)$$

Again divide both sides by $x+1$

$$\therefore \frac{8x-10}{x+1} = \frac{C}{x+1}+D$$

$$\left[\begin{array}{l} 8-10 \\ 0-\ 8 \\ \hline 8-18 \end{array}\right] -1 \text{ is the operator}$$

$$\therefore 8-\frac{18}{x+1} = \frac{c}{x+1,}+D$$

$\therefore C = -18$ (the remainder in the 3rd division) and $D=8$.

$\therefore$ the given expression

$$= 2+9(x+1)-18(x+1)^2+8(x+1)3$$

Thus A, B, C, D, etc., are the successive remainders got by dividing the given expression by $x+1$ and the successive quotients by $x+1$.

Or shortly thus :—

$$\begin{array}{l|l} & 8+6-3+1 \\ x+1 & -8+2+1 \\ \hline & 8-2-1+ \;\;|\; 2 \text{ units} \\ x+1 & -8+10 \\ \hline & 8-10 \;\;|\; +9(x+1) \\ x+1 & -8 \\ \hline & 8 \;\;|\; -18\,(x+1)^2 \end{array}$$

$\therefore 8x^3+6x^2-3x+1 \equiv 8(x+1)^3-18(x+1)^2+9(x+1)+2.$

Ex 3. Express $2x^3+3x^2-x+4$ as a function of $x-1$

The following method also may be adopted :—

Let $2x^3+3x^2-x+4=f(x)$

and let $x-1=y$

Then $x=y+1$

$\therefore f(y+1)=2(y+1)^3+3(y+1)^2-(y+1)+4$

$=2y^3+9y^2+11y+8$

$\therefore f(x)=2(x-1)^3+9(x-1)^2+11(x-1)+8$

EXERCISE 60.

(1) If $f(n)=n^2+8n-3$

find $f(1)$, $f(2)$, $f(3)$, $f(-1)$, $f(-2)$

$f(-3)$, $f(-\frac{1}{2})$, $f(\frac{1}{n})$, $f(n)$, $f(-n)$.

(2) If $f(y)=y^2-3y+2$

find $f(1)$, $f(0)$, $f(y+1)$, $f(y-1)$,

$f(y+\frac{1}{2})$, $f(y-\frac{3}{4})$, $f(y+2)+f(y-2)$.

(3) If $f(n)=n^3+3n^2+3n+1$

$\phi(n)=n^3-3n^2+3n-1$

find $f(1)+\phi(-1)$, $f(2)-\phi(-2)$

$f(3)-\phi(\frac{1}{2})$, $f(4)+\phi(-\frac{1}{4})$,

$f(n+h)+\phi(n-h)$, $f(n-h)-\phi(n+h)$

(4) If $f(n)=(n+1)(n+2)(n+3)(n+4)$

find $f(5)$, $f(-1)$, $f(-4)$, $f(2)$

$f(0)$; show that $f(n)-f(-n)$

$=20n^3+64n$

(5) $y=f(n)=n^2+3n-5$
find $f(y)$ in terms of n
$f(y+1)$,,
$f(y-1)$,,

(6) $f(m)=(m+1)m(m-1)$
find $f(m+1)$, $f(m-1)$, $f(m+2)$ and $f(m-2)$

(7) $f(n)=\dfrac{n(n+1)(2n+1)}{6}$
find $f(10)$, $f(6)$, $f(2n)$, $f(2n+1)$, $f(2n-1)$.

(8) $f(n)=n^3+3n$, $\phi(n)=3n^2+1$
show that $\{f(n)+\phi(n)\}\{f(n)-\phi(n)\}$
$=(n^2-1)^3$

(9) In the above question find
$\left\{f\left(\frac{1}{n}\right)+\phi\left(\frac{1}{n}\right)\right\}\quad\left\{f\left(\frac{1}{n}\right)-\phi\left(\frac{1}{n}\right)\right\}$

(10) If $z=f(y)=(y+1)(y+2)$ find $f(z)$ in term of y.

(11) If $y=f(x)=\dfrac{px+q}{rx-p}$, show that $f(y)=x$.

EXERCISE 61.

Find the remainder in each of the following without actual division :—

1. ax^3+bx^2+cx when divided by $x-1$.
2. ax^3+bx^2+cx+d by $x+p$.
3. px^5+qx^4+rx+4 by $x+a$ and by $x-a$.
4. $ax^4+bx^3+cx^2+dx+e$ by $x-2$ and by $x+2$.
5. $ma^5+na^4+pa^3+qa^2+ra+s$ by $a-b$ and by $a+b$.
6. $3y^5-2y^4+3y^3+4y^2-6y+3$ by $y+1$ and $y-1$.

What value of x will make each of the following expressions divisible by the divisor opposite to it ?—

7. x^6-2x^3+x-3 by x^2-2x+1.
8. $x^3-5x^2+4x+34$ by x^2-3x-7.
9. $x^6-2x^5-3x^4-29x^3-33x^2+20x+15$ by x^2-2x+3.
10. $x^3-3x^2a+5xa^2-9a^3$ by $x-4a$ and by $x-2a$
11. Find c so that x^2+7x+c may be divisible by $x+4$.

Shew that each of the following is divisible by the divisor opposite to it without remainder :—

12. $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $a-b$, $b-c$ and $c-a$.

13. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$ by $a+b$, $b+c$ and $c+a$.

14. $a(b^2-c^2)b)+b(c^2-a^2)+c(a^2-b^2)$ by $a-b$.

15. $(a+b+c)^3-a^3-b^3-c^3$ by $a+b$, $b+c$ and $c+a$.

16. x^3-3x+2 by $x-1$ 17. x^3+3x+4 by $x+1$.

18. x^2x-y^2n by $x+y$ and $x-y$.

19. $an(b-c)+bn(c-a)+cn(a-b)$ by $a-b$, $b-c$ and $c-a$.

20. $(x+y+z)^5-x^5-y^5-z^5$ by $x+y$, $y+z$ and $z+x$.

Write down the quotients of :—

21. x^8-y^8 by x^2-y^2. 22. $x^{15}+y^{15}$ by x^3+y^3.

23. $x^{14}+y^{14}$ by x^2+y^2. 24. $x^{20}-y^{20}$ by x^2+y^2.

25. Shew that $(x^3+y^3)n+\{3xy(x+y)\}^n$ is divisible by $(x+y)^3$ if n be an odd integer.

26. Shew that $(x+4)^{3n}-(7x+22)^n$ is divisible by $(x+2)(x+3)(x+7)$.

27. $(a^2+ab+b^2)^n+(a^2-ab+b^2)^n$ by $2(a^2+b^2)$ if n is odd.

28. $a^{2n+1}+b^{2n+1}$ be exactly divisible by $a^{21}+b^{21}$, shew that $n-10$ is a multiple of 21

When n is any positive integer :—

29. Prove that 4^n-3n-1 is divisible by 9.

30. Shew that $5(5^n-1)-4n$ is divisible by 16.

31. Shew that $11^n-10n-1$ ends in two cyphers.

When n is an *even* integer, shew that—

32. 4^n+5n-1 is divisible by 25.

33. 7^n+8n-1 is divisible by 64,

34. $9^n+10n-1$ ends in two cyphers.

35. Find a and b in order that x^3-ax^2+bx-6 may be divisible by x^2+2x+3.

36. If x^2-3x+3 is a factor of x^4+ax^2+bx-9, find a and b.

37. Find p and q in order that x^4+1 may be divisible by x^2+px+q.

38. Find what numerical values must be given to a and b so that the expression $x^4+2x^3+ax^2-bx+3$ may be divisible by x^2+3x-4.

39. If $A(x+1)^2+B(x-1)(x+1)+C(x-1)^2\equiv 5x^2$, find A, B and C.

40. If $4x^2+5x+5\equiv a(x+1)(x+2)+b(x+1)+c$, find a, b and c.

41. Express $8x^3+9x^2-3x+4$ as a function of $x-1$ and also as a function of $2x-1$.

42. Express x^3 in the form $A(x-1)^3+B(x-1)^2+C(x-1)+D$.

43. Express $2x^3+3x^2+5x-6$ in the form $A+A_1(x+1)+A_2(x+1)(x+2)+A_3(x+1)(x+2)(x+3)$.

44. Express $(x^2+3x+4)^3$ as a function of z if $z=x+2$.

133. Homogeneity and Symmetry.

Definition: *An expression is said to be homogeneous when the degree of every term in it is the same.*

$Ax+By$ is the form of a *homogeneous* expression of the *first* degree in x and y;

$Ax^2+Bxy+Cy^2$, of the *second* degree in x and y;

and $Ax^3+Bx^2y+Cxy^2+Dy^3$, of the *third* degree in x and y.

For three quantities x, y, z, the corresponding expressions are:

$Ax+By+Cz$.

$Ax^2+By^2+Cz^2+Dyz+Exz+Fxy$;

and $Ax^3+By^3+Cz^3+mxy^2+m_1x^2y+nyz^2+n_1y^2z+pzx^2+p_1z^2x+rxyz$.

Law of Homogeneity. *The product of two homogeneous expressions, of the mth and nth degrees respectively, is a homogeneous expression of the $(m+n)$th degree.*

Corollary. The factors of a homogeneous expressions are also homogeneous.

Thus $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=x^3+y^3+z^3-3xyz$.

$(a+b+c)(a+b-c) \quad =a^2+b^2-c^2+2ab$.

$a^2b+b^2a+b^2c+bc^2+c^2a+ca^2+2abc=(a+b)(b+c)(c+a)$.

This law is of great use in testing the accuracy of algebraical work in Multiplication, Division and Factorization.

Definition: *An expression is said to be symmetrical with respect to certain letters when the interchange of any two of these letters throughout the expressions would have the value of the expression unaltered.*

Symmetrical expressions involving two quantities x and y of the 1st, 2nd, and 3rd degrees are:—

$Ax+Ay$; $Ax^2+Bxy+Ay^2$; $Ax^3+Bxy^2+Bx^2y+Ay^3$.

The corresponding expressions involving x, y, z are:—

$Ax+Ay+Az$;

$Ax^2+Bxy+Ay^2+Bxz+Az^2+Byz$;

$Ax^3+Ay^3+Az^3+B(xy^2+x^2y+yz^2+y^2z+zx^2+z^2x)+Cxyz$

Note.—The terms that have the same co-efficient are those that are derivable from each other by interchanges of the letters.

Rule of Symmetry.—*The algebraical sum, product or quotient of two symmetrical expressions is a symmetrical expression.*

134. To resolve a homogeneous expression of the second degree.

Ex. 1. $15p^2+14pq-8q^2+14qr+4rp-3r^2$.

First Method.—

Put the letters involved in the expression successively equal to 0, and combine the factors of the resulting expressions.

i. Suppose $q=0$;

then the expression $=15p^2+4pr-3r^2=(3p-r)(5p+3r)$.

ii. Suppose $r=0$;

then the expression $=15p^2+14pq-8q^2=(3p+4q)(5p-2q)$.

Examining these two results, we find

that $+4q$ and $-r$ go with $+3p$ while $+3r$ and $-2q$ go with $+5p$.

Hence the expression $=(3p+4q-r)+(5p-2q+3r)$.

Second Method.—

Arrange the expression as a quadratic in any one letter and resolve it into factors.

The expression,

$=15p^2+p(14q+4r)-(8q^2-14qr+3r^2)$

$=15p^2+p(14q+4r)-(4q-r)(2q-3r)$,

$=15p^2+5p(4q-r)-3p(2q-3r)-(4q-r)(2q-3r)$

$=5p(3p+4q-r)-(2q-3r)(3p+4q-r)$

$=(3p+4q-r)(5p-2q+3r)$

The method of verification used in Multiplication and Division may also be advantageously used to test the correctness of the process of Factorization.

Verification :

$15+14-8+14+4-3=(3+4-1)(5-2+3)$

$36=6\times6$

Ex. 2. $7a^2-ab-5ac-6b^2-8bc-2c^2$.

First Method.—

i. Suppose $a=0$:

then the expression $=-6b^2-8bc-2c^2=2(3b+c)(b+c)$.

ii. Suppose $b=0$:

then the expression $=7a^2-5ac-2c^2=-(7a+2c)(a-c)$.

Making the two results uniform we have

(i) $(6b+2c)(-b-c)$.

(ii) $(7a+2c)(a-c)$.

Examining these two we find

that $+7a$, and $+6b$ go with $+2c$ while $-b$ and $+a$ go with $-c$.

Hence the expression $=(7a+6b+2c)(a-b-c)$.

Second Method.—

The expression

$=7a^2-a(b+5c)-(6b^2+8bc+2c^2)$

$=7a^2-a(b+5c)-2(3b+c)(b+c)$

$=7a^2-7a(b+c)+2a(3b+c)-2(3b+c)(b+c)$

$=7a(a-b-c)+2(3b+c)(a-b-c)$

$=(a-b-c)(7a+6b+2c)$.

Verification :

$7-1-5-6-8-2=(7+6+2)(1-1-1)$

$-15=\qquad 15\times -1$

Note.—If the expression involves four letters, suppose *two of them at a time to be zero* ; if it involves five letters, suppose *three of them at a time to be zero* ; and so on.

135. To resolve an expression of the type $ax^2+fxy+by^2+gx+fy+c$.

Ex. $2x^2-xy-y^2+4x+5y-6$.

If the factors are of the form $(ax+by+c)(a^1x+b^1y+c^1)$, then the terms of the second degree will be obtained from the first two terms of each factor. They must therefore be so chosen that their product will give the terms of the second degree.

$\therefore 2x^2-xy-y^2\equiv(x-y)(2x+y)$

$\therefore 2x^2-xy-y^2+4x+5y-6=(x-y+k)(2x+y+l)$.

By applying the Principle of Indeterminate co-efficients, we have

$l+2k=4$ (i)

$k-l=5$ (ii)

$kl=-6$ (iii)

By adding (i) and (ii) we get $3k=9$

$\therefore k=3$

Putting $k=3$ in (ii) we get $l=-2$ These values satisfy the third equation $kl=-6$

$\therefore$ The factors are $(x-y+3)(2x+y-2)$.

The factors can be got by the following method also :—

$$
\begin{aligned}
&2x^2 - xy - y^2 + 4x + 5y - 6 \\
&= 2x^2 + x(4 - y) - (y^2 - 5y + 6) \\
&= 2x^2 + x(4 - y) - (y - 3)(y - 2) \\
&= 2x^2 + x\{(y - 2) - 2(y - 3)\} - (y - 3)(y - 2) \\
&= 2x^2 + x(y - 2) - 2x(y - 3) - (y - 3)(y - 2) \\
&= x(2x + y - 2) - (y - 3)(2x + y - 2) \\
&= (x - y + 3)(2x + y - 2)
\end{aligned}
$$

EXERCISE 62.

Resolve into factors :—

1. $6x^2 + 21xy + 18y^2 + 28z^2 + 45yz + 26xz$.
2. $6a^2 + 13ab - 5ac - 5b^2 + 13bc - 6c^2$.
3. $2x^2 + 7xy + 7xz + 6y^2 + 11yz + 3z^2$.
4. $10a^2 - 8ab - 11ac - 24b^2 - 53bc - 35c^2$.
5. $a^2 - 2b^2 + c^2 + ab + 2ac + bc$.
6. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
7. $2x^2 + 2y^2 - z^2 + 5xy + xz - yz$.
8. $2a^2 + 2b^2 + c^2 + 5ab + 3ac + 3bc$.
9. $2a^2 - b^2 + 2c^2 + ab + 5ac - bc$.
10. $3x^2 + 4z^2 - 6xy - 8xz + 8yz + 3y^2$.
11. $2x^2 - 9ax + 10x - 18a^2 + 45a - 28$.
12. $3x^2 + 7xy - 4x + 4y^2 - 7y - 15$.
13. $6a^2 + 11ab - 13a - 10b^2 + 34b - 28$.
14. $3x^2 - 3xy + 25x - 6y^2 + 40y - 50$.
15. $2a^2 + 2b^2 - 9 + 5ab - 3a + 3b$.
16. $14x^2 + 33xy + 46x + 18y^2 + 42y + 12$.
17. $3x^2 - 2y^2 - 6z^2 - 5xy - 3xz - 8yz$.
18. $xy + 2x^2 - 3y^2 + 4yz + xz - z^2$.
19. $2x^2 - 9xz - 5xy + 4z^2 - 8yz - 12y^2$.
20. $a^2 + b^2 + c^2 + d^2 + 2ab + 2ac - 2ad + 2bc - 2bd - 2cd$.

136. To resolve an expression into factors by means of the Remainder Theorem.

It has already been deduced from the Remainder Theorem that an expression in x is exactly divisible by $x - a$, if it vanishes when $+a$ is substituted for x ; and by $x + a$, if it vanishes when $-a$ is substituted for x.

Hence $x+a$ will be a factor of an expression in x according as it vanishes when $+a$ is substituted for x.

Take $x-1$ as a trial factor of the expression

$$px^4+qx^3+rx^2+sx+t.$$

Then $x-1$ will be one of its factors

if substituting 1 for x, $p+q+r+s+t=0$.....(1)

Similarly $x+1$ will be a factor of $px^4+qx^3+rx^2+sx+t$

if substituting -1 for x, $p-q+r-s+t=0$

i.e., if $p+r+t=q+s$...(2)

Examining these two results we have the following rule :—

Rule—i. "*If the algebraic sum of the co-efficients of an expression in* x *be equal to* 0, *then it has a factor* x—1."

ii "*If the sum of the co-efficients of all the odd terms of an expression in* x = *the sum of the co-efficients of all the even terms, it has a factor* x+1."

Ex. 1. $4x^3+3x^2-7$.

Here the sum of the co-efficients $=4+3-7=0$;

therefore it will have $x-1$ as a factor.

Now $4x^3+3x^2-7=4x^3-4+3x^2-3$

$=4(x^3-1)+3(x^2-1)$

$=(x-1)\{4(x^2+x+1)+3(x+1)\}$.

Ex. 2. $3x^3+5x^2+6x+4$.

Here $3+6$ which is the sum of the odd co-efficients

$=5+4$ which is the sum of the even ones ;

$\therefore x+1$ is a factor of the expression,

Now $3x^3+5x^2+6x+4=3x^2(x+1)+2x(x+1)+4(x+1)$

$=(x+1)(3x^2+2x+4)$.

Note.—(1) If any powers of x are missing, they must be considered as present with zero co-efficients.

(2) An expression in x will have x^2-1 for a factor, if the sum of the odd co-efficients and the sum of the even ones be each equal to to 0.

Similarly factors other than $x\pm1$ can be found.

Ex. 3. $6x^3+11x^2-19x+6$.

The trial factors are $x\pm1$, $x\pm2$, $x\pm3$,

where 1, 2, 3 are the factors of 6.

Substitute -3 for x.

$6(-27)+11\,.\,9.-19(-3)+6=-162+162=0.$

$\therefore$ $x+3$ is a factor.

$\therefore$ the expression $=(x+3)(6x^2-7x+2)$

$=(x+3)(2x-1)(3x-2).$

Where an expression involves brackets it can be factorised in a simpler manner.

Ex. 4. $(x+1)(x+3)(x+5)(x+7)+15.$

Multiply $x+1$ and $x+7$ together as well as $x+3$ and $x+5$ together, so as to get two factors which have as many terms common as possible.

Then the expression

$=(x^2+8x+7)(x^2+8x+15)+15.$

$=a(a+8)+15$ (putting a for x^2+8x+7.)

$=a^2+8a+15.$

$=(a+3)(a+5)$

$=(x^2+8x+10)(x^2+8x+12)$

$=(x^2+8x+10)(x+2)(x+6).$

EXERCISE 63.

Resolve into factors :—

1. $3a^3-2a-1.$
2. $2x^3-5x^2+7x-4.$
3. $2x^3+x^2-2x-1.$
4. $3x^4-7x^3+9x^2-12x+7.$
5. $x^3-3x^2+3x-1.$
6. $3a^3-5a^2+3a-1.$
7. $3x^4-2x^3+3x^2-8.$
8. $4a^3-6a^2+5a-3.$
9. $11x^3+7x^2-5x-1.$
10. $7x^4+4x^3-2x^2-x-2.$
11. $x^3-x^2+x-1.$
12. $ax^3+bx^2+cx-(a+b+c).$
13. $x^4-x^3+7x+5.$
14. $x^3+3x^2+5x+3.$
15. $x^3+15x^2+74x+120.$
16. $2x^3+7x^2+5x-4.$
17. $6x^3-11x^2-4x+4.$
18. $18x^4-9x^3-32x^2+9x+14.$
19. $6x^4+11x^3-13x^2-11x+7.$
20. $6x^4-9x^3-12x^2+9x+6.$
21. $6x^4-7x^3-34x^2-3x+18.$
22. $4x^4+x^3-25x^2-37x-15.$
23. $x^4+5x^3+2x^2-20x-24.$
24. $x^4-x^3-15x^2+9x+54.$
25. $x(x-3)(x-1)(x+2)-16.$

26. $(x+2)(x+4)(x+8)+12$.

27. $(x-2)(2x-5)(3x-8)(6x-19)-3$.

28. $(x-4)(x+5)(x-7)(x+2)-324$

29. $(x+2)^2(x+1)(x+3)-12$.

$\sum a$ means $a+b+c$ if there are three quantities a, b, c and $a+b+c+d$ if there are four quantities a, b, c and d. Similarly $\sum_{abc} a^2$ means $a^2+b^2+c^2$; $\sum a^2 b$ means $a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$.

$\sum abc$ $abcd$ means $abc+bcd+cda+dab$. $\sum_{abc}$ notation is used to denote the sum of all terms similar to the one placed after the symbol $\sum$; similarly $\pi a = abc$, if there are three quantities and $\pi a = abcd$, if there are four ; $\pi(a+b)$ means $(a+b)(b+c)(c+a)$.

Symmetrical Expressions.

Ex. 1 $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

Substitute b for a, then

the expression

$=b(b^2-c^2)+b(c^2-b^2)+c(b^2-b^2)$

$=b(b^2-c^2)-b(b^2-c^2)+c(b^2-b^2)$

$=0$.

$\therefore a-b$ is a factor of the expression.

From cyclic symmetry $b-c$ and $c-a$ must also be factors.

Hence there are in all three algebraical factors and no more, as the expression is of the third degree.

But there may be some numerical factor Let it be k.

Then $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=k(a-b)(b-c)(c-a)$

To find the value of k give any values to a, b and c.

Let $a=1$, $b=2$, and $c=3$.

Then $1(4-9)+2(9-1)+3(1-4)=k(-1)(-1)(2)$

$2=2k$.

$\therefore k=1$

$\therefore$ the expression $=(a-b)(b-c)(c-a)$.

The value of k may also be found by comparing the co-efficients of similar terms on both sides.

e.g., in the expression on the one side there is the term $-a^2b$

and on the other there is the term $k(a)(b)(-a)$ *i.e.*, $-ka^2b$

As these two must be equal , $k=1$,

Note. The expression can be factorised also by either of the following methods :—

(1) By arranging it according to powers of some contained letter or quantity.

Arranging it in descending powers of a, we get.

$$\begin{aligned}\Sigma a(b^2-c^2) &\equiv a^2(c-b)-a(c^3-b^2)+bc(c-b)\\ &=(c-b)\{a^2-a(c+b)+bc\}\\ &=(c-b)\,(a-b)\,(a-c)\\ &=(a-b)\,(b-c)\,(c-a)\end{aligned}$$

(2) By splitting up one of the terms.

$$\begin{aligned}\Sigma a(b^2-c^2) &= a(b^2-c^2)+b\{-(b^2-c^2)-(a^2-b^2)\}+c(a^2-b^2)\\ &=a(b^2-c^2)-b(b^2-c^2)-b(a^2-b^2)+c(a^2-b^2)\\ &=(b^2-c^2)(a-b)-(a^2-b^2)(b-c)\\ &=(a-b)(b-c)(b+c-a-b)\\ &=(a-b)(b-b)(c-a)\end{aligned}$$

Ex. 2. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)$.

Substitute c for b, then

the expression $=a(c^3-c^3)+c(c^3-a^3)+c(a^3-c^3)=0$.

$\therefore b-c$ is a factor.

Similarly or from symmetry $c-a$ and $a-b$ must also be factors.

Now as the product of these three factors will be only of the third degree and as the given expression is of the fourth degree there must be one more algebraical factor of the first degree and symmetrical with respect to a, b and c.

Let this factor along with the numerical factor which may in all cases be present, be $k(a+b+c)$.

Then $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3)\equiv k(a-b)(b-c)(c-a)(a+b+c)$.

Let $a=1$, $b=2$, $c=0$.

Then $1\cdot 8+2(-1)+0=k\,(-1)(2)(-1)\cdot 3)$

$$6=6k.$$

$$\therefore k=1$$

or comparing like terms

$$-ac^3=ka(-c)(c)(c)=kac^3.$$

$$\therefore k=1$$

$\therefore$ the expression $=(a-b)(b-c)(c-a)(a+b+c)$.

Ex. 3 Resolve :—$(a+b+c)^3-a^3-b^3-c^3$.

We find that $a+b$ is a factor of the expression for it is equal to $(-b+b+c)^3+b^3-b^3-c^3$ *i.e.*, 0 if we put $-b$ for b, similarly $b+c$ and

$c+a$ are factors because it is symmetrical with respect to a, b and c $\therefore$ it can have no fourth algebraical factor because the expression is of the third degree but it may have a numerical factor, let it be k.

Then $(a+b+c)^3-a^3-b^3-c^3 \equiv k(a+b)(+c)(c+b)$. Put $a=1$, $b=1$, $c=1$ in this identity.

$\therefore 3^3-1^3-1^3-1^3=k\,(2)(2)(2)$

i,e.. $24=8k \therefore k=3$.

Thus $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$

a formula which you already know

Ex. 4. Resolve $a^3+b^3+c^3-3abc$. *w—h* c we find by putting $-b$ for a that the expression $= -b^3+b^3+c^3+3b^2c$ which is not zero.

We similarly conclude that $b+c$, $c+a$ are not factors. Similarly we know that $a-b$, $b-c$, $c-a$ are also not factors. Therefore it must have a trinomial factor of the first degree symmetrical with respect to a, b and c. This evidently is $a+b+c$. Now for a put $-(b+c)$, then the resulting expression

$-(b+c)^3+b^3+c^3+3(b+c)bc.$

$= -(b+c)^3+(b+c)^3=0.$

Thus by the remainder theorem we see $a+b+c$ is a factor, the other factor should be a homogeneous expression of the second degree, at the same time symmetrical with respect to a,b and c.

$\therefore a^3+b^3+c^3-3abc=(a+b+c)\{k(a^2+b^2+c^2)+l(ab+ac+bc)\}$

Where k and l are numerical quantities to be determined. Give two sets of values to a. b and c, *viz.*, 1, 0, 0, 1, 1, 0, respectively.

Then $1^3+0^3+0^3-3.1.0.0$

$=(1+0+0)\{(k1^2+0^2+0^2)+l(1.0+1.0+0)\}$

i.e. $1=1\,k$, 1 *i.e.* $1=k$.

Again $1^3+1^3+0^3-3.1.1.0$

$=(1+1+0)\{k(1^2+1^2+0^2)+l(1.0+1.0+1.1)\}$

i.e. $2=2\,(2k+l)$

i.e. $1=2k+l$ and k has already been found to be.

$+1=2+l$ or $1-2=l$ or $-1=l$

Thus $a^3+b^3+c^3-3\,abc$

$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

EXERCISE 64.

Resolve in factors :—

1. $a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$
2. $a(b+c)^2+b(b+a)^2+c(a+b)^2-4abc.$
3. $ab(a-b)+bc(b-c)+ca(c-a).$
4. $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2).$
5. $a^3(b-c)+b^3(c-a)+c^3(a-b).$
6. $a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4).$
7. $ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2).$
8. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2).$
9. $a^4(b-c)+b^4(c-a)+c^4(a-b).$
10. $(a-b)^5+(b-c)^5+(c-a)^5.$
11. $a^2b^2(a^2-b^2)+b^2c^2(b^2-c^2)+c^2a^2(c^2-a^2).$
12. $a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3).$
13. $a^3(b^4-c^4)+b^3(c^4-a^4)+c^3(a^4-b^4).$
14. $(a+b)^2(b-a)-(b+c)^2(c-b)+(c+b)^2(a-c).$
15. $b^2(1-b^2)+b^2c^2(b^2-c^2)+c^2(c^2-1).$
16. $\frac{a}{c}\left(\frac{a}{b}-\frac{b}{c}\right)+\frac{b}{a}\left(\frac{b}{c}-\frac{c}{a}\right)+\frac{c}{b}\left(\frac{c}{a}-\frac{a}{b}\right).$
17. $a^3(b-c)+b^3(c-a)+c^3(a-b).$
18. $a(b-c)^3+b(c-a)^3+c(a-b)^3.$
19. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2).$
20. $a^4(b^2+c^2)+b^4(c^2+a^2)+c^4(a^2+b^2)+2a^2b^2c^2.$
21. $x^2(y^2+z^2)^2+y^2(z^2+x^2)^2+z^2(x^2+y^2)^2-4x^2y^2z^2.$
22. $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)-1.$
23. $\left(x^3+\frac{1}{x}+\frac{1}{x^3}\right)\left(x+x^3+\frac{1}{x^3}\right)-1.$
24. $4c^2(a+b-c)+4b^2(c+a-b)+4a^2)(b+c-a)$
$-4(a+b-c)(b+c-a)(c+a-b)$
25. $s^3-(s-a)^3-(s-b)^3-(s-c)^3$, if $2s=a+b+c.$
26. $a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2.$
27. $(x+y+z)^3+(x+y-z)^3+(x+z-y)^2+(y+z-x)^2.$
28. $\sum(a+b)(a^3-b^3).$
29. $(a+b+c)^5-a^5-b^5-c^5.$
30. $(bc-a^2)^2+(ca-b^2)^2+(ab-c^2)^2$ given that there are two quadratic factors.

CHAPTER XVI.

HIGHEST COMMON FACTOR.

137. Definitions. A **Common Factor** of two or more algebraical expressions is an expression which will exactly divide each of them.

Thus p is a common factor of ap and bp.

The **Highest Common Factor** of two or more, algebraical expressions is the expression of highest dimensions which will exactly divide each of them.

Thus abc is the higest common factor of ab^2c and a^2bc.

The abbreviation H. C. F. is often used instead of Highest Common Factor.

It should be noted that the H. C. F. of two algebraical expressions is not necessarily the G. C. M. of the values of the expressions.

Thus, if $a=6$ and $b=4$

$$a^2b=6^2.4=(6\cdot4\cdot2)\ 3.$$

$$ab^2=6\cdot4^2=(6\cdot4\cdot2)\ 2.$$

The G. C. M of the numerical values of a^2b and ab^2 is therefore $6\cdot4\cdot2$, while the H. C. F. of a^2b and ab^2 is ab or $6\cdot4$.

138. H. C. F. of Monomial Expressions.

Rule.—"Find the H. C. F. of the numerical co-efficients, and prefix it as a co-efficient to the algebraical H. C. F which is obtained by taking each letter which is common to all the expressions to the lowest power in which it occurs.

Ex. Find the H. C. F. of $5x^3yz^3$, $15x^3y^2z$ and $20x^2y^2z^2$.

The H. C. F. of 5, 15 and 20 is 5;

the letters common to the expressions are x, y and z;

the lowest powers in which they occur are x^2, y and z;

therefore the H. C. F. is $5x^2yz$.

EXERCISE 65.

Find the H. C. F. of :—

1. $2x^2b^3$ and x^3b^2.
2. $15x^3b$ and $20x^2z^3$.
3. $9ab^2c^3$ and $24a^3b^4$.
4. $18p^2q^4$ and $45p^5q^3$.
5. $2x^3y^2$, $3x^2y^3$ and $4x^4y^4z$.
6. $17a^5b^2c^3$, $51a^4b^3c^4$ and $68a^6b^3c^4$.

7. $36a^2b^3c^4x^5$, $54a^5c^3x^4$ and $81a^4b^3c^5$.

8. $21x^{p+2q}y^{q+2r}$ and $35x^{2p+q}y^{2q+r}$.

9. $14a^{m+n}b^{p+q}$, $70a^{m+1}b^{p+q-1}$ and $140a^{m+2}b^{p+q-2}$.

10. $30x^{a+2b+3c}y^{3p+\frac{1}{2}q+r}$, $72x^{b+2c+3a}y^{3q+2r+p}$ and $210x^{c+2a+3b}y^{3r+2p+q}$.

139. H. C. F. of Compound or Multinomial Expressions which can be readily resolved into factors.

Rule.—"*Find the H. C. F. of the numerical factors and annex to it all the factors which are common to all the expressions, taking each factor to the lowest power in which it occurs in any of the expressions.*"

Ex. 1. Find the H. C. F. of :—

a^2-b^2, $a^2-2ab+b^2$ and a^3-b^3.

Here $a^2-b^2=(a+b)(a-b)$.

$a^2-2ab+b^2=(a-b)(a-b)$.

$a^3-b^3=(a-b)(a^2+ab+b^2)$.

We see at once that $a-b$ is the only common factor ;

∴ the H. C. F. is $a-b$.

Ex. 2. Find the H. C. F. of :—

x^2+4x+3, x^2-2x-3 and x^2+2x+1.

Here $x^2+4x+3=(x+1)(x+3)$,

$x^2-2x-3=(x+1)(x-3)$

And $x^2+2x+1=(x+1)^2$.

∴ the H. C. F. is $x+1$, because it is the only common factor.

Note—It is enough if one of the given expressions can be readily resolved into factors ; for we may factorise the others by taking these as trial factors.

Ex. 3. Find the H. C. F. of :—

x^3+8, $2x^3+3x^2+x+6$, and $3x^3+5x^2+4$.

Here $x^3+8=(x+2)(x^2-2x+4)$.

Now x^2-2x+4 cannot be the H. C. F. as it cannot be a factor of $2x^3+3x^2+x+6$; for the last term 6 of the latter expression is not divisible by the last term 4 of the former. Hence take $x+2$ as the trial factor.

Thus $2x^3+3x^2+x+6=(x+2)(2x^2-x+3)$

And $3x^3+5x^2+4=(x+2)(3x^2-x+2)$.

∴ the H. C. F. is $x+2$, because it is the only common factor.

EXERCISE 66.

Find the H. C. F. of :—

1. a^3+b^3, a^2-ab+b^2 and $a^4+a^2b^2+b^4$.
2. a^3+1, a^2+4a+3 and a^2+6a+5.
3. x^2-1, x^2+3x+2 and x^3+3x^2+3x+1.
4. $x^2+7x-18$, x^3-8 and x^2-4.
5. $x^2-(b+c)x+bc$, $x^2-(a+b)x+ab$ and $x^2-2bx+b^2$.
6. $x^2-7xy+6y^2$ and x^3-xy^2.
7. x^4+4 and x^2+2x+2.
8. x^3-x, $2x^2-4x+2$ and x^3+x^2-2x.
9. $4x^3+12x^2+9x$ and $4x^2-2x-12$.
10. $21x^3-26x^2+8x$ and $6x^2-x-2$.
11. $a^2+3a-10$ and $a^3-a^2-14a+24$.
12. $a^2+ab+ac+bc$, $a^2+2ac+c^2$ and a^3+c^3.
13. $x^3+y^3+z^3-3xyz$ and $x^2+2xy+y^2-z^2$.
14. $3x^3-19x+16$ and $3x^3-19x^2+16$.
15. $x(6x^2-8y^2)-y(3x^2-4y^2)$ and $2xy(2y-x)+4x^3-2y^3$.
16. $6x^3-7x^2+1$ and $3x^4-8x^3-12x^2+1$.
17. x^3+3x^2-9x+5 and $x^3-19x+30$.
18. $x^3-6x^2+12x-8$ and $3x^3-3x^2-11x+10$.
19. x^4-x^3+7x+5 and x^4-x^3+2x+2.
20. $4x^3-8ax^2-20a^2x+24a^3$ and $6x^3+24ax^2+6a^2x-36a^3$.

140. H. C. F. of two Compound Expressions whose factors cannot be readily found.

Rule.—"*(a) Arrange the expressions according to descending or ascending powers of some common letter.*

(b) Take as divisor that one whose degree in the common letter is not higher than the degree of the other, and the remaining one as the dividend.

(c) Perform the division till you get a remainder of lower degree than the divisor.

(d) Consider this remainder as a new divisor, and the last divisor as a new dividend.

(e) Then consider the new remainder arising from this division as another new divisor, and the last divisor as another new dividend.

(f) Continue this process till there is no remainder.

(g) The last divisor is the H. C. F. required."

141. The truth of the rule depends upon the following principles :—

(1) **If p divides A, then it will divide mA.**

For since p divides A, we may suppose $A = xp$.

Then $mA = mxp$.

Thus p divides mA.

(2) **If p divides A and B, then it will divide mA ± nB.**

For since p divides A and B, we may suppose

$A = xp$, and $B = yp$;

then $mA \pm nB = mxp \pm nyp = (mx \pm ny)p$

Thus p divides $mA \pm nB$.

142. Proof of the rule given in Art. 140.

Let A and B be two expressions arranged according to descending powers of some common letter, and let the index of the highest power of that letter in A be not less than the index of the highest power of that letter in B.

Divide A by B ; let p be the quotient and C the remainder.

Divide B by C ; let q be the quotient and D the remainder.

Divide C by D, and suppose that there is no remainder and let r denote the quotient.

$$\begin{array}{r|l} B & A(p \\ & pB \\ \hline C & B(q \\ & qC \\ \hline D & C(r \\ & rD \\ \hline & 0 \end{array}$$

Thus we have the following results :—

$$A = pB + C,\ B = qC + D \text{ and } C = rD.$$

First, we shall show that D is a common factor of A and B.

Since $C = rD$, D divides C.

$\therefore$ D divides qC and also $qC + D$. (Art. 141.)

That is, D divides B ; for $B = qC + D$.

Again, since D divides B and C, it divides $pB + C$. (Art. 141.)

That is, D divides A ; for $A = pB + C$.

Therefore D divides A and B ; that is, D is a common factor of A and B.

Secondly, we shall show that D is the H. C. F. of A and B.

For, every expression which divides A and B divides $A - pB$, or C. (Art. 141

Thus every expression which is a common factor of A and B is a common factor of B and C.

Again, every expression which divides B and C divides $B - qC$ or D. (Art. 141.)

Thus every expression which divides B and C divides C and D.

Therefore every expression which is a common factor of A and B is a factor of D.

But no expression higher than D can divide D exactly.

Therefore D is the H. C. F. of A and B.

Note.—Every common factor of A and B is a factor of their H. C. F.

143. The process of finding the H. C. F. is often simplified by the following artifices :—

(1) We may remove any factor of either of the expressions which is not a factor of the other.

(2) We may multiply either of the expressions by any quantity which does not introduce a common factor.

(3) Simple factors of the expressions may be removed ; the H.C. F. of the expressions is the product of the H. C. F. of these factors and the H. C. F. of the quotients.

(4) We may treat the divisor and the dividend, at any stage of the process, in the same way as the original expressions.

Ex. 1. Find the H. C. F. of :—

$9a^4+9a^3+9a^2-27a$ and $6a^5+18a^4+30a^3+18a^2$

These expressions are equal to

$9a(a^3+a^2+a-3)$ and $6a^2(a^3+3a^2+5a+3)$ respectively

By the third rule of this Article, we may remove the factors $9a$ and $6a^2$ whose H. C. F. is $3a$; we must now find the H. C. F. of a^3+a^2+a-3 and a^3+3a^2+5a+3, and multiply it by $3a$.

$$a^3+a^2+a-3)a^3+3a^2+5a+3(1$$
$$\underline{a^3+a^2+a-3}$$
$$2\,2a^2+4a+6$$

(Divide by 2 which is not a factor of the divisor.)

$$a^2+2a+3)a^3+a^2+a-3(a-1$$
$$\underline{a^3+2a^2+3a}$$
$$-a^2-2a-3$$
$$\underline{-a^2-2a-3}$$

$\therefore$ a^2+2a+3 is the H. C. F. of a^3+a^2+a-3 and a^3+3a^2+5a+3.

Hence the H. C. F. of the given expressions $= 3a(a^2+2a+3)$.

Ex. 2. Find the H. C. F. of:—

$2x^3-15x+14$ and $x^4-15x^2+28x-12$.

To avoid fractions in the quotient we multiply the latter by 2, since 2 is not a factor of the other expression.

$$\begin{array}{l} x^4-15x^2+28x-12 \\ 2 \\ \hline 2x^3-15x+14)2x^4-30x^2+56x-24(x \\ \quad 2x^4-15x^2+14x \\ \hline \quad -3\ -15x^2+42x-24 \\ \quad\quad 5x^2-14x+8 \end{array}$$

(Divide by—3 which is not a factor of the divisor).

Multiply each term of $2x^3-15x+14$ by 5 which is not a factor of $5x^2-14x+8$.

$$\begin{array}{l} 5x^2-14x+8)10x^3-75x+70\ (2x \\ \quad 10x^3-28x^2+16x \\ \hline \quad 7\ 28x^2-91x+70 \\ \quad\quad 4x^2-13x+10. \end{array}$$

(Divide by 7).

Multiply $5x^2-14x+8$ by 4 to avoid fractions in the quotient.

$$\begin{array}{l} 4x^2-13x+10)20x^2-56x+32(5 \\ \quad 20x^2-65x+50 \\ \hline \quad 9\ |\ 9x-18 \\ \quad\quad x-2)4x^3-13x+10(4x-5 \\ \quad\quad\quad 4x^3-8x \\ \hline \quad\quad\quad -5x+10 \\ \quad\quad\quad -5x+10 \end{array}$$

(Divide by 9).

Therefore the H. C. F. is $x-2$.

Ex. 3. Find the H. C. F. of :—

$4x^4-9x^2+6x-1$ and $6x^3-7x^2+1$.

In this case, it is advantageous to arrange the expressions according to ascending powers of x.

$$\begin{array}{l} 1-7x^2+6x^3)-1+6x-9x^2+4x^4(-1+3 \\ \quad -1+7x^2-6x^3 \\ \hline 2x\ 6x-15x^2+6x^3+4x^4 \\ \quad 3-8x+3x^2+2x^3 \\ \quad 3\quad\quad -21x^3+18x^2 \\ \hline -8x\ -8x+24x^2-16x^3 \\ 1-3x+2x^2)\ 1-7x^2+6x^3(1+3x \\ \quad 1-3x+2x^2 \\ \hline \quad\quad 3x-9x^2+6x^3 \\ \quad\quad 3x-9x^2+6x^3 \end{array}$$

(Divide by $2x$).

(Divide by $-8x$).

Therefore the H. C. F. is $1-3x+2x^2$.

Ex. 4. Find the H. C. F. of :—

$4x^4+9x^3+2x^2-2x-4$ and $3x^3+5x^2-x+2$.

The process of successive division can be conveniently arranged thus :—

$$
\begin{array}{r|l|l|l}
3x & 3x^3+5x^2-x+2 & 4x^4+9x^3+2x^2-2x-4 & \\
 & & 3 & \\
\hline
 & & 12x^4+27x^3+6x^2-6x-12 & 4x \\
 & & 12x^4+20x^3-4x^2+8x & \\
\hline
 & & 7x^3+10x^2-14x-12 & \\
 & & 3 & \\
\hline
 & & 21x^3+30x^2-42x-36 & 7 \\
 & & 21x^3+35x^2-7x+14 & \\
\hline
 & & -5 \mid -5x^2-35x-50 & \\
\hline
 & 3x^3+21x^2+30x & x^2+7x+10 & x. \\
\hline
-16 & -16x^2-31x+2 & & \\
 & -16x^2-112x-160 & & \\
\hline
 & 81 \mid 81x+162 & & \\
\hline
 & x+2 & x^2+2x & \\
\hline
 & & 5x+10 & 5. \\
 & & 5x+10 & \\
\hline
\end{array}
$$

$\therefore$ the H. C. F. is $x+2$.

Ex. 5. Find the H.C.F. of $10a^3+9a^2-19a+6$ and $6a^4+a^3-6a^2+26a-12$.

$$
\begin{array}{r|l|l|l}
5a & 10a^3+9a^2-19a+6 & 6a^4+a^3-6a^2+26a-12 & \\
 & 10a^3+15a^2-10a & 5 & \\
\hline
 & -6a^2-9a+6 & 30a^4+5a^3-30a^2+130a-60 & 3a \\
 & -6a^2-9a+6 & 30a^4+27a^3-57a^2+18a & \\
\hline
 & & -22a^3+27a^2+112a-60 & \\
 & & 5 & \\
\hline
 & & -110a^3+135a^2+560a-300 & -11 \\
 & & -110a^3-99a^2+209a-66 & \\
\hline
 & & 117 \mid 234a^2+351a-234 & \\
\hline
 & & 2a^2+3a-2 & \\
\hline
\end{array}
$$

$\therefore$ The H. C. F. is $2a^2+3a-2$.

Note.—The process can still further be simplified by the Method of Detached Co-efficients.

144. H. C. F. of more than two expressions.

If we have *three* expressions A, B, C, first find the H. C. F. of two of them, A and B ; let D be the H. C. F.;

then the H. C. F. of D and C is the H. C. F. of A, B and C.

In the same way, we may find the H. C. F. of *four* algebraical expressions.

Ex. Find the H. C. F. of :—

a^2-3a+2, a^2-4a+3 and $a^3-6a^2+11a-6$.

The H. C. F. of the first two is $a-1$, which is found to be a factor of the last ; hence $a-1$ is the H. C. F. of the three expressions.

EXERCISE 67.

Find the H. C. F. of:—

1. x^3+x^2+x-3 and x^3+3x^2+5x+3.
2. $x^4-3x^3+2x^2+x-1$ and x^3-x^2-2x+2.
3. $2x^5-11x^2-9$ and $4x^5+11x^4+81$.
4. $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$.
5. $6x^4+x^3-x$ and $4x^3-6x^2-4x+3$.
6. $3x^5-10x^3+15x+8$ and $x^5-2x^4-6x^3+4x^2+13x+6$.
7. $11x^4+24x^3+125$ and $x4+24x+55$.
8. $1-4x^3+3x^4$ and $1+x-x^2-5x^3+4x^4$.
9. $x^4-px^3+(q-1)x^2+px-q$ and $x^4-px^3+(p-1)x^2+qx-p$.
10. $25x^4+5x^3-x-1$ and $20x^4+x^2-1$.
11. x^3-7x+6 and $6x^3-7x^2+1$.
12. $5x^4-x^3-x^2-5x+2$ and $5x^4-x^3-7x^2+5x-2$.
13. $ax^3+x^2(b-a)-x(a^2+b+ab)+a(a+b)$
and $bx^3-x^2(a+b-ab)-ax(a+b-1)+a^2$.
14. $1+a^2+a^4$ and $1+2a+a^2-a^4$.
15. $3x^3-7x^2-18x-8$ and $2x^3-3x^2-17x-12$.
16. $4x^5-8x^3a^2+28x^2a^2-24xa^4+24a^5$
and $6x^4+24x^3a-12x^2a^2-24xa^3+96a^4$.
17. $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$
and $x^4-(a-b)x^3+(a-b)b^2x-b^4$.
18. $3x^2+(4a-2b)x-2ab+a^2$ and $x^3+(2a-b)x^2-(2ab-a)x-a^2b$.
19. $a^3+3a^2b+3ab^2+b^3$, $a^3+a^2b+ab^2+b^3$ and $a^3+a^2b-ab^2-b^3$.
20. $x^3-9x^2+26x-24$, $x^3-10x^2+31x-30$,
and $x^3-11x^2+38x-40$.
21. $x^3+3x-70$, $x^3-39x+70$ and $x^3-48x+7$.
22. $1-5x^2+4x^4$, $1+2x-3x^3-6x^4$ and $1+2x-4x^3-8x^4$.
23. $x^3-2x^2-13x-10$, $x^3-4x^2-11x+30$ and x^3-6x^2-x+30.
24. $8x^3-18x^2-51x-20$, $6x^3-35x^2+48x-16$
and $3x^3-17x^2+22x-8$.

145. To find the H. C. F. of two expressions by destroying the highest and lowest terms alternately.

Ex. 1. Find the H. C. F. of :—

a^4+2a^3+2a+1 and a^3-a^2-7a+3.

Let $A=a^4+2a^3+2a+1$

and $B=a^3-a^2-7a+3$.

The H. C. F. of A and B is a factor of

$A-B\times a=(a^4+2a^3+2a+1)-a(a^3-a^2-7a+3)$

$=3a^3+7a^2-a-1=C$ say.

The H. C. F. of A and B is a factor of

$C-3B=(3a^3+7a^2-a-1)-3(a^3-a^2-7a+3)$

$=10(a^2+2a-1)$ . . . . . (i)

Again the H. C. F. of A and B is a factor of

$B+3C=(a^3-a^2-7a+3)+3(3a^3+7a^2-a-1)$

$=10a^3+20a^2-10a$

$=10a(a^2+2a-1)$ . . . . . (ii)

We have shown that the H. C. F. of A and B is a factor of $10(a^2+2a-1)$ and of $10a(a^2+2a-1)$.

Neither 10 nor a is a common factor of a and b and a^2+2a-1 cannot be further resolved.

$\therefore a^2+2a-1$ is their H. C. F.

Ex. 2 Find the H. C. F. of :—

$2x^5-11x^2-9$ and $4x^5+11x^4+81$.

If D be their H. C. F., then D is a factor of

$(4x^5+11x^4+81)-2(2x^5-11x^2-9)$,

i.e., of $11(x^4+2x^2+9)$,

i.e., of $(x^4+2x^2+9$ . . . . . (i)

Again, D is a factor of $9(2x^5-11x^2-9)+(4x^5+11x^4+81)$

i.e., of $11x^2(2x^3+x^2-9)$,

i.e., of $(2x^3+x^2-9)$, ... (ii)

$\therefore D$ is a factor of (i) and (ii),

$\therefore$ it is a factor of their sum, *i.e.*, of $x^2(x^2+2x+3)$,

i.e., of x^2+2x+3.

Since x^2+2x+3 cannot be further resolved, $D=x^2+2x+3$.

146. Miscellaneous examples.

Ex. 2, What value of a will make

$2(a^2+a)x^2+(11a-2)x+4$ and

$2(a^3+a^2)x^3+(11a^2-2a)x^2+(a^2+5a)x+5a-1$ have a common factor?

Divide the one expression by the other as in finding their H.C.F.

$$\left.\begin{matrix}2(a^2+a)x^2+\\(11a-2)x+4\end{matrix}\right\}\ \begin{matrix}2(a^3+a^2)x^3+(11a^2-2a)x^2+(a^2+5a)x+5a-1\\ 2(a^3+a^2)x^3+(11a^2-2a)x^2+4ax\end{matrix}\ \left\{ ax$$

$$(a^2+a)x+5a-1$$

$$(a^2+a)x+5a-1\,]\,2(a^2+a)x^2+(11a-2)x+4[\,2x$$

$$2(a^2+a)x^2+10ax-2x$$

$$ax+4$$

$$ax+4\,]\,(a^2+a)x+5a-1[\,a+1$$

$$(a^2+a)x+4a+4$$

$$a-5$$

In order that the expressions may have a common factor the remainder $a-5$ must be equal to 0. $\therefore a=5$.

Ex. 2. If $x+p$ be a common factor of ax^2+bx+c and cx^2+bx+a, find the value of p and the relation between, a, b and c.

Dividing ax^2+bx+c by $x+p$, the remainder is ap^2-bp+c.

Since $x+p$ is a factor of ax^2+bx+c,

this remainder $ap^2-bp+c=0$ . . . . . (i)

Again, dividing cx^2+bx+a by $x+p$,

the remainder is cp^2-bp+a.

For the same reason, $cp^2-bp+a=0$. . . . . (ii)

From (i) and (ii), we have $p^2(a-c)+c-a=0$

$$\therefore p^2=\frac{a-c}{a-c}=1\ \therefore p=\pm 1.$$

$\therefore x\pm 1$ is a factor of each of the given expressions.

$\therefore a+b+c=0$, or $a+c=b$. . . . . (Art. 136).)

Ex. 3. Find the condition that ax^2+bx+c and dx^2+ex+f should have a common factor of the form $x+p$.

The common factor of ax^2+bx+c and dx^2+ex+f is a factor of $d(ax^2+bx+c)-a(dx^2+cx+f)$, *i.e.*, of $x(bd-ae)+cd-af$. . . . . (i)

Again, the common factor of ax^2+bx+c and dx^2+ex+f is a factor of $f(ax^2+bx+c)-c(dx^2+ex+f)$,

i.e., of $x^2(af-cd)+x(bf-ec)$.

Rejecting the factor x which is not a common factor of the given expressions, we have the common factor of the given expressions, as a factor of $x\ (af-cd)+bf-ec$. . . . . (11)

It is clear that if the given expressions have a factor of the form $x+p$, it must be a factor of

$x(bd-ae)+cd-af$ and $x(af-cd)+bf-ec$,

i. e., of $(bd-ae)\left(x+\frac{cd-af}{bd-ae}\right)$ and $(af-cd)\left(x+\frac{bf-ec}{af-cd}\right)$

Therefore $x+p=\left(x+\frac{cd-af}{bd-ae}\right)=x+\left(\frac{bf-ec}{af-cd}\right)$

$\therefore \frac{cd-af}{bd-ae}=\frac{bf-ec}{af-cd}$, or $(cd-bf)^2=(bf-ec)(ae-bd)$ which is the condition required.

EXERCISE 68.

1. If $x+a$ be the H. C. F. of x^2+px+q and $x^2+p'x+q'$, prove that $(p-q')a=q-q'$.

2. If x^2+px+q and $x^2+p'x+q'$ have a common factor of the form $x+a$, shew that it is also a factor of $px^2+(q-p')x-q'$.

3. If $x+a$ be a common factor of x^2+px+q and x^2+rx+s, find the value of a in terms of p, q, r and s.

4. If $x+a$ be the H. C. F. of x^2+px+q and x^2+qx+p, then either $a+1=0$ or $p=q$.

5. If x^3-px^2+qx-r and $x^3-p'x^2+q'x-r$ have a common factor of the form $x-a$, then $a=\frac{q-p'}{p-q'}$.

6. If the expressions $ax^3-(3a+b)cx^2+(a^3+bc^2)x+d$ and $bx^3+(a-b)cx^2+a(c^2-a^2)x-d$ have a common factor of the form px^2+qx+r, prove that this factor is an exact square.

7. Find the relation between a, b and c in order that ax^3+bx+c and cx^3+bx^2+a may have a common factor of the form x^2+x+1.

8. If $x+p$ be the H. C. F. of $x^2+ax+ab$ and $x^2+cx+bc$, shew that $p=b$.

9. If $x+p$ be the H. C. F. of x^2+ax+b and $x^2+2bx+3a$, shew that $p=\frac{b-ba}{a-2b}$.

10. If $x+a$ be the H. C. F. of x^2+mx+1 and x^2+nx+2, shew that $a=\frac{1}{n-m}$.

11. If ax^2+bx+c and $ax^2+(a+b)x+a$ have a common factor of the form $x+p$, shew that each of the expressions $(a-c)(b+c)$ and $(a-b)(c-a)$ is an exact square.

12. If $x+p$ be a common factor of x^2+bx+c and x^2+ax+d shew that it is a factor of $bx^2+(c-a)x-d$.

13. If x^2-3x+a and $x^2+2x+6a$ have a common factor find it.

14. What value of m will make

$x^3-(m-6)x^2-2mx+24$ and $x^3-(m+3)x^2-(2m-45)x-3$, have a common factor?

15. For what value of a will

$x^3-ax^2+19x-a-4$ and $x^3-(a+1)x^2+23x-a-7$ have a common factor?

16. If x^2+px+q and $x^2+p'x+q'$ have a common factor, then $(q'-q)^2+(p'-p)^2(pq'-p'q)=0$.

CHAPTER XVII.

Lowest Common Multiple.

147. A Common Multiple of two or more algebraical expressions is an expression which is divisible by each of them without remainder.

The Lowest Common Multiple of two or more algebraical expressions is the expression of lowest dimensions which is exactly divisible by each of them.

The term **Least Common Multiple** is not appropriate in Algebra ; for when numerical values are assigned to the letters of the expressions, the numerical value of their L. C. M. is not necessarily the L. C. M. of the values of the expressions. Therefore, the term *Lowest Common Multiple* should be preferred.

148. L. C. M. of Simple Expressions and of such Compound Expressions as can be easily resolved into factors.

Rule :—"*Find the L. C. M. of the numerical factors and after it write down all the factors that occur in the different expressions, raising each factor to the highest power in which it occurs in any of the expressions.*"

Ex. 1. Find the L. C. M. of $18a^3b^2c$, $6a^2b^3d$ and $7a^4d^5$.
The L. C. M. of 18, 6 and 7 is 126.
The different factors that occur are a, b, c, and d ;
the highest powers in which they occur are a^4, b^3, c and d^5.
$\therefore$ $126a^4b^3cd^5$ is the L. C. M.

Ex. 2. Find the L. C. M. of :—
$7(x^3-y^3)^2$, $5(x-y)^3$ and $21(x^3-y^3)$.
The L. C. M. of 7, 5 and 21 is 105 ;
and $(x^3-y^3)^2=(x+y)^2(x-y)^2$;
and $x^3-y^3=(x-y)(x^2+xy+y^2)$.
The different factors are $x+y$, $x-y$ and x^2+xy+y^2.
Their highest powers are $(x+y)^2, (x-y)^3, x^2+xy+y^2$.
$\therefore$ the L. C. M. $=105(x+y)^2(x-y)^3(x^2+xy+y^2)$.

EXERCISE 69.

Find the L. C. M. of :—

1. $3x^2a$, $6x^2y^2z^2$ and $27\ x^4y^4z^4$.
2. $8a^3b^2c$, $12ab^3c^2$ and $20a^2bc^3$.
3. $x^3b - xb^3$ and $x^3b^3 + x^2b^3$.
4. $x^2 + 4x + 3$ and $x^2 - 2x - 15$.
5. $l^2 - (2m - 3n)l - 6mn$ and $l^2 - 4ml + 4m^2$.
6. $3x^2 + 7xy - 6y^2$ and $2x^3 + 7x^2y + 3xy^2$.
7. $x^3 - (l + m + n)x^2 + (lm + mn + nl)x - lmn$.
8. $(lx + my)^2 - (l - m)(x + z)(lx - my) + (l - m)^2xz$
   and $(lx - my)^2 - (l + m)(x + z)(lx - my) + (l + m)^2xz$.
9. $x^2 - a^2$, $x^3 - a^3$ and $x^4 - a^4$.
10. $x^2 + 5x + 4$, $x^2 + 2x - 8$ and $x^2 + 7x + 12$.
11. $x^2 + (a + b)x + ab$, $x^2 + (b + c)x + bc$ and $x^2 + (c + a)x + ca$.
12. $x^2 + (b - c)x - bc$, $x^2 + (c - a)x - ca$ and $x^2 + (a - b)x - ab$.
13. $2x^2 - x - 1$, $2x^2 + 3x + 1$ and $4x^4 - 5x^2 + 1$.
14. $2x^2 + xy - y^2$, $3x^2 + 2xy - y^2$ and $4x^2 + 3xy - y^2$.
15. $3x^2 - 5x + 2$, $5x^2 - 9x + 4$ and $4x^3 - 4x^2 - x + 1$.
16. $6x^2 - x + 1$, $3x^2 + 7x + 2$ and $2x^2 + 3x - 2$.
17. $9x^4 - 28x^2 + 3$, $27x^4 - 12x^2 + 1$, $27x^4 + 6x^2 - 1$ and $x^4 - 6x^2 + 9$.
18. $(a^2 - b^2)^3$, $(a^3 - b^3)^2$ and $(a^4 - b^4)(a - b)^3$.
19. $a^2(b - c) + b^2(c - a) + c^2(a - b)$ and $(a - b)^3 + (b - c)^3 + (c - a)^3$.
20. $a^5 - b^5$, $a^4 + a^2b^2 + b^4$ and $(a + b + c)(ab + ac + bc) - abc$.

149 L C M. of two Compound Expressions.

Let A and B be the two given expressions, and D their H. C. F.;

then $A = pD$ and $B = qD$, where p and q have no common factor, since D is the highest common factor of A and B.

The L. C. M. of pD and qD is clearly pqD ;

but $$pqD = p \times qD = pB = \frac{A}{D} \times B = \frac{A \times B}{D},$$

or $$pqD = q \times pD = qA = \frac{B}{D} \times A = \frac{A \times B}{D}.$$

Hence we have the following rule :—

Rule.—"*Divide either of the expressions by their H. C. F., and multiply the quotient by the other.*"

or "Divide the product of the expressions by their H. C. F., and the quotient is their L. C. M."

If A and B are any two expressions, the relation between them, their H.C.F. and their L. C. M. are given by the following formulæ:—

$$\text{L. C. M.} = \frac{A \times B}{\text{H. C. F.}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\text{H. C. F.} = \frac{A \times B}{\text{L. C. M.}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$$A \times B = \text{H. C. F.} \times \text{L. C. M.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$A = \frac{\text{H. C. F.} \times \text{L. C. M.}}{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

$$B = \frac{\text{H. C. F.} \times \text{L. C. M.}}{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

Ex. 1. Find the L. C. M. of :—

$x^3 - 3x^2 + 3x - 1$ and $x^3 - x^2 - x + 1$.

First expression $= (x-1)^3$.

Second expression $= x^2(x-1) - (x-1)$

$= (x-1)(x^2-1)$

$= (x-1)^2(x+1)$

Therefore their H.C.F. is $(x-1)^2$.

Hence the L. C. M. $= \dfrac{(x-1)^3(x-1)^2(x+1)}{(x-1)^2} = (x-1)^3(x+1)$.

Ex. 2. Find the L. C. M. of:—

$x^3 + 3x^2 - 4x - 12$ and $x^3 + 2x^2 - x - 2$.

First expression $= x^2(x+3) - 4(x+3) = (x^2-4)(x+3)$

$= (x+2)(x-2)(x+3)$.

Second expression $= x^2(x+2) - (x+2) = (x^2-1)(x+2)$

$= (x+1)(x-1)(x+2)$.

Their H. C. F. is $x+2$.

Hence the L. C. M. $= \dfrac{(x^2-4)(x+3)(x^2-1)(x+2)}{x+2}$

$= (x^2-4)(x+3)(x^2-1)$.

150. Every Common Multiple of two Algebraical expressions is a Multiple of their Lowest Common Multiple.

Let A and B be the two expressions, and M their L. C. M., and let N denote any other common multiple.

Suppose, if possible, that when N is divided by M there is a remainder R, q being the quotient.

Then $R = N - qM$.

Since A and B are factors of M and N, they are also factors of $N - qM$, *i.e.*, of R . . . . . (Art. 141.)

Thus R is a multiple of A and B ;

but R is of *lower* dimensions than M, which, by supposition is their L. C. M.

This is absurd ; hence there can be no remainder R ;

that is, N is a multiple of M.

151. L. C. M. of three or more Compound Expressions.

Let A, B, C, D.....be the expressions.

Find the L. C. M. of two of them say A and B.

If M be this L. C. M., then the L. C. M. of M and C is the L. C. M. of A, B, and C.

If N be this L. C. M., then the L. C. M. of N and D is the L. C. M. of A, B, C and D ; and so on.

We can easily deduce from the foregoing Articles that

(1) **The L. C. M. of two expressions is the H. C. F. of all their commom multiples.**

(2) **The H. C. F of two expressions is the L. C. M. of all their commom factors.**

152. Miscellaneous Examples.

Ex. 1. If $x+p$ be the H. C. F. of x^2+ax+b and x^2+cx-d, shew that their L. C. M. is equal to $x^3+(a+c-p)x^2+(ac-p^2)x+p(a-p)(c-p)$.

Dividing x^2+ax+b by $x+p$,

the quotient is $x+a-p$ and the remainder is p^2-ap+b.

Since $x+p$ is a factor of x^2+ax+b,

the remainder p^2-ap+b must be equal to 0,

$\therefore b=ap-p^2=p(a-p)$ . . . . . (1

Again, dividing x^2+cx+d by $x+p$,

the quotient is $x+c-p$ and the remainder is p^2-cp+d.

For the same reason, $p^2-cp+d=0$.

$\therefore d=cp-p^2=p(c-p)$

We have $x^2+ax+b=(x+p)(x+a-p)$

and $x^2+cx+d=(x+p)(x+c-p)$.

$\therefore$ The L. C. M. of the two expressions

$$=\frac{(x^2+ax+b)(x^2+cx+d)}{x+p}$$

$=(x+a-p)(x^2+cx+d)$

$=(x+a-p)(x^2+cx+cp-p^2) \because d=p(c-p)$

$=x^3+(a+c-p)x^2+(ac-p^2)x+p(a-p)(c-p).$

Ex. 2. The H. C. F. of two expression is $x-4$;
and their L. C. M. is $x^3-9x^2+26x-24$. Find the expressions

The L. C. M. $=x^3-9x^2+26x-24=(x-2)(x-3)(x-4)$.

The H. C. F. $=x-4$.

The product of the expressions

$=$H. C. F. $\times$ L. C. M.

$=(x-4)(x-2)(x-3)(x-4)$.

It is plain that each expression must contain $x-4$ as a factor, and that the other factors of the expressions must have no common factor.

$\therefore$ the expressions are $(x-4)(x-2)$ and $(x-4)(x-3)$:

or $x-4$ and $(x-4)(x-2)(x-3)$.

EXERCISE 70.

Find the L. C. M. of :—

1. $6a^3-11a^2+5a-3$ and $9a^3-9a^2+5a-2$.
2. $x^3-9a^2+26x-24$ and $x^3-13x^2+31x-30$.
3. $2x^3+(2a-3b)x^2-(2b^2+3ab)x+3b^3$ and $2x^2-(3b-2c)x-3bc$.
4. $4x^4+4x^3+7x^2+11x+4$ and $6x^4+7x^3+4x^2+5x+2$.
5. $4a^4+8a^3+21a^2+18a+27$ and $3a^4+6a^3+17a^2+16a+24$.
6. $6x^2-11x+3$, $4x^2-4x-3$ and $6x^2+25x-9$.
7. $6a^2-19a+13$, $12a^2-11a+2$ and $8a^2+10a-3$.
8. $a^4-3a+20$, $a^3-5a+12$ and $3a^3-5a^2+16$.
9. x^5-2x^4-x-1 and x^5+2x^3+x+1.
10. $ab+2a^2-3b^2-4bc-ac-c^2$

and $9ac+2a^2-5ab+4c^2+8bc-12b^2$.

11. If the H. C. F. and L. C. M. of A and B be x and y respectively, and if $x+y=ma+\frac{b}{m}$,

shew that $x^3+y^3=m^3a^3+\frac{y^3}{m^3}$.

12. The L. C. M. of two quantities is $x^4-5a^2x^3+4a^4$ and their H. C. F. is x^2-a^2 ;

one of the quantities is $x^3-2ax^2-a^2x+2a^3$; find the other.

13. If $x+d$ be the H. C. F. of x^3+ax+b and $x^3+a^1x+b^1$, prove that their L. C. M. is equal to

$$x^5+(a+a^1-d)x^3-(aa^1-d^2)x+(a-d)(a^1-d)d.$$

14. If x be the H. C. F. of a and b, and y be their L. C. M. shew that $a^3+b^3+x^3+y^3$ is the sum of two squares.

15. If $x+p$ be the H. C. F. of x^2+ax+b and x^2+cx+b, their L. C. M. will be $x^3+x^2(2a-p)+x\left\{\frac{b(a+p)}{p}\right\}+\frac{b^2}{p}$.

16. If x^2+nx+1 and x^3+nx^2+1 have a common factor, determine n; and then shew that the L. C. M. of these quantities is $x^4-3x^3+2x^2+x+1$.

17. If the H. C. F. of two expressions be $x-5$ and their L. C. M. be $x^3-13x^2+52x-60$, find the expressions.

18. If the H. C. F. of two expressions be $x+4$ and their L. C. M. be $x^3-4x^2-17x+60$, find the exprssions.

19. If $x+1$ be the H. C. F. of two expressions and $x^3+6x^2+11x+6$ be their L. C. M., find the expressions.

20. If $x+d$ be the H. C. F. of $x^2+ax+ad$ and $x^2+cx+dc$, their L. C. M. will be $x^3+x^2(a+c)+acx$.

21. If x^2+ax+b and x^2+px+q have a common factor of the first degree, then their L. C. M. is equal to

$$x^3+\frac{ab-pq}{b-q}x^2+\left\{ap-\left(\frac{b-q}{a-p}\right)^2\right\}x+bq.\frac{x-p}{b-q}.$$

22. If H_1, H_2, H_3, be the H. C. F. of A and B, B and C, C and A respectively, then the L. C. M. of A, B and C

$$=\frac{ABC}{H_1H_2H_3}\times\text{H. C. F. of } A, \text{ B and } C.$$

23. If the H. C. F. of $ax^2+2bx+c$ and $bx^2+2ax+c$ is of the form $x+p$, shew that their L. C. M. is $(x-2)(ax+1)(bx+1)$.

24. If x^2+nx+1 and x^3+nx^2+1 have a common factor of the first degree shew that their L. C. M. is $x^4-3x^3+2x^2+x-1$.

25. If $p+q+r=0$, the three expressions px^2+qx+r, qx^2+rx+p, and rx^2+px+q have a common binomial factor $x-1$, and their L.C.M. is $(x-1)(px+p+q)(qx+q+r)(rx+r+p)$.

CHAPTER XVIII.

FRACTIONS.

153. The fraction $\frac{a}{b}$ denotes that the unit is divided into b equal parts, and that a parts of these are taken ;

the fraction $\frac{a-x}{a+x}$ denotes that the unit is divided into $(a+x)$ equal parts, and that $(a-x)$ of these are taken.

In the fraction $\frac{a}{b}$, a is called the **numerator** and b the **denominator**.

Every integer may be considered as a fraction with unity for its denominator ; thus $p=\frac{p}{1}$.

154. The value of a fraction is not altered if the numerator and the denominator be divided or multiplied by the same quantity.

Thus $\frac{a}{b}=\frac{am}{bm}=\frac{an}{bn}=\frac{\frac{a}{m}}{\frac{b}{m}}=\frac{\frac{a}{n}}{\frac{b}{n}}$

It follows from this that we may change the signs of both numerator and denominator of a fraction without changing its value.

Thus $\frac{a}{b}=\frac{a\times(-1)}{b\times(-1)}=\frac{-a}{-b}$;

$$\frac{a-x}{2a-x}=\frac{(a-x)(-1)}{(2a-x)(-1)}=\frac{-a+x}{-2a+x}=\frac{x-a}{x-2a}.$$

155. To reduce a fraction to its simplest Form—A fraction is said to be in its *simplest form*, when its numerator and denominator have no common factor.

And if two or more expressions are divided by their H. C. F. the quotients cannot therefore have any more common factor.

Hence we have the following rule :--

Rule.—" *Divide both the numerator and the denominator by their H. C. F.; the quotients are respectively the numerator and the denominator of the reduced fraction.*

Ex. 1. Reduce $\frac{2ab(a^2-b^2)}{6(a^2b+ab^2)}$ to its lowest terms.

The H. C. F. of the numerator and the denominator is equal to $2ab\,(a+b)$.

Hence $\frac{2ab(a^2-b^2)}{(6a^2b+ab^2)}=\frac{2ab(a+b)(a-b)}{6ab(a+b)}=\frac{a-b}{3}$.

Ex. 2. Reduce $\frac{11a^4+24a^3+125}{a^4+24a+55}$ to its lowest terms.

The H. C. F. of the numerator and denominator is a^2+4a+5. Dividing both by it, the quotients are

$11a^2-20a+25$ and $a^2-4a+11$.

Hence the fraction in its lowest terms $=\frac{11a^2-20a+25}{a^2-4a+11}$.

EXERCISE 71.

Reduce to its lowest terms :—

1. $\frac{20a^2bc^3}{35ab^2c}$.
2. $\frac{15x^2y^3z^4}{25xy^7z^3}$.
3. $\frac{12x^4y^3z^3}{16x^3y^4z^4}$.
4. $\frac{a^5x^4y^2z^3}{ab^3x^2y^3z^2}$.
5. $\frac{x^2-a^2}{x^3-ax}$.
6. $\frac{a^3-3a}{9a-a^3}$.
7. $\frac{4a^2-9b^2}{4a^2+6ab}$.
8. $\frac{3ab-12a^2}{b^2-16a^2}$.
9. $\frac{a^4+a^2b^2+b^4}{a^3+b^3}$.
10. $\frac{x^4+x^2y^2+y^4}{x^6-y^6}$.
11. $\frac{x^2-4x+3}{x^2-7x+12}$.
12. $\frac{6x^2-7x-20}{4x^3-27x+5}$.
13. $\frac{x^2-3x-4}{x^2-4x-5}$.
14. $\frac{x^3-6x^2+11x-6}{x^3-3x+2}$.
15. $\frac{a^3-3a^2b+3ab^2-b^3}{a^2-2ab+b^2}$.
16. $\frac{6x^3-5x^2+4}{2x^3-x^2-x+2}$.
17. $\frac{3x^3-16x2+23x-6}{2x^3-11x^2+17x-6}$.
18. $\frac{x^3-x^2y+xy^2-y^3}{x^4+2x^2y^2+y^4}$.
19. $\frac{x^3-3x-2}{2x^3+3x^2-1}$.
20. $\frac{x^3-39x+70}{x^2-3x+70}$.
21. $\frac{x^3-19x^2+119x-245}{3x^2-31x+119}$.
22. $\frac{2x^3+8x^2y+16xy^2+16y^3}{8x^2+4xy-24y^2}$.

23. $\dfrac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{a^2+b^2+2ab-c^2}$.

24. $\dfrac{a^3+b^3+c^3-3abc}{(a-b)^2+(b-c)^2+(c-a)^2}$.

25. $\dfrac{(a-b)^3+(b-c)^3+(c-a)^3}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

26. $\dfrac{a^2+ab+ac+bc}{a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc}$.

27. $\dfrac{(2a+b)^3-(a-b)^3-(a+2b)^3}{(2a+b)(a+2b)}$.

28. $\dfrac{(a+b+c)^3-a^3-b^3-c^3}{(a+b+c)(ab+ac+bc)-abc}$.

29. $\dfrac{(ab-1)^2+(a+b-2)(a+b-2ab)}{(ab+1)^2-(a+b)^2}$.

30. $\dfrac{ab+2a^2-3b^2+4bc+ac-c^2}{2a^2-9ac-5ab+4c^2-8bc-12b^2}$.

156. To reduce fractions having different denominators to equivalent fractions having the same denominator.

Rule.—"*Take as the common denominator the L. C. M. of all the denominators; divide this L. C. M. by each denominator in turn, and multiply the quotients by the corresponding numerators the products so obtained are the numerators of the equivalent fractions.*"

Note.—Compare the corresponding rule in Arithmetic.

Ex. 1. Reduce $\dfrac{1}{2x}$, $\dfrac{3}{4xy}$ and $\dfrac{5}{6y^2}$ to equivalent fractions having a common denominator.

The L. C. M. of the denominators is $12xy^2$.

Dividing this by each denominator in turn, the quotients are $6y^2$, $3y$ and $2x$;

Multiplying these by the corresponding numerators, we get $6y^2$, $9y$ and $10x$.

Therefore the fractions are $\dfrac{6y^2}{12xy^2}$, $\dfrac{9y}{12xy^2}$ and $\dfrac{10x}{12xy^2}$.

Ex. 2. Reduce $\dfrac{2}{b^2-a^2}$, $\dfrac{3a}{(a+b)^2}$ and $\dfrac{4b^2}{(a-2)^3}$ to equivalent fractions having a common denominator.

The first fraction may be written $\frac{-2}{a^2-b^2}$ by changing the signs of both the numerator and the denominator.

The L. C. M. of the denominators is $(a+b)^2(a-b)^2$: dividing this by the corresponding denominators,

we get a^2-b^2, $(a-b)^2$ and $(a+b)^2$.

Therefore the required fractions are :—

$$\frac{-2(a^2-b^2)}{(a+b)^2(a-b)^2}, \quad \frac{3a(a-b)^2}{(a+b)^2(a-b)^2} \text{ and } \frac{4b^2(a+b)^2}{(a+b)^2(a-b)^2}.$$

157. Addition and Subtraction of Fractions.

Rule.—"*Transform the fractions into equivalent ones having a common denominator; then add or subtract the numerators, as the case may be, retaining the common denominator.*"

Ex. 1. Find the sum of $\frac{1}{a}$ and $\frac{1}{b}$.

The common denominator is ab.

$$\therefore \frac{1}{a}=\frac{b}{ab} \text{ and } \frac{1}{b}=\frac{a}{ab} \therefore \text{the sum } =\frac{b}{ab}+\frac{a}{ab}=\frac{b+a}{ab}.$$

Ex. 2. Find the sum of $\frac{a}{a-b}$ and $\frac{a^2}{a^2-b^2}$.

The common denominator is a^2-b^2.

$$\therefore \frac{a}{a-b}+\frac{a^2}{a^2-b^2}=\frac{a(a+b)}{a^2-b^2}+\frac{a^2}{a^2-b^2}=\frac{a^2+ab+a^2}{a^2-b^2}=\frac{2a^2+ab}{a^2-b^2}.$$

Ex. 3. Simplify $\frac{x+a}{x-a}-\frac{x^2-a^2}{x^2+a^2}+\frac{x^2+a^2}{x^2-a^2}$.

Here the L. C. M. of the denominators

$$=(x^2+a^2)(x^2-a^2)=x^4-a^4 :$$

therefore the expression

$$=\frac{(x+a)(x^2+a^2)(x+a)-(x^2-a^2)(x^2-a^2)+(x^2+a^2)(x^2+a^2)}{x^4-a^4}$$

$$=\{(x^4+a^4+2x^2a^2+2ax^3+2a^3x)-(x^4+a^4-2x^2a^2)$$
$$+(x^4+a^4+2x^2a^2)\}+(x^4-a^4).$$

$$=\frac{x^4+2ax^3+2a^3x+6x^2a^2+a^4}{x^4-a^4}$$

Ex. 4. Simplify $\frac{a+c}{(a-b)(x-a)}+\frac{b+c}{(b-a)(x-b)}$

Here $+\dfrac{b+c}{(b-a)(x-b)}=-\dfrac{b+c}{(a-b)(x-b)}$

$\therefore$ the L. C. M. of the denominators is $(a-b)(x-a)(x-b)$;

$\therefore$ the expression $=\dfrac{(a+c)(x-b)-(b+c)(x-a)}{(a-b)(x-a)(x-b)}$

$=\dfrac{ax-ab+cx-bc-(bx-ab+cx-ac)}{(a-b)(x-a)(x-b)}$

$=\dfrac{ax-bx-bc+ac}{(a-b)(x-a)(x-b)}$

$=\dfrac{x(a-b)+c(a-b)}{(a-b)(x-a)(x-b)}$

$=\dfrac{(a-b)(x+c)}{(a-b)(x-a)(x-b)}$

$=\dfrac{x+c}{(x-a)(x-b)}$

EXERCISE 72.

Reduce to equivalent fractions having a common denominator:—

1. $\dfrac{xy}{z}, \dfrac{yz}{x}, \dfrac{xz}{y}$

2. $\dfrac{a}{bc}, \dfrac{b}{ca}, \dfrac{c}{ab}$

3. $\dfrac{1}{a+1}, \dfrac{3}{2a+2}, \dfrac{5}{a^2-1}$

4. $\dfrac{1}{x^2-3x+2}, \dfrac{1}{x^2-5x+6}, \dfrac{1}{x^2-4x+3}$

5. $\dfrac{1}{x-y}, \dfrac{1}{y-z}, \dfrac{1}{z-x}$

6. $\dfrac{1}{a^2+ab+b^2}, \dfrac{1}{a^2-ab+b^2}, \dfrac{1}{a^6-b^6}$

7. $\dfrac{x}{x^2+xy+xz+yz}, \dfrac{y}{y^2+xy+xz+yz}, \dfrac{z}{z^2+xy+xz+yz}$

Simplify :—

8. $\dfrac{1}{(a-b)(a-c)}-\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(b-c)(c-a)}$

9. $\dfrac{1+x}{1+x+x^2}+\dfrac{1-x}{1-x+x^2}$.

10. $\dfrac{2a+3b}{2a-3b}+\dfrac{2a-3b}{2a+3b}$.

11. $\dfrac{2}{2x+1}-\dfrac{8}{4x+3}+\dfrac{3}{3x+2}$

12. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a-b}{a+b}$.

13. $\dfrac{a-x}{x}+\dfrac{2a}{a-x}+\dfrac{a^2+a^2x}{x^2-a^2x}$.

14. $\dfrac{1}{x}-\dfrac{2}{x+1}+\dfrac{1}{x+2}$.

15. $\dfrac{x^2}{x^2-1}-\dfrac{x}{1-x}+\dfrac{x}{1+x}$.

16. $\frac{1}{a+x}+\frac{1}{a-x}+\frac{2a}{a^2+x^2}$.

17. $\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$.

18. $\frac{2}{x}+\frac{3}{1-2x}-\frac{2x-3}{4x^2-1}$.

19. $\frac{a^2}{(a-b)^3}+\frac{ab}{(a-b)^2}+\frac{a}{a-b}$.

20. $\frac{1}{(a-2)(a-3)}+\frac{2}{(a-1)(3-a)}+\frac{1}{(a-1)(a-2)}$.

21. $\frac{x^3}{a^3+x^3}+\frac{c}{a+x}+\frac{a}{b}$.

22. $\frac{x+y}{y}-\frac{x}{x+y}-\frac{x^3-x^2y}{x^2y-y^3}$.

23. $\frac{1}{2x+y}+\frac{1}{2x-y}-\frac{3x}{4x^2-y^2}$.

24. $\frac{x}{1-x}+\frac{x^2}{x^2-1}+\frac{x^3}{1+x^3}$.

25. $\frac{4}{9(a-2)}+\frac{5}{9(a+1)}-\frac{1}{3(a+1)^2}$.

26. $\frac{1}{x^2-2}-\frac{2}{x^2-1}+\frac{2}{x^2+1}-\frac{1}{x^2+2}$.

27. $\frac{x-1}{(x+2)(x+5)}-\frac{2(x+2)}{(x+5)(x-1)}+\frac{x+5}{(x-1)(x-2)}$.

28. $\frac{a^2-bc}{(a+b)(a+c)}+\frac{b^2-ca}{(b+c)(b+a)}+\frac{c^2-ab}{(c+a)(c+b)}$.

29. $\frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}$.

30. $\frac{1}{x^2-3x+2}+\frac{1}{x^2-5x+6}+\frac{1}{x^2-8x+15}$.

31. $\frac{1-x+x^2}{1+x+x^2}-\frac{1+x+x^2}{1-x+x^2}+\frac{4x(1+x^2)}{1+x^2+x^4}$.

32. $\frac{(a-b)^2}{(b-c)(c-a)}+\frac{(b-c)^2}{(c-a)(a-b)}+\frac{(c-a)^2}{(a-b)(b-c)}$.

33. $\frac{3+2a}{2-a}+\frac{2a-3a}{(x-a)^2}+\frac{a^2}{(x-a)}$.

34. $\frac{3+2a}{2-a}-\frac{2-3a}{2+a}-\frac{16a-a^2}{a^2-4}$.

35. $\frac{1}{1+a+a^2+a^3}+\frac{1}{1-a-a^2+a^3}-\frac{2a^3}{1-a^2-a^4+a^6}$.

158. Multiplication and Division of Fractions.

I. To Multiply two or more fractions :—

Rule —"*Multiply the numerators of the fractions together for a new numerator and the denominators together for a new denominator and reduce the resulting fraction to its lowest terms.*"

Ex. 1. Multiply $\frac{a^2-b^2}{4}$ by $\frac{a+b}{(a-b)^2}$.

$$\frac{a^2-b^2}{4}\times\frac{a+b}{(a-b)^2}=\frac{(a^2-b^2)(a+b)}{4(a-b)^2}=\frac{(a+b)(a+b)}{4(a-b)}=\frac{(a+b)^2}{4(a-b)}.$$

Ex. 2. Multiply $\frac{a^3-b^3}{a^3+b^3}$ by $\frac{(a+b)^2}{(a-b)^2}$.

$$\text{The product}=\frac{(a^3-b^3)(a+b)^2}{(a^3+b^3)(a-b)^2}=\frac{(a-b)(a^2+ab+b^2)(a+b)^2}{(a+b)(a^2-ab+b^2)(a-b)^2}$$

$$=\frac{(a^2+ab+b^2)(a+b)}{(a^2-ab+b^2)(a-b)}$$

$$=\frac{a^3+2a^2b+2ab^2+b^3}{a^3-2a^2b+2ab^2-b^3}$$

II. To divide one fraction by another :—

Rule.—"*Invert the Divisor and proceed as in Multiplication.*"

Ex. 1. Divide $\frac{a^2-b^2}{a+2b}$ by $\frac{a-b}{3a+6b}$.

$$\text{The quotient}=\frac{a^2-b^2}{a+2b}\times\frac{3a+6b}{a-b}=\frac{(a+b)(a-b)3(a+2b)}{(a+2b)(a-b)}$$

$$=3(a+b).$$

Ex. 2. Divide $\frac{a^3+b^3}{a^3-b^3}$ by $\frac{a^2-ab+b^2}{a-b}$.

$$\text{The quotient}=\frac{a^3+b^3}{a^3-b^3}\times\frac{a-b}{a^2-ab+b^2}$$

$$=\frac{(a+b)(a^2-ab+b^2)(a-b)}{(a-b)(a^2+ab+b^2)(a^2-ab+b^2)}$$

$$=\frac{a+b}{a^2+ab+b^2}.$$

Note—$\frac{\frac{a}{b}}{c}=\frac{a}{b}\div c=\frac{a}{b}\times\frac{1}{c}=\frac{a}{bc}$

and $\frac{a}{\frac{b}{c}}=a\div\frac{b}{c}=\frac{a}{1}\times\frac{c}{b}=\frac{ac}{b}$.

159. Mixed and complex fractions.

The expression $x^2+1-\frac{2}{x+1}$ is a mixed fraction.

To simplify this fraction, we have

$$x^3+1-\frac{2}{x+1}=\frac{x^3+1}{1}-\frac{2}{x+1}$$
$$=\frac{(x^3+1)(x+1)-2}{x+1}$$
$$=\frac{x^3+x^2+x+1-2}{x+1}$$
$$=\frac{x^3+x^2+x-1}{x+1}$$

Again $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$ is a *complex* fraction.

To simplify this fraction we have

$$\frac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}=\frac{a}{b+\dfrac{c}{\dfrac{df+e}{f}}}=\frac{a}{b+\dfrac{cf}{df+e}}=\frac{a}{\dfrac{bdf+be+cf}{df+e}}=\frac{a(df+e)}{bdf+be+cf}$$

$$=\frac{adf+ae}{bdf+be+cf}.$$

160. To find the H. C. F. and the L. C. M. of two or more fractions.—

Rule.—"*The H. C. F. of two or more fractions in their lowest terms.*"

$$=\frac{\textit{the H. C. F. of numerators}}{\textit{the L. C. M. of denominators}}$$ as in Arithmetic.

and the L. C. M. of two or more fractions in their lowest terms

$$=\frac{\textit{the L. C. M. of numerators}}{\textit{the H. C. F. of denominators}}$$ as in Arithmetic.

Ex. Find the H. C. F. and L. C. M. of $\dfrac{a^2}{bc}, \dfrac{b^2}{ca}, \dfrac{c^2}{ab}$.

Here the fractions are in their lowest terms.

Their H. C. F. $=\dfrac{\text{the H. C. F. of } a^2, b^2 \text{ and } c^2}{\text{the L. C. M. of } bc, ca \text{ and } ab}=\dfrac{1}{abc}$

Their L. C. M. $=\dfrac{\text{the L. C. M. of } a^2, b^2 \text{ and } c^2}{\text{the H. C. F. of } bc, ca, \text{ and } ab}=\dfrac{a^2b^2c^2}{1}=a^2b^2c^2.$

EXERCISE 73.

Simplify :—

1. $\dfrac{a^4-b^4}{(a+b)^3} \div \dfrac{a-b}{ab+b^2}$.

2. $\dfrac{a^4-b^4}{a^2-2ab+b^2} \div \dfrac{a^2+b^2}{a-b}$

3. $\dfrac{x^2-9x+20}{x^2-6x} \times \dfrac{x^2-13x+42}{x^2-5x}$.

4. $\left(1-\dfrac{a-b}{a+b}\right) \times \left(2+\dfrac{2b}{a-b}\right)$.

5. $\dfrac{a^2-x^2}{a+b} \times \dfrac{a^2-b^2}{ax+x^2} \times \left(a+\dfrac{ax}{a-x}\right)$.

6. $\dfrac{x(a-x)}{a^2+2ax+x^2} \times \dfrac{x(a+x)}{a^2-2ax+x^2}$.

7. $\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right) \div \left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)$

8. $\left(a+\dfrac{b-a}{1+ab}\right) \div \left(1-a\dfrac{b-a}{1+ab}\right)$.

9. $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right) \div \left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

10. $\left(\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x^2-y^2}{x^2+y^2}\right) \div \left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)$.

11. $\left\{\left(\dfrac{1}{a}+\dfrac{1}{b+c}\right) \div \left(\dfrac{1}{a}-\dfrac{1}{b+c}\right)\right\} \left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$.

12. $\dfrac{1}{x+\dfrac{1}{y+\dfrac{1}{z}}} \times \dfrac{1}{x+\dfrac{1}{y}}$.

13. $\left(\dfrac{2a+b}{a+b}-1\right) \div \left(1-\dfrac{a}{a+b}\right)$

14. $\left(1-\dfrac{2ab}{a^2+b^2}\right) \div \left(\dfrac{a^3-b^3}{a-b}-3ab\right)$.

15. $\left\{a^2-\dfrac{(a^2+b^2-c^2)^2}{4b^2}\right\} \div \left\{\left(\dfrac{a+c}{b}\right)^2-1\right\}$

16. $\left\{1-\dfrac{(a^2+b^2-c^2+d^2)^2}{4(ab+cd)^2}\right\} \div \left\{(a+b)^2-(c-d)^2\right\}$

17. $\left\{\dfrac{a^2+x^2}{a^2-x^2}-\dfrac{a^3+x^3}{a^3-x^3}\right\} \div \left\{\dfrac{a^3-x^3}{a^3+x^3}-\dfrac{a-x}{a+x}\right\}$.

18. $\left(\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}\right)\left(\dfrac{a^2}{b^2}-1+\dfrac{b^2}{a^2}\right) \div \left(\dfrac{a^4}{b^4}+1+\dfrac{b^4}{a^4}\right)$.

19. $\left(\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}\right) \div \left\{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}\right\}$.

20. $\dfrac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+c}} \div \dfrac{1+\dfrac{a^2-b^2}{a^2+b^2}}{1-\dfrac{a^2-b^2}{a^2+b^2}}$

21. $\dfrac{\dfrac{a+b+c}{a+b-c}+\dfrac{c+a-b}{b+c-a}}{\dfrac{a+b-c}{c+a-b}+\dfrac{b+c-a}{a+b+c}} \div \dfrac{\dfrac{a+b+c}{a+b-c}}{\dfrac{b+c-a}{c+a-b}}$.

22. $\dfrac{(a+b)^2+(a-b)^2}{\dfrac{b-a}{1}-\dfrac{-(a-b)}{1}} + \dfrac{(a+b)^3+(b-a)^3}{(a+b)^2-(a-b)^2}$

23. $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}-\dfrac{b}{a+b}} \div \dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}} \times \dfrac{a^2}{a^2+b^2}$

24. $\dfrac{\dfrac{2(a-b)}{2c^2}}{2(a-b)+\dfrac{\dfrac{c^2b}{b^3-a^2}}{c+\dfrac{c^2b}{b^3-a^2}}}$

25. $\dfrac{\dfrac{1}{1}}{x+\dfrac{\dfrac{x+3}{1+2-x}}{1}}$

26. $1+\dfrac{\dfrac{m-n}{m-2n+\dfrac{m+n}{m-2n}}}{1+\dfrac{m+n}{n-2m}}$

Find the H. C. and L. C. M. of :—

27. $\dfrac{a^2-b^2}{a+b}$, $\dfrac{a^3-b^3}{a-b}$ and $\dfrac{a^4-b^4}{a^2+b^2}$.

28. $\dfrac{x^3+8}{(x+2)^3}$, $\dfrac{x^3-4}{(x+1)^2}$ and $\dfrac{x^2+2x^3}{x^3-4x-12}$.

29. $\dfrac{a^4-1}{(a^2+1)^3}$, $\dfrac{a^3-2a+1}{a^3+1}$ and $\dfrac{a^4-a}{a^3+2a^2+a}$.

30. $\dfrac{a^3(b-c)}{bc(a+b)(b+c)}$, $\dfrac{b^3(c-a)}{ac(b+c)(c+a)}$ and $\dfrac{c^3(a-b)}{ab(a+b)(c+a)}$,

161. The following results are important, and the student is required to work out and verify each.

1. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}=0$

2. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}=0$

3. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}=1.$

4. $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}=1.$

5. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}=a+b+c$

6. $\dfrac{a^4}{(a-b)(a-c)}+\dfrac{b^4}{(b-c)(b-a)}+\dfrac{c^4}{(c-a)(c-b)}$
$=a^2+b^2+c^2+ab+ac+bc$

Ex. 1. Simplify :—

$$\frac{b+c}{(a-b)(a-c)}+\frac{c+a}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}.$$

The expression

$$=\frac{a+b+c-a}{(a-b)(a-c)}+\frac{a+b+c-b}{(b-c)(b-a)}+\frac{a+b+c-c}{(c-a)(c-b)}$$

$$=(a+b+c)\left\{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}\right\}$$
$$-\left\{\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}\right\}$$

$=(a+b+c)(0)-(0).$

$=0$.....(using 1 and 2).

Ex. 2. Simplify :—

$$\frac{1}{(x-a)(a-b)(a-c)}+\frac{1}{(x-b)(b-c)(b-a)}+\frac{1}{(x-c)(c-a)(c-b)}$$

The expression

$$=\frac{1}{(x-a)(x-b)(x-c)}\left\{\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-a)(x-c)}{(b-c)(b-a)}\right.$$
$$\left.+\frac{(x-a)(x-b)}{(c-a)(c-b)}\right\}$$

$$=\frac{1}{(x-a)(x-b)(x-c)}\left[x^2\left\{\frac{1}{(a-b)(a-c)}+\text{two similar two terms}\right\}\right.$$

$$-x\left\{\frac{b+c}{(a-b)(a-c)}+\text{two similar terms}\right\}$$
$$+\left\{\frac{bc}{(a-b)(a-c)}+\text{two similar terms}\right\}$$
$$=\frac{1}{(x-a)(x-b)(x-c)}\{x^2(b+c-x-b-c)+1\}\quad\text{.....(using 1, 4 and Ex. 1)}$$
$$=\frac{1}{(x-a)(x-b)(x-c)}.$$

Ex. 3. Simplify
$$\frac{(a+1)^3}{(a-b)(a-c)}+\frac{(b+1)^3}{(b-c)(b-a)}+\frac{(c+1)^3}{(c-a)(c+b)}.$$

The expression $=\left\{\frac{a^3}{(a-b)(a-c)}+\text{two similar terms}\right\}$
$$+3\left\{\frac{a^2}{(a-b)(a-c)}+\text{two similar terms}\right\}$$
$$+3\left\{\frac{a}{(a-b)(a-c)}+\text{two similar terms}\right\}$$
$$+\left\{\frac{1}{(a-b)(a-c)}+\text{two similar terms}\right\}$$
$$=(a+b+c)+3\times1+3\times0+0$$
$$=a+b+c+3\text{.....(using 1, 2, 3, 5).}$$

EXERCISE 74.

Simplify :—

1. $\dfrac{(a+1)^3}{(a-b)(a-c)}+\dfrac{(b+1)^3}{(b-c)(b-a)}+\dfrac{(c+1)^3}{(c-a)(c-b)}.$
2. $\dfrac{a^3-bc}{(a-b)(a-c)}+\dfrac{b^3-ac}{(b-c)(b-a)}+\dfrac{c^3-ab}{(c-b)(c-b)}.$
3. $\dfrac{a^3+1}{(a-b)(a-c)}+\dfrac{b^3+1}{(b-c)(b-a)}+\dfrac{c^3+1}{(c-a)(c-b)}.$
4. $\dfrac{na^3+mbc}{(a-b)(a-c)}+\dfrac{nb^3+mca}{(b-c)(b-a)}+\dfrac{nc^3+mab}{(c-a)(c-b)}.$
5. $\dfrac{b^3+c^3}{(a-b)(a-c)}+\dfrac{c^3+a^3}{(b-c)(b-a)}+\dfrac{a^3+b^3}{(c-a)(c-b)}.$
6. $\dfrac{a(b+c)}{(a-b)(a-c)}+\dfrac{b(c+a)}{(b-c)(b-a)}+\dfrac{c(a+b)}{(c-a)(c-b)}.$
7. $\dfrac{(a+2)(a+3)}{(a-b)(a-c)}+\dfrac{(b+2)(b+3)}{(b-c)(b-a)}+\dfrac{(c+2)(c+3)}{(c-a)(c-b)}$

8. $\dfrac{a^2+a+1}{(a-b)(a-c)}+\dfrac{(b^2+b+1)}{(b-c)(b-a)}+\dfrac{c^2+c+1}{(c-a)(c-b)}$.

9. $\dfrac{b+c}{a^2-a(b+c)+bc}+\dfrac{c+a}{b^2-b(c+a)+ca}+\dfrac{a+b}{c^2-c(a+b)+ab}$.

10. $\dfrac{a^2+x^2}{(a-b)(a-c)}+\dfrac{b^2+x^2}{(b-c)(b-a)}+\dfrac{c^2+x^2}{(c-a)(c-b)}$.

11. $\dfrac{b^2+c^2-2a^2}{(a-b)(a-c)}+\dfrac{c^2+a^2-2b^2}{(b-c)(b-a)}+\dfrac{a^2+b^2-2c^2}{(c-a)(c-b)}$.

12. $\dfrac{a^2-(b-c)^2}{(a-b)(a-c)}+\dfrac{b^2-(c-a)^2}{(b-c)(b-a)}+\dfrac{c^2-(a-b)^2}{(c-a)(c-b)}$.

13. $\dfrac{(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{(b+c)(b+a)}{(b-c)(b-a)}+\dfrac{(c+a)(c+b)}{(c-a)(c-b)}$.

14. $\dfrac{(a-m)(a+n)}{(a-b)(a-c)}+\dfrac{(b-m)(b+n)}{(b-c)(b-a)}+\dfrac{(c-m)(c+n)}{(c-a)(c-b)}$.

15. $\dfrac{(b+c)(x^2+a^2)}{(c-a)(a-b)}+\dfrac{(c-a)(x^2+b^2)}{(a-b)(b-c)}+\dfrac{(a+b)(x^2+c^2)}{(b-c)(c-a)}$.

16. $\dfrac{(b+c)^2}{(a-b)(a-c)}+\dfrac{(c+a)^2}{(b-c)(b-a)}+\dfrac{(a+b)^2}{(c-a)(c-b)}$.

17. $\dfrac{(a-h)(a-k)}{(a-b)(a-c)bc}+\dfrac{(b-h)(b-k)}{(b-c)(b-a)ca}+\dfrac{(c-h)(c-k)}{(c-a)(c-b)ab}$.

18. $\dfrac{a^3+1}{(a-b)(a-c)}+\dfrac{b^3+1}{(b-c)(b-a)}+\dfrac{c^3+1}{(c-a)(c-b)}$.

19. $\dfrac{a^4+1}{(a-b)(a-c)}+\dfrac{b^4+1}{(b-c)(b-a)}+\dfrac{c^4+1}{(c-a)(c-b)}$.

20. $\dfrac{1}{(a-1)(a-b)(a-c)}+\dfrac{1}{(b-1)(b-c)(b-a)}+\dfrac{1}{(c-1)(c-a)(c-b)}$

21. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$.

22. $\dfrac{1}{(a+x)(a-b)(a-c)}+\dfrac{1}{(b+x)(b-c)(b-a)}+\dfrac{1}{(c+x)(c-a)(c-b)}$.

23. $\dfrac{a}{(a-1)(a-b)(a-c)}+\dfrac{b}{(b-1)(b-c)(b-a)}+\dfrac{c}{(c-1)(c-a)(c-b)}$.

24. $\dfrac{a^2}{(a-1)(a-b)(a-c)}$ + the other two similar terms.

25. $\dfrac{a}{(a-x)(a-b)(a-c)}$ + the other two similar terms.

26. $\dfrac{a^3}{(a-x)(a-b)(b-c)}$ + the other two similar terms.

27. $\dfrac{a^3}{(a-1)(a-b)(a-c)}$ + the other two similar terms.

28. $\dfrac{a^3-a+1}{(a-b)(a-c)(x-a)}$ + the other two similar terms.

29. $\dfrac{(x+1)^3}{(a-x)(a-b)(a-c)}$ + the other two similar terms.

30. $\dfrac{ma^2+na+p}{(a+x)(a-b)(a-c)}$ + the other two similar terms.

162. Miscellaneous Examples.

The methods employed in the following examples will help the student in simplifying many difficult expressions.

1. Simplify $\dfrac{a-b}{(c-a)(x-b)}+\dfrac{b-c}{(x-b)(x-c)}+\dfrac{c-a}{(x-c)(x-a)}$.

Now $\dfrac{a-b}{(x-a)(x-b)}=\dfrac{-(x-a)+(x-b)}{(x-b)(x-a)}=-\dfrac{1}{x-b}+\dfrac{1}{x-a}$.

$\dfrac{b-c}{(x-b)(x-c)}=\dfrac{-(x-b)+(x-c)}{(x-b)(x-c)}=\dfrac{1}{x-c}+\dfrac{1}{x-b}$.

$\dfrac{c-a}{(x-c)(x-a)}=\dfrac{-(x-c)+(x-a)}{(x-c)(x-a)}=-\dfrac{1}{x-a}+\dfrac{1}{x-c}$.

∴ the whole expression

$$=-\frac{1}{x-b}+\frac{1}{x-a}-\frac{1}{x-c}+\frac{1}{x-b}-\frac{1}{x-a}+\frac{1}{x-c}$$
$$=0.$$

Note.—Each fraction is resolved into two partial fractions.

2. Simplify $\dfrac{\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}+3}{\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}}$.

The numerator $=\dfrac{a}{b+c}+1+\dfrac{b}{c+a}+1+\dfrac{c}{a+b}+1$

(Splitting 3 into 3 ones.)

$$=\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}$$
$$=(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$$
$$(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$$

$\therefore$ the expression $= \dfrac{a+b+c}{\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}}$

3. Simplify $\dfrac{a^2+(b-c)^2}{(c+a)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$

The expression

$$=\frac{(a+b-c)(a-b+c)}{(a+c+b)(a+c-b)}+\frac{(b+a-c)(b-a+c)}{(a+b+c)(a+b-c)}+\frac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}$$

$$=\frac{a+b-c}{a+b+c}+\frac{b-a+c}{a+b+c}+\frac{c+a-b}{a+b+c}$$

$$=\frac{a+b-c+b-a+c+c+a-b}{a+b+c}$$

$$=\frac{a+b+c}{a+b+c}$$

$=1.$

4. Simplify :—

$$(a+b)\left(\frac{x^2+b^2-c}{2b}\right)+(b+c)\left(\frac{b^2+c^2-a^2}{2bc}\right)+(c+a)\left(\frac{c^2+a^2-b^2}{2ac}\right)$$

The expression

$$=\left(\frac{1}{2a}+\frac{1}{2b}\right)(a^2+b^2-c^2+\left(\frac{1}{2b}+\frac{1}{2c}\right)+(b^2+c^2-a^2)+\left(\frac{1}{2c}+\frac{1}{2a}\right)(c^2+a^2-b^2)$$

$$=\frac{1}{2a}(a^2+b^2-c^2+c^2+-b^2)+\frac{1}{2b}(a^2+b^2-c^2+b^2+c^2-a^2)+\frac{1}{2c}(b^2+c^2-a^2+c^2+a^2-b^2)$$

$$=\frac{1}{2a}(2a^2)+\frac{1}{2b}(2b^2)+\frac{1}{2c}(2c^2)=a+b+c.$$

5. Simplify $\dfrac{a^2(cy-bz)^3+b^2(az-cx)^2+c^2(bx-ay)}{bcx^2(cy-bz)+acy^2(az-cx)+abz^2(bx+ay)}$

The numerator $=(acy-abz)^3+(abz-bcx)^3+(bcx-acy)^3$

Since $(acy-abz)+(abz-bcx)+(bcx-acy)=0$

$$(acy+abz)^3+(abz-bcx)^3+(bcx-acy)^3$$
$$=3(acy-abz)(abz-bcx)(bcx-acy)$$
$$=3abc(cy-bz)(az-cx)(bx-ay).$$

The denominator

$$= a^3b^2c^2x^2\left(\frac{cy-bz}{a^2bc}\right) + a^2b^2c^2x^2y^2\left(\frac{az-cx}{b^2ac}\right) + a^2b^2c^2z^2\left(\frac{bx-ay}{c^2ab}\right)$$

$$= a^2b^2c^2\left\{\frac{x^2}{a^2}\left(\frac{y}{b}-\frac{z}{c}\right) + \frac{y^2}{b^2}\left(\frac{z}{c}-\frac{x}{a}\right) + \frac{z^2}{c^2}\left(\frac{x}{a}-\frac{y}{b}\right)\right\}$$

$$= a^2b^2c^2\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{y}{b}-\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)$$

$$\left[\because p^2(q-r)+q^2(r+p)+r^2(p-q) = (p-q)(q-r)(p-r)\right] = \frac{(bx-ay)(cy-bz)(cx-az)}{abc}$$

$$\therefore \text{the given expression} = \frac{3abc(cy-bz)(az-cx)(bx-ay)}{(bx-ay)(cy-bz)(cx-az)}$$

$$= -3abc.$$

6. Simplify :—

$$\frac{1}{(a+1)(a-2)} + \frac{1}{(a+2)(a+3)} + \frac{1}{(a+3)(a+4)} + \&c.$$

Taking the first two, we have

$$\frac{1}{(a+1)(a+2)} + \frac{1}{(a+2)(a+3)} = \frac{a+3+a+1}{(a+1)(a+2)(a+3)}$$

$$= \frac{2(a+2)}{(a+1)(a+2)(a+3)}$$

$$= \frac{2}{(a+1)(a+3)}$$

Taking $\frac{2}{(a+1)(a+3)}$ and $\frac{1}{(a+3)(a+4)}$ we have

$$\frac{2}{(a+1)(a+3)} + \frac{1}{(a+3)(a+4)} = \frac{2a+8+a+1}{(a+1)(a+3)(a+4)}$$

$$= \frac{3(a+3)}{(a+1)(a+3)(a+4)}$$

$$= \frac{3}{(a+1)(a+4)}$$

Taking this and the fourth term, *viz.*, $\frac{1}{(a+4)(a+5)}$

we can shew that the sum $= \frac{4}{(a+1)(a+5)}$.

Similarly the sum of five terms $= \frac{5}{(a+1)(a+6)}$.

and the sum of n terms $= \frac{n}{(a+1)(a+n+1)}$.

EXERCISE 75.

Simplify. –

1. $\dfrac{a-b}{(a+2)(b+2)}+\dfrac{b-c}{(b+2)(c+2)}+\dfrac{c-a}{(c+2)(a+2)}$.

2. $\dfrac{a^2-b^2}{(a^2-1)(b^2+1)}+\dfrac{b^2-c^2}{(b^2+1)(c^2+1)}+\dfrac{c^2-a^2}{(c^2+1)(a^2+1)}$.

3. $\dfrac{2x-3}{(x-1)(x-2)}+\dfrac{2x-5}{(x-2)(x-3)}+\dfrac{2x-4}{(x-3)(x-1)}$.

4. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

5. $\dfrac{a-c}{(a-b)(b-c)}-\dfrac{b-a}{(b-c)(c-a)}+\dfrac{a-b}{(c-a)(a-b)}$.

6. $\dfrac{4-a-b}{(2-a)(2-b)}+\dfrac{a+c-4}{(2-a)(2-c)}+\dfrac{c-b}{(2-b)(2-c)}$.

7. $\dfrac{\dfrac{a^2}{x-a}+\dfrac{b^2}{x-b}+\dfrac{c^2}{x-c}+a+b+c}{\dfrac{a}{x-a}+\dfrac{b}{x-b}+\dfrac{c}{x-c}}$.

8. $\dfrac{\dfrac{x+y}{x-y}+\dfrac{y+z}{y-z}+\dfrac{z+x}{z-x}+3}{\dfrac{x}{x-y}+\dfrac{y}{y-z}+\dfrac{z}{z-x}}$.

9. $\dfrac{\dfrac{1}{x-c}+\dfrac{1}{a-x}+\dfrac{1}{x-b}}{\dfrac{c}{x-c}-\dfrac{a}{x-a}+\dfrac{b}{x-b}+1}$.

10. $\dfrac{\dfrac{a+b+c}{a+b+c+d}+\dfrac{a+b}{a+b+c}+\dfrac{a}{a+b}-3}{\dfrac{d}{a+b+c+d}+\dfrac{c}{a+b+c}+\dfrac{b}{a+b}}$.

11. $\dfrac{\dfrac{b}{b-a}+\dfrac{c}{c-b}+\dfrac{d}{d-c}+\dfrac{a}{a-d}-4}{\dfrac{a}{b-a}+\dfrac{b}{c-b}+\dfrac{c}{d-c}+\dfrac{d}{a-d}}$.

12. $\dfrac{\dfrac{a_1}{x+a_1}+\dfrac{a_2}{x+a^2}+\dfrac{a_3}{x+a_3}+\ldots+\dfrac{a_n}{x+a_n}-n}{\dfrac{1}{x+a_1}+\dfrac{1}{x+a^2}+\dfrac{1}{x+a_3}+\ldots+\dfrac{1}{x+a^n}}$.

13. $\dfrac{x^4-(x-1)^2}{(x^2+1)^2-a^2}+\dfrac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\dfrac{x^2(x-1)^2-1}{x^2-(x+1)^2}$.

14. $\dfrac{x^2-(a-b)^2}{(x+b)^2-a^2}+\dfrac{a^2-(x-b)^2}{(x-a)^2-b^2}+\dfrac{b^2-(x-a)^2}{(a+b)^2-x^2}$.

15. $\dfrac{(x+2y)^2-y^2}{(x+y)^2-4y^2}+\dfrac{(x-y)^2-4y^2}{(x-2y)^2-y^2}+\dfrac{(2x+3y)^2-y^2}{(2x+y)^2-8y^2}$

16. $\dfrac{(a+b)}{ab}(a^2+b^2-c^2)+\dfrac{b+c}{bc}(b^2+c^2-a^2)+\dfrac{c+x}{ca}(c^2+a^2-b^2)$

17. $\dfrac{a+b}{2ab}(a^3+b^3-c^3)+\dfrac{b+c}{2bc}(b^3+c^3-a^3)+\dfrac{c+a}{2ca}(c^3+a^3-b^3)$.

18. $\dfrac{a^2+b^2}{a^2b^2}(a^2+b^2-c^2)+\dfrac{b^2+c^2}{b^2c^2}(b^2+c^2-a^2)+\dfrac{c^2+a^2}{c^2a^2}(c^2+a^2-b^2)$.

19. $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{a^3(b-c)+b^3(c-a)+c^3(a-b)}$

20. $\dfrac{a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3}{a^3(b-c)+b^3(c-a)+c^3(a-b)}$

21. $\dfrac{(b+c-2a)^2}{(a+b-2c)(c+a-2b)}+\dfrac{(c+a-2b)^2}{(a+b-2c)(b+c-2a)}+\dfrac{(a+b-2c)^2}{(c+a-2b)(b+c-2a)}$.

22. $\dfrac{a^3\left(\frac{1}{b}-\frac{1}{c}\right)+b^3\left(\frac{1}{c}-\frac{1}{a}\right)+c^3\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}$

23. $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{ab(a-b)+bc(b-c)+ca(c-a)}$

24. $\dfrac{1}{(x+1)(2x+1)}+\dfrac{1}{(2x+1)(3x+1)}+\dfrac{1}{(3x+1)(4x+1)}+\dfrac{1}{(4x+1)(5x+1)}$.

25. $\dfrac{1}{(a-2b)(2a-3b)}+\dfrac{1}{(2a-3b)(3a-4b)}+\dfrac{1}{(3a-4b)(4a-5b)}+\dfrac{1}{(4a-5b)(5a-6b)}$.

What is the sum of n terms of the above?

CHAPTER XIX.

IDENTITIES.

163. When two quantities are equal to each other for all values of the letters involved in them, they form an **identity**; the sign $\equiv$ is placed between them.

For example, $a^2 - b^2 \equiv (a+b)(a-b)$,

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y),$$

and $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$ are **perfect identities.**

But $a^3 + b^3 + c^3 = 3abc$ when $a + b + c = 0$ is a **conditional identity**; for it is true only when the condition $a + b + c = 0$ is given.

To prove that an identity is true,

(*a*) Take the left side expression and show by successive transformations that it is equal to the other; or

(*b*) Take the right side expression and show by successive transformations that it is equal to the other; or

(*c*) Take the difference between the two expressions and show by grouping up terms that it is equal to 0; or

(*d*) Reduce the left side expression to some form and show that the right side expression can also be reduced to the same form.

The following are the *axioms* on which the operations with identities are based.

1. If equals be *added to* or *subtracted from equals* the *sums* or the *remainders* are equal.

Thus if $a = b$, $c = d$, then $a + c = b + d$ and $a - c = b - d$.

2. If equal quantities be *multiplied* or *divided* by the same or equal quantities, the *products* or the *quotients* are equal.

Thus, if $a = b$, $c = d$, then $ac = bd$ and $\frac{a}{c} = \frac{b}{d}$.

3. Any quantity may be transposed from one side of the identity to the other by changing its sign.

Thus if $a + b = c$ then $a = c - b$ for $a + b - b = c - b$ from (1).

6. If two quantities are equal, their roots or their powers are also equal.

Thus, if $a=b$, then $\sqrt{a}=\sqrt{b}$ and $a^2=b^2$.

164. Examples worked out.

Ex. 1. If $a+b+c=0$, shew that—

(i) $(ab+ac+bc)^2=a^2b^2+b^2c^2+c^2a^2$.

(ii) $a^2-bc=b^2-ca=c^2-ab$

$=a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2=-(ab+ac+bc)=\frac{a^2+b^2+c^2}{2}$.

(iii) $a^4+b^4+c^4=2(ab+ac+bc)^2$.

(i) $(ab+ac+bc)^2=a^2b^2+a^2c^2+b^2c^2+2ab(ac+bc)+2ac\times bc$

.....by formula,

$=a^2b^2+a^2c^2+b^2c^2+2abc(a+b+c)$

$=a^2b^2+a^2c^2+b^2c^2+2abc\times 0$

$\equiv a^2b^2+a^2c^2+b^2c^2$.

(ii) Since $a+b+c=0$, $a+b=-c$.

$\therefore (a+b)(a-b)=-c(a-b)$.

$\therefore a^2-b^2=-ac+bc$.

$\therefore a^2-bc=b^2-ac$.

Again, since $a+b+c=0$, $b+c=-a$.

$\therefore (b+c)(b-c)=-a(b-c)$.

$\therefore b^2-c^2=-ab+ac$.

$\therefore b^2-ac=c^2-ab$.

But it has been shewn that $a^2-bc=b^2-ac$.

$\therefore a^2-bc=b^2-ca=c^2-ab$.

$a^2-bc=a\times a-bc$

$=(a\times -(b+c)-bc$

$\because a=-(b+c)$

$=-(ab+ac+bc)$

Similarly $b^2-ac=b\times b-bc$

$=b\times -(a+c)=ac$

$=-(ab+ac+bc)$

Similarly $c^2-ab=-(ab+ac+bc)$

$a^2+ab+b^2=a\times a+ab+b\times b$

$=-a(b+c)+ab-b(a+c)$

$=-ab-ac+ab-ab-bc$

$=-(ab+ac+bc)$

Similarly b^2+bc+c^2, c^2+ca+a^2
may be each shown to be equal to $-(ab+ac+bc)$
or thus $a^2+ab+b^2=(a+b)^2-ab$

But $a+b=-c$

$\therefore\ a^2+ab+b^2=(-c)^2-ab=c^2-ab$

But $c^2-ab=-(ab+ac+bc)$

$\therefore a^2+ab+b^2=b^2+bc+c^2=c^2+ac+a^2$

$=-(ab+ac+bc)$

Again $a+b+c=0$

$\therefore (a+b+c)^2=0$

$\therefore a^2+b^2+c^2=-2(ab+ac+bc)$

$\therefore \dfrac{a^2+b^2+c^2}{2}=-(ab+ac+bc)$

$\therefore a^2-bc=b^2-ac=c^2-ab=a^2+ab+b^2$

$=b^2+bc+c^2=c^2+ac+a^2=-(ab+ac+bc)$

$=\dfrac{a^2+b^2+c^2}{2}$.

(iii) Since $a+b+c=0$,

$(a+b+c)^2 \qquad =0$

$\therefore\ a^2+b^2+c^2 \qquad =-2(ab+ac+bc)$

$\therefore\ (a^2+b^2+c^2)^2 \qquad =4(ab+ac+bc)^2$

$\therefore\ a^4+b^4+c^4+2(a^2b^2+b^2c^2+a^2c^2)$

$=4(ab+ac+bc)^2$

$\therefore\ a^4+b^4+c^4+2(ab+ac+bc)^2 \qquad =4(ab+ac+bc)^2$ ...by (i)

$\therefore\ a^4+b^4+c^4 \qquad =2(ab+ac+bc)^2$.

Ex. 2. If $3s=a+b+c$, shew that

$(s-a)^3+(s-b)^3+(s-c)^3=3(s-a)(s-b)(s-c)$.

Let $s-a=x$, $s-b=y$ and $s-c=z$; then

$x+y+z=3s-a+s-b+s-c=3s-(a+b+c)$

$=a+b+c-(a+b+c)=0$.

If $x+y+z=0$, then $x^3+y^3+z^3=3xyz$.

$\therefore\ (s-a)^3+(s-b)^3+(s-c)^3 \qquad =3(s-a)(s-b)(s-c)$

Ex. 3. Prove that

$s^3-(s-a)^3-(s-b)^3-(s-c)^3=3abc$, if $2s=a+b+c$.

$s=3s-2s=3s-(a+b+c)=s-a+s-b+s-c$.

$\therefore s^3-(s-a)^3-(s-b)^3-(s-c)^3$

$= (s-a+s-b+s-c)^3 - (s-a)^3 - (s-b)^3 - (s-c)^3$
$- 3\{s-a+s-b\}\{s-b+s-c\}\{s-c+s-a\}$.
$[\because (x+y+z)^3 - x^3 - y^3 - z^3 = 3(x+y)(y+z)(z+x)]$
$= 3(2s-a-b)(2s-b-c)(2s-c-a)$
$= 3(a+b+c-a-b)(a+b+c-b-c)(a+b+c-c-a)$
$= 3abc$.

Note. The above identity may also be proved by the method of factorisation (Art. 111).

We leave it as an exercise for the student.

Ex. 4. If $ab+ac+bc=1$, shew that
$(1+a^2)(1+b^2)(1+c^2) = (a+b)^2(b+c)(c+a)^2 = (a+b+c-abc)^2$.

Since $ab+ac+bc=1$.
$\therefore 1+a^2 = a^2+ab+ac+bc = a(a+b)+c(a+b) = (a+b)(a+c)$.
Similarly $1+b^2 = (b+c)(b+a)$;
and $1+c^2 = (c+a)(c+b)$
$\therefore (1+a^2)(1+b^2)(1+c^2)$
$= (a+b)(b+c)(b+c)(b+a)(c+a)(c+b)$
$= (a+b)^2(b+c)^2(c+a)^2$
$= \{(a+b+c)(ab+ac+bc) - abc\}^2$
$= (a+b+c-abc)^2$.

Ex. 5. If $a^3+b^3-c^3 = 3abc$, shew that
(i) $a+b+c=0$ or (ii) $a=b=c$.

Since $a^3+b^3+c^3 = 3abc$ $\therefore a^3+b^3+c^3-3abc = 0$
$\therefore$ $(a+b+c)(a^2+b^2+c^2-ab-ac-bc) = 0$
If $x \times y = 0$, then either $x=0$ or $y=0$
$\therefore$ either $a+b+c = 0$, ...(i)
or $a^2+b^2+c^2-ab-ac-bc = 0$
$\therefore$ $2a^2+2b^2+2c^2-2ab-2ac-2bc = 0$
$\therefore$ $\{(a-b)^2+(b-c)^2+(c-a)^2\} = 0$

Each of $(a-b)^2$, $(b-c)^2$, $(c-a)^2$ is positive.

The sum of 3 positive quantities cannot be equal to zero unless each is equal to zero.

$\therefore$ $(a-b)^2 = 0$ *i.e.* $\therefore$ $a-b=0$ *i.e.* $\therefore$ $a=b$
$(b-c)^2 = 0$ *i.e.* $\therefore$ $b-c=0$ *i.e.* $\therefore$ $b=c$
$(c-a)^2 = 0$ i.e. $\therefore$ $c-a=0$ *i.e.* $\therefore$ $c=a$
$\therefore$ $a=b=c$ .....(ii)

Ex. 6. (*a*) If $x+\frac{1}{x}=1$,

shew that $x^2+\frac{1}{x^2}=x^4+\frac{1}{x^4}=x^8+\frac{1}{x^8}=x^{16}+\frac{1}{x^{16}}$ &c.

$$x^2+\frac{1}{x^2}=x^2+\frac{1}{x^2}+2-2=\left(x+\frac{1}{x}\right)^2-2=1-2=-1.$$

$$x^4+\frac{1}{x^4}=x^4+\frac{1}{x^4}+2-2=\left(x^2+\frac{1}{x^2}\right)^2-2=(-1)^2-2=-1.$$

$$x^8+\frac{1}{x^8}=x^8+\frac{1}{x^8}+2-2=\left(x^4+\frac{1}{x^4}\right)^2-2=(-1)^2-2=-1.$$

Similarly it may be shewn that $x^{16}+\frac{1}{x^{16}}=-1$, &c.

(*b*) If $x+y=a$, and $xy=b$

find the value of x^2+y^2, x^3+y^3, x^4+y^4 and x^5+y^5.

$$x^2+y^2=(x+y)^2-2xy$$
$$=a^2-2b \quad \ldots\ldots(1)$$

$$x^3+y^3=(x+y)^3-3xy(x+y)$$
$$=a^3-3b(a)=a^3-3ab \quad \ldots\ldots(2)$$

$$x^4+y^4=(x^2+y^2)^2-2x^2y^2$$
$$=(a^2-2b)^2-2b^2 \quad \ldots\ldots(3)$$

$$x^5+y^5=(x^3+y^3)(x^2+y^2)-(x^3y^2+x^2y^3)$$
$$=(a^3-3ab)(a^2-2b)-x^2y^2(x+y)$$
$$=(a^3-3ab)(a^2-2b)-b^2a \quad \ldots\ldots(4)$$

Ex. 7. Express $(a^2+b^2)(c^2+d^2)$ as the sum of two perfect squares.

$$(a^2+b^2)(c^2+d^2)$$
$$=a^2c^2+a^2d^2+b^2c^2+b^2d^2$$
$$=a^2c^2+b^2d^2+a^2d^2+b^2c^2+2abcd-2abcd.$$
$$=(a^2c^2+b^2d^2+2abcd)+(a^2d^2+b^2c^2-2abcd)$$
$$=(ac-bd)^2+(ad-bc)^2$$
$$=(ac-bd)^2+(ad+bc)^2$$

Ex. 8. Shew that $(x+1)(x+2)(x+3)(x+4)+1$ is a perfect square.

$$(x+1)(x+2)(x+3)(x+4)+1$$
$$=\{(x+1)(x+4)\}\{(x+2)(x+3)\}+1$$
$$(x^2+5x+4)(x^2+5x+6)+1$$
$$=y(y+2)+1 \quad \ldots\ldots[\text{putting } y \text{ for } x^2+5x+4]$$
$$=y^2+2y+1$$
$$=(y+1)^2$$
$$=(x^2+5x+5)^2.$$

Ex. 9. Express $(2a+3b+4c)(4a+3b+2c)$ as the difference of two perfect squares.

First Method:—

We know that $(x+y)(x-y)=x^2-y^2$.

Let $x+y=2a+3b+4c$; and $x-y=4a+3b+2c$.

Then $x=\frac{1}{2}\{(x+y)+(x-y)\}=3(a+b+c)$

And $y=\frac{1}{2}\{(x+y)-(x-y)\}=c-a$.

$\therefore (2a+3b+4c)(4a+3b+2c)=\{3(a+b+c)\}^2-(c-a)^2$.

Second Method:—

We know that $xy=\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2$.

Let $x=2a+3b+4c$; and $y=4a+3b+2c$.

Then $\frac{x+y}{2}=3(a+b+c)$; and $\frac{x-y}{2}=c-a$.

$\therefore (2a+3b+4c)(4a+3b+2c)=\{3(a+b+c)\}^2-(c-a)^2$.

Ex. 10. If $x=a^2-bc$, $y=b^2-ac$ and $z=c^2-ab$, shew that $(a+b+c)(x+y+z)=ax+by+cz$

$$\begin{aligned}(a+b+c)(x+y+z)&=(a+b+c)(a^2-bc+b^2-ac+c^2-ab)\\&=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)\\&=a^3+b^3+c^3-3abc\\&=(a^3-abc)+(b^3-abc)+(c^3-abc)\\&=a(a^2-bc)+b(b^2-ac)+c(c^2-ab)\\&=ax+by+cz.\end{aligned}$$

Ex. 11. Shew that

$$x^3+y^3+z^3-3xyz=\tfrac{1}{2}(\Sigma x)\Sigma(x-y)^2$$

$$\begin{aligned}x^3+y^3+z^3-3xyz&=(x+y+z)\\&\qquad(x^2+y^2+z^2-xy-x^2+y^2)\\&=\tfrac{1}{2}(x+y+z)\{2x^2+2y^2+2z^2-2xy-2x^2-2y^2\}\\&=\tfrac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}\\&=\tfrac{1}{2}\Sigma x(x-y)^2\end{aligned}$$

Ex. 12. Shew that

$$\begin{aligned}&(x+2y+3z)^3+(y+2z+3x)^3+(z+2x+3y)^3\\&-3(x+2y+3z)(y+2z+3x)(z+2x+3y)\\&=18(x^3+y^3+z^3-3xyz).\end{aligned}$$

Let $x+2y+3z=a$, $y+2z+3x=b$, $z+2x+3y=c$

Then the left hand side expression.

$$=a^3+b^3+c^3-3abc$$
$$=\tfrac{1}{2}\sum a,\ \sum(a-b)^2 \qquad \text{[See Ex. 11}$$
$$=\tfrac{1}{2}(x+2y+3z+y+2z+3x+z+2x+3y)$$
$$\{(x+2y+3z-y-2z-3x)^2+\text{Similar terms.}\}$$
$$=\tfrac{1}{2}\times(6x+6y+6z)\{(y+z-2x)^2+\text{Similar terms.}\}$$
$$=\tfrac{1}{2}(6x+6y^2+6z)\ (6y+6z^2+6x^2-6xy-6xz-6yz)$$
$$=\tfrac{1}{2}\ 6\ (x+y+z)\ 6\ (x^2+y^2+z^2-xy-xz-yz)$$
$$=18\ (x^3+y^3+z^3-3xyz).$$

EXERCISE 76.

Show that—

1. $(a-b)^2-(b-c)(c-a)=(b-c^2)-(c-a)(a-b)$
$$=(c-a)^2-(a-b)(b-c)$$

(a) $(a-b)(a+b-c)+(b-c)(b+c-a)(c-a)(c+a-b)=0$

(b) $\sum x(y-z)(zx+xy-zy)=0$

(c) $(a+b+c)^2+\sum a(b+c-a)=4\sum ab.$

(d) $(a+b+c)^2+\sum(a+b-c)(b+c-a)$
$$=4\sum ab.$$

(e) $(a+b+c)x+y+z)+(a+b-c)(x+y-z)$
$$+(b+c-a)(y+z-x)+(c+a-b)(z+x-y).$$
$$=4(ax+by+cz).$$

(f) $(2x+y)(2y+z)(2z+x)+$
$$(y+2z)(z+2x)(x+2y)=6\ (x+y+z)$$
$$(xy+xz+yz).$$

2. $(a-b)^3+(b-c)^3+(c-a)^3.$
$$=2\{(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)\}^3.$$

(a) $\sum(a-b)(b+c)(c+a)=-(a-b)(b-c)(c-a).$

(b) $\sum(x-y)(z^2+xy)=-2(x-y)(y-z)(z-x).$

(c) $\sum x(y-z)(2x+y+z)=-(x-y)(x-z)(z-x).$

(d) $\sum(a-b)(x-a)(x-b)=-(a-b)(b-c)(c-a).$

(e) $\sum_{pqr}(p^3-p+1)(q^2-r^2)=-(p-q)(q-r)(r-p).$

3. $\dfrac{(a^3-b^3)^3+(b^3-c^3)^3+(c^3-a^3)^3}{(a-b)^3+(b-c)^3+(c-a)^3}=(a+b)\,(b+c)\,(c+a).$

(a) $\sum_{abc}(a+b)\,(a-b)^3=2(a-b)\,(b-c)\,(c-a)\sum a.$

(b) $\sum_{abc}(a-b)\,(p-c)^2=(a-b)\,(b-c)\,(c-a)\,(a+b+c-3p)$

(c) $(\gamma-2p-q)\,(\gamma-2q-r)\,(\gamma-2r-p)-(\gamma-p-2q)\,(\gamma-q-2r)$
$(\gamma-r-2p)=(p-q)\,(q-r)\,(r-p).$

4. $(3a-2b-c)^3+(3b-2c-a)^3+(3c-2a-b)^3=3(3a-2b-c)$
$(3b-2c-a)\,(3c-2a-b).$

5. $\dfrac{(a^3-b^3)^3+(b^3-c^3)^3+(c^3-a^3)^3}{(a^2+ab+b^2)(b^2+bc+c^2)(a^2+ca+c^2)}$
$=3(a-b)\,(b-c)\,(c-a).$

If $a+b+c=0$, shew that—

6. $a^3+ab+b^3=b^2+bc+c^2=c^2+ca+a^2.$

7. $(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)\,(2b-c)\,(2c-a).$

8. $a(b+c)^2+b(c+a)^2+c(a+b)^2=3abc.$

9. $\dfrac{a^3+b^3+c^3}{a^5+b^5+c^5}=-\dfrac{2}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

10. $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^2=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}.$

11. $\dfrac{a^5+b^5+c^5}{5}=\dfrac{a^3+b^3+c^3}{3}\times\dfrac{a^2+b^2+c^2}{2}.$

12. $\dfrac{a^2+b^2+c^2}{2}\times\dfrac{a^5+b^5+c^5}{5}=2\times\dfrac{a^3+b^3+c^3}{3}$
$\times\dfrac{a^4+b^4+c^4}{4}.$

If $2s=a+b+c$, shew that—

13. $s^2+(s-a)^2+(s+b)^2+(s-c)^2=(a^2+b^2+c^2).$

14. $4a^2b^2-(a^2+b^2-c^2)^2=16s(s-a)\,(s-b)\,(s-c).$

15. $a(s-b)(s-c)+b(s-c)\,(s-a)+c(s-a)\,(s-b)+2(s-a)\,(s-b)$
$(s-c)=abc$

16. $(s-a)^2+(s-b)^2+(s-c)^2+2(ab+ac+bc)=3s^2.$

17. If $3s=2(a+b+c)$, then

(i) $(2a-s)^3+(2b-s)^3+(2c-s)^3=3(2a-s)\,(2b-s)\,(2c-s)$;

(ii) $(s-a-b)^3+(s-b-c)^3+(s-c-a)^3=3(s-a-b)\,(s-b-c)$
$(s-c-a).$

18. Prove that $(s-a)^3+(s-b)^3+3(s-a)\,(s-b)c=c^3$, if $2s=a+b+c$

19. Show that
$(x-a)(x-b)(a-b)+(x-b)(x-c)(b-c)+(x-c)(x-a)(c-a)=(a-b)(b-c)(a-c)$

20. If $x^2-yz=a^2$, $y^2-xz=b^2$, $z^2-xy=c^2$,
prove that $(a^2x+b^2y+c^2z)=(x+y+z)(a^2+b^2+c^2)$.

21. If $x+y=a$, and $xy=b^2$, shew that
$x^3+y^3=a^3-3b^2a$ and $x^4+y^4=a^4-4a^2b^2+2b^4$.

22. Shew that.
$a^4+b^4+c^4-2a^2b^2-2b^2c^2-2a^2c^2=0$, if $a+b=c$.

23. If $x+\frac{1}{x}=a$,
shew that $x^3+\frac{1}{x^3}+x^2+\frac{1}{x^2}+x+\frac{1}{x}=(a+1)(a^2-2)$

24. If $ab=xy$ and $ma+\frac{b}{m}=x+y$,
shew that $x^3+y^3=m^3a^3+\frac{b^3}{m^3}$.

25. If $a+b+c=ab+ac+bc$,
prove that $(a+b)(b+c)(c+a)+abc$ is a perfect square

26 Shew that (i) $(x+2)(x+4)(x+6)(x+8)+16$ and
(ii) $(x+a)(x+2a)(x+3a)(x+4a)+a^4$ are
perfect squares.

27. Express $(a+b+c-d)(a+b-c+d)$ as the difference of two squares.

28. Shew that $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2+(ay-bx)^2$
$+(cx-az)^2+(cy-bz)^2$.

29. Shew that
$a^5+b^5+c^5+5abc(ab+ac+bc)=0$, if $a+b+c=0$.

30. Shew that $4^{2n-1}+4^{n+1}-4^n+9$ is a perfect square.

31. Shew that
(i) $3(2a-b-c)(c-a)$ the difference of two squares
(ii) $2(x^2+y^2+z^2-xy-yz-zx)$=the sum of 3 squares.

Prove that—

32. $(a+b+c+d)^4+(a-b-c+d)^4+(a-b+c-d)^4+(a+b-c-d)^4$
$=4(a^2+b^2+c^2+d^2)$.

33. $a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3=a^2(b^3-c^3)+b^2(c^3-a^3)$
$+c^2(a^3-b^3)$.

34. $(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)\ (b+c)\ (c+a)$
$=2(a^3+b^3+c^3-3abc).$

35. [illegible]

36 $(a+b)^5-a^5-b^5=5ab(a+b)\ (a^2+ab+b^2).$

37. If $x+y=a$, $xy=b$,
shew that $(1+x^2)\ (1+y^2)=a^2+(1-b)^2.$

38. $a^2(b+c)+b^2(c+a)+c^2(a+b)-a^3-b^3-c^3-2abc.$
$=(b+c-a)\ (c+a-b)\ (a+b-c).$

39. $(a^2-bc)\ (b-c)+(b^2-ca)\ (c-a)+(c^2-ab)\ (a-b)=0$

40. [illegible]
$=(x^3+y^3+z^3-3xyz)^2$
$=(x^2+2yz)^3+(y^2+2xz)^3+(z^2+2xy)^3$
$-3(x^2+2yz)\ (y^2+2xz)\ z^2+2xy).$

41. If $a=y+z-x$, $b=z+x-y$, $c=x+y-z$,
then $a^3+b^3+c^3-3abc=4(x^3+y^3+z^3-3xyz).$

42. Shew that $a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2$
$+(b+c-a)\ (c+a-b)\ (a+b-c)=4abc.$

43. Prove that $2\{(a-b)^4+(b-c)^4+(c-a)^4\}$
$=\{(a-b)^2+(b-c)^2+(c-a)^2\}^2.$

44. [illegible]
$-4+\left(\frac{b}{c}+\frac{c}{b}\right)\ \left(\frac{c}{a}+\frac{a}{c}\right)\ \left(\frac{a}{b}+\frac{b}{a}\right)$

45. Shew that
$\frac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}=\frac{3}{2}(a+b+c+d).$

46. Shew that $(a^2-a+1)\ (b-c)$
$+(b^2-b+1)\ (c-a)+c^2-c+1)\ (a-b.)$
$=(a^2-a+1)\ (b^2-c^2)+(b^2-b+1)\ (c^2-a^2)$
$+(c^2-c+1)\ (a^2-b^2)$

47. Prove that
$(ma-nb)^3+(mb-nc)^3+(mc-na)^3$
$-3(ma-nb)\ (mb-nc)\ mc-na)$
$=(m^3-n^3)\ (a^3+b^3+c^2-3abc).$

48. If $2s=a+b+c$, shew that
$s(s-a)(s-b)+s(s-b)(s-c)+s(s-c)(s-a)$
$=(s-a)(s-b)(s-c)+abc.$

49. If $s=a+b+c$, prove that
$(s-3a)^2+(s-3b)^2+(s-3c)^2$
$=3\{(a-b)^2+(b-c)^2+(c-a)^2\}.$

50. If $2s=a+b+c$
$a^2(s-a)^2+b^2(s-b)^2+c^2(s-c)^2+2s(s-a)(s-b)(s-c)$
$=2\{(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\}^2.$

51. If $2s=a+b+c$ prove that
$a^2(s-a)+b^2(s-b)+c^2(s-c)=abc+4(s-a)(s-b)(s-c)$

52. If $2s=a+b+c$ prove that
$a(s-a)^2+b(s-b)^2+c(s-c)^2=abc+2(s-a)(s-b)(s-c)$

53. If $x+y=a$, $xy=b$
find the value of x^4+y^4, x^6+y^6,

54. If $x+y+z=a$, $xy+xz+yz=b$, and $xyz=c$
find the value of (1) $x^2+y^2+z^2$.
(2) $x^3+y^3+z^3$
(3) $x^4+y^4+z^4$
(4) $x^5+y^5+z^5$

CHAPTER XX.

IDENTITIES AND FRACTIONS.

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Ex. 1. To resolve $\dfrac{1}{(x-1)(x-3)}$ into two partial fractions.

Let $\dfrac{2}{(x-1)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-3}$, where A and B do not contain x.

Then $\dfrac{2}{(x-1)(x-3)} = \dfrac{A(x-3)+B(x-1)}{(x-1)(x-3)}$.

$\therefore 2 = A(x-3)+B(x-1)$.

Since this is true for all values of x, it must be true when $x=1$

$\therefore 2 = A(1-3)$, $\therefore 2 = -2A$, $\therefore A = -1$.

Similarly when $x=3$, $B=1$.

$\therefore \dfrac{2}{(x-1)(x-3)} = -\dfrac{1}{x-1} + \dfrac{1}{x-3}$.

Ex. 2. To resolve $\dfrac{x^2}{(x-1)(x-2)(x-3)}$ into three partial fractions.

Let $\dfrac{x^2}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$,

where A, B and C do not contain x.

Then $x^2 = A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$.

Since this is true for all values of x,

it must be true when $x=1$,

$\therefore 1 = A(1-2)(1-3)$, $\therefore 1 = 2A$, $\therefore A = \frac{1}{2}$.

Similarly, when $x=2$, $4 = B(2-1)(2-3)$, $\therefore B = -4$,

And when $x=3$, $9 = C(3-1)(3-2)$, $\therefore C = \frac{9}{2}$.

$\therefore \dfrac{x^2}{(x-1)(x-2)(x-3)} = \dfrac{1}{2(x-1)} - \dfrac{4}{x-2} + \dfrac{9}{2(x-3)}$.

EXERCISE 77.

Resolve each of the following into two partial fractions :—

1. $\dfrac{a-b}{(x-a)(x-b)}$.

2. $\dfrac{3x}{(x-4)(x+2)}$.

3. $\dfrac{12}{(x+6)(x+2)}$.

4. $\dfrac{x^2-4x}{x^2+8}$.

5. $\dfrac{x}{(2x-1)(3x-1)}$ 6. $\dfrac{2x}{x^2-(a+b)}$ 7. $\dfrac{fx+q}{(x+a)(x+b)}$

Resolve each of the following into three partial fractions.

8. $\dfrac{2x+5}{2(2x+1)(x+2)}$. 9. $\dfrac{3}{(a+1)(a+2)(a+3)}$.

10. $\dfrac{x^2}{(x+1)(x+2)(x+3)}$. 11. $\dfrac{3x-5}{(x-2)(x-3)(x-4)}$.

12. $\dfrac{ax^2+bx+c}{(x-p)(x-q)(x-r)}$. 13. $\dfrac{10y^2-41y+37}{(y-1)(y-2)(y-3)}$

14. $\dfrac{2x^2-10x+14}{(x-2)(x-3)(x-4)}$. 15. $\dfrac{12x^2+73x+106}{(x+2)(x+4)(x+3)}$

166. Examples worked out.

Ex 1 Find the value of $\dfrac{(x-a)(x-b)}{(x-a-a)^2}$ when $x=\dfrac{a^2+ab+b^2}{a+b}$

$$x=\frac{a^2+ab+b^2}{a+b}=\frac{a(a+b)+b^2}{a+b}=a+\frac{b^2}{a+b}.$$

$$\therefore x-a=\frac{b^2}{a+b} \quad \ldots\ldots\ldots\ldots \quad (i)$$

Again $x=\dfrac{b(a+b)+a}{a+b}=b+\dfrac{a^2}{a+b}$ $\therefore x-b=\dfrac{a^2}{a+b}$ (ii)

Once again $x=\dfrac{a^2+2ab+b^2-ab}{a+b}=a+b-\dfrac{ab}{a+b}$

$$\therefore x-a-b=-\frac{ab}{a+b} \quad \therefore (x-a-b)^2=\frac{a^2b^2}{(a+b)^2} \quad \ldots\ldots \quad (iii)$$

From (i), (ii) and (iii) $\dfrac{(x-a)(x-b)}{(x-a-b)^2}=\dfrac{a^2}{a+b}\times\dfrac{b^2}{a+b}\div\dfrac{a^2b^2}{(a+b)^2}$

$$=\frac{a^2}{(a+b)}\times\frac{b^2}{(a+b)}\times\frac{(a+b)^2}{a^2b^2}$$

$$=1$$

Ex. 2. If $b=\sqrt{ac}$, shew that $\dfrac{b+3a}{c+3b}=\dfrac{a+3b}{b+3c}$.

Now $\dfrac{b+3a}{c+3b}=\dfrac{\sqrt{ac}+3a}{c+3\sqrt{ac}}=\dfrac{\sqrt{a}\sqrt{c}+3\sqrt{a}}{\sqrt{c}(\sqrt{c}+3\sqrt{a})}=\dfrac{\sqrt{a}}{\sqrt{c}}$ (i)

Again $\dfrac{a+3b}{b+3c}=\dfrac{a+3\sqrt{ac}}{\sqrt{ac}+3c}=\dfrac{\sqrt{a}(\sqrt{a}+3\sqrt{c})}{\sqrt{c}(\sqrt{a}+3\sqrt{c})}=\dfrac{\sqrt{a}}{\sqrt{c}}$ (ii)

Since $\dfrac{b+3a}{c+3b}$ and $\dfrac{a+3b}{b+3c}$ are each equal to $\dfrac{\sqrt{a}}{\sqrt{c}}$,

$$\frac{b+3a}{c+3b}=\frac{a+3b}{b+3c}.$$

Ex 3. If $\frac{1}{c}=\frac{a+b}{b-ab}$, shew that $\frac{1}{b}=\frac{a+c}{b-ac}$ and $\frac{1}{a}=\frac{b+c}{b-bc}$.

Now $\frac{1}{c}=\frac{a+b}{b-ab}$ $\therefore\ b-ab=c(a+b).\quad =ac+bc.$

$\therefore\ b\quad =ab+ac+bc$

$b-ac=ab\times bc=b(a+c)$

$b\quad =\frac{b-ac}{a+c}$

$\therefore\ \frac{1}{b}=\frac{a+c}{b-ac}$.

Again since $b=ab+ac+bc$, $\therefore\ b-bc=ab+ac=a(b+c)$

$a=\frac{b-bc}{b+c}$

$\therefore\ \frac{1}{a}=\frac{b+c}{b-bc}$

Ex. 4. If $x=\frac{a-b}{1+ab}$, $y=\frac{b-c}{1+bc}$ and $z=\frac{c-a}{1+ca}$,

shew that $x+y+z=xyz$.

Now $x+y+z$

$=\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}$

$=\frac{a-b}{1+ab}+\frac{b-a+a-c}{1+bc}+\frac{c-a}{1+ca}$

$=(a-b)\left\{\frac{1}{1+ab}-\frac{1}{1+bc}\right\}+(c-a)\left\{\frac{1}{1+ca}-\frac{1}{1+bc}\right\}$

$=(a-b)\frac{b(c-a)}{(1+ab)(1+bc)}+(c-a)\frac{c(b-a)}{(1+ca)(1+bc)}$

$=\frac{(a-b)(c-a)}{(1+bc)}+\left\{\frac{b}{1+ab}-\frac{c}{1+ca}\right\}$

$=\frac{(a-b)(c-a)}{1+bc}\left\{\frac{b-c}{(1+ab)(1+ca)}\right\}$

$=\frac{(a-b)(b-c)(c-a)}{(1+ab)(1+bc)(1+ca)}$

$=xyz$.

EXERCISE 78.

Find the values of :—

1. $\frac{x(x-a)+x(x-b)}{(x-a)(x-b)}$ when $x=\frac{a+b}{2ab}$

2. $\dfrac{x-a}{x-b}-\left(\dfrac{x-2a}{x-2b}\right)^2$ when $x=\dfrac{ab}{a+b}$.

3. $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$ when $x=\dfrac{a+b}{2}$.

4. $\dfrac{a(a-c)}{(a-x)(a+c)}$ when $x=\dfrac{2ac}{a+c}$.

5. $\left(\dfrac{x-2a}{x-2b}\right)^3-\dfrac{x-4a+2b}{x+2a-4b}$ when $x=a+b$.

6. $\dfrac{(a-x)(b-x)}{(c+x)(d+x)}$ when $x=\dfrac{ab-cd}{a+b+c+d}$.

7. $\dfrac{1}{x-a}+\dfrac{1}{x-b}-\dfrac{1}{b}$ when $x=\dfrac{2ab}{a+b}$.

8. $\dfrac{(x-a)(x-b)(x+2a+2b)}{(x+2a)(x+2b)(x-a-b)}$ when $x=2(a+b)$

9. $\dfrac{(x-a)(x-b)(a-b)}{(x-a-b)(b-a)}$ when $x=\dfrac{a^2+ab+b^2}{a+b}$.

10. $\dfrac{x+a}{x+b}\left\{\dfrac{2x+b+c}{2x+a+c}\right\}^2$ when $x=\dfrac{c^2-ab}{a+b-2c}$.

11. $\dfrac{bx-a}{ax-b}$ when $x=\dfrac{a^2+b^2}{2ab}$.

12. $\left(\dfrac{x+2c}{x+2d}\right)\left(\dfrac{x-c}{x-d}\right)^2$ when $x=\dfrac{2(c^2+cd+d^2)}{3(c+d)}$.

13. $\dfrac{x^2-y^2+x}{y^2-x^2+y}$ when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$.

14. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}+\dfrac{4ab}{x^2-4b^2}$ when $x=\dfrac{ab}{a+b}$.

15. $\dfrac{1}{x-a}+\dfrac{1}{x-b}-\dfrac{a-b}{x^2-ab}$ when $x=\dfrac{1}{\dfrac{1}{2a}+\dfrac{1}{2b}}$.

16. $\dfrac{(a+b)(x-a)(b-x)}{ab(2x-a-b)}$ when $x=\dfrac{ab}{a+b}$.

17. $\dfrac{x^2+a^2-b^2}{2ax}+\dfrac{a^2+b^2-x^2}{2ab}+\dfrac{b^2+x^2-a^2}{2bx}$

when $x=a+b$.

18. $\dfrac{x+y-1}{x-y+1}$ when $x=\dfrac{a+1}{ab+1}$ and $y=\dfrac{ab+a}{ab+1}$.

19. $\dfrac{x^2-3abx-2b^2}{x^2-ab}+\dfrac{x^2-4ab}{x-2a}$ when $x=a+b$.

20. $\left(\frac{x}{x-1}\right)^2\left(\frac{x}{x+1}\right)^2$ when $x=\sqrt{\frac{n-1}{n+1}}$

Shew that—

21. $\frac{2bc+(b^2+c^2-a^2)}{2bc-(b^2+c^2-a^2)}=\frac{s(s-a)}{(s-b)(s-c)}$ if $2s=a+b+c$

22. $\frac{ab}{(x-a)(x-b)}+\frac{bc}{(x-b)(x-c)}+\frac{ca}{(x-c)(x-a)}=0$ if $\frac{1}{x}=\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

23. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\frac{x}{a}=\frac{p^2-\frac{1}{p^2}}{p^2+\frac{1}{p^2}}$ and $\frac{b}{y}=\frac{1}{2}\left(p^2+\frac{1}{p^2}\right)$.

24. $\frac{1}{m(n-x)}+\frac{1}{n(a-x)}+\frac{1}{m(x-a)}=0$ if $x=\frac{n}{m}(m-n+a)$.

25. $\frac{(x-a)(x-b)}{(y-a)(y-b)}=\frac{x}{y}$ if $x=\frac{a+b}{2}$ and $y=\frac{2ab}{a+b}$.

26. If $x=\frac{a-b}{m-c}, y=\frac{b-c}{m-a}$ and $z=\frac{c-a}{m-b}$

find the value of $x+y+z+xyz$.

27. If $x=\frac{(2c-b)^2-ab}{a+3b-4c}$, shew that $\sqrt{\frac{x+a}{x+b}}=\frac{a+b-2c}{2(b-c)}$.

167. Examples worked out

Ex. 1 If $a+b+c=0$, shew that—

(i) $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}=0$;

(ii) $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$;

(iii) $\frac{1}{-a^2+b^2-c^2}+\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}=0$

(iv) $\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab}=0$.

(i) $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{c+a+b}{abc}=\frac{0}{abc}=$. $\because a+b+c=0$.

(ii) $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$

$\because \frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}=0$. [by (i)]

From this we may deduce

$$\left(\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}\right)^2=\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}.$$

(iii) $a+b+c=0 \quad \therefore a+b=-c$

$\therefore (a+b)^2=c^2.$

$\therefore a^2+b^2+2ab=c^2$

$\therefore a^2+b^2-c^2=-2ab.$

Similarly $b^2+c^2-a^2=-2bc$

and $c^2+a^2-b^2=-2ca$

Hence $\dfrac{1}{a^2+b^2-c^2}+\dfrac{1}{b^2+c^2-a^2}+\dfrac{1}{c^2+a^2-b^2}$

$$=-\frac{1}{2ab}-\frac{1}{2bc}-\frac{1}{2ca}$$

$$=-\frac{1}{2}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$=0 \quad \because \dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}=0$ . . . . . [by (i)]

(iv) $a+b+c=0, \quad \therefore a = -\times(b+c).$

$\therefore a^2 = -a(b+c)$

$\therefore a^2+a^2+bc=a^2+bc-a(b+c).$

$\therefore 2a^2+bc=a^2+bc-ab-ac=a(a-b)-c(a-b)=(a-b)(a-c).$

Similarly $2b^2+ca=(b-a)(b-c)$ and $2c^2+ab=(c-a)(c-b).$

Hence $\dfrac{a}{2a^2+bc}+\dfrac{b}{2b^2+ca}+\dfrac{c}{2c^2+ab}$

$$=\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$$

= . . . , ... [by (2) of Art. 161].

Ex. 2. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, then show that

(i) $a=-b$, or $b=-c$, or $c=-a$,

(ii) $\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}=\dfrac{1}{a^3+b^3+c^3}$

(iii) $\dfrac{1}{a^5}+\dfrac{1}{b^5}+\dfrac{1}{c^5}=\dfrac{1}{a^5+b^5+c^5}$,

and generally

(iv) $\dfrac{1}{a^{2n+1}}+\dfrac{1}{b^{2n+1}}+\dfrac{1}{c^{2n+1}}=\dfrac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{1}{a+b+c}$

$\therefore \frac{ab+bc+ca}{abc} = \frac{1}{a+b+c}$

$\therefore (a+b+c)(ab+ac+bc) = abc$

$\therefore (a+b+c)(ab+ac+bc)-abc = 0$

$\therefore (a+b)(b+c)(c+a) = 0$

$\therefore a+b=0.$ *i.e.*, $a=-b$;

or $b+c=0.$ *i. e.*, $b=-c$;

or $c+a=0,$ *i. e.*, $c=-a$

(ii) We have shewn that $a=-b \quad \therefore a^3=-b^3$

$\therefore \frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} = -\frac{1}{b^3}+\frac{1}{b^3}+\frac{1}{c^3} = \frac{1}{c^3} = \frac{1}{a^3+b^3+c^3} \because a^3+b^3=0.$

(iii) $a=-b \quad \therefore \quad a^5=-b^5.$

$\therefore \frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5} = -\frac{1}{b^5}+\frac{1}{b^5}+\frac{1}{c^5} = \frac{1}{a^5+b^5+c^5} \because a^5+b^5=0$

(iv) $a=-b. \quad \therefore a^{2n+1}=-b^{2n+1} \quad \because 2n+1$ is odd.

$\therefore \frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}} = -\frac{1}{b^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}$

$= \frac{1}{c^{2n+1}}$

$= \frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}$

$\because a^{2n+1}+b^{2n+1}=0.$

Ex. 3. If $ab+ac+bc=1$, shew that

(i) $\frac{a^2+1}{(a+b)(a+c)}+\frac{b^2+1}{(b+c)(b+a)}+\frac{c^2+1}{(c+a)(c+b)} = 3(ab+ac+bc).$

(ii) $\frac{a+b}{1-ab}+\frac{b+c}{1-bc}+\frac{c-a}{1+ca} = \frac{(a+b)(b+c)(c+a)}{(1-ab)(1-bc)(1-ca)}.$

(i) $1=ab+ac+bc,$

$1+a^2=a^2+ab+ac+bc=a(a+b)+c(a+b)=(a+b)(a+c)$

Similarly $1+b^2=(b+c)(b+a)$; and $1+c^2=(c+a)(c+b).$

Hence the given expression

$= \frac{(a+b)(a+c)}{(a+b)(a+c)}+\frac{(b+c)(b+a)}{(b+c)(b+a)}+\frac{(c+a)(c+b)}{(c+a)(c+b)}$

$=1+1+1$

$=3$

$=3(ab+bc+ca).$

(ii) $ab+ac+bc=1$, $\therefore 1-ab=ac+bc=c(a+b)$.

$$\therefore \frac{a+b}{1-ab} = \frac{1}{c}.$$

Similarly $\frac{b+c}{1-bc}=\frac{1}{a}$ and $\frac{c+a}{1-ca}=\frac{1}{b}$

$$\therefore \frac{a+b}{1-ab}+\frac{b+c}{1-ac}+\frac{c+a}{1-ca}=\frac{1}{c}+\frac{1}{a}+\frac{1}{b}$$

$$=\frac{ab+ac+bc}{abc}$$

$$=\frac{1}{abc}$$

$$=\frac{1}{c}\cdot\frac{1}{a}\cdot\frac{1}{b}.$$

Ex. 4. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, then shew that

(i) $\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)}=3$;

(ii) $\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}+a+b+c=0$.

(i) Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, $\frac{ab+ac+bc}{abc}=0$.

$\therefore ab+ac+bc=0$

$\therefore a^2+ab+bc+ca=a^2$

$\therefore a(a+b)+c(a+b)=a^2$

$\therefore (a+b)(a+c)=a^2$.

Similarly, by adding b^2 and c^2 to both sides of $ab+ac+bc=0$, we can get $(b+c)(b+a)=b^2$ and $(c+a)(c+b)=c^2$

$$\therefore \frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)}$$

$$=\frac{a^2}{a^2}+\frac{b^2}{b^2}+\frac{c^2}{c^2}$$

$$=1+1+1$$

$$=3$$

(ii) $\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}+a+b+c$

$$=\frac{ab}{a+b}+c+\frac{bc}{b+c}+a+\frac{ca}{c+a}+b$$

$$= \frac{ab+ac+bc}{a+b} + \frac{bc+ab+ac}{b+c} + \frac{cbc+ab+a}{c+a}$$
$$= \frac{0}{a+b} + \frac{0}{b+c} + \frac{0}{c+a}$$
$$= 0.$$

Ex. 5 If $a+b+c+d=0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0$,

prove that $a^3+b^3+c^3+d^3=0$.

Since $a+b+c+d=0$, $a+b=-(c+d)$.

Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0$, $\frac{1}{a}+\frac{1}{b}+=-\left(\frac{1}{c}+\frac{1}{d}\right)$

i.e., $\frac{a+b}{ab}=-\frac{c+d}{cd}$

$\therefore \frac{1}{ab}=\frac{1}{cd}$ $\because$ $a+b=-(c+d)$

$\therefore ab=cd.$

Again $(a+b)^3=-(c+d)^3$

$\therefore a^3+b^3+3ab(a+b)=-c^3-d^3-3cd(c+d)$

$\therefore a^3+b^3+c^3+d^3) = -3ab(a+b)-3cd(c+d)$

$= -3cd(a+b)+3cd(a+b)$

$=0.$

Ex 6. If $a+b+c=abc$,

shew that $\frac{a+b}{ab-1}+\frac{b+c}{bc-1}+\frac{c+a}{ca-1}=abc.$

Since $a+b+c=abc$, $a+b=abc-c=c(ab-1)$

Similarly, $b+c=a(bc-1)$

and $c+a=b(ac-1)$.

Hence $\frac{a+b}{ab-1}+\frac{b+c}{bc-1}+\frac{b+a}{ca-1}=\frac{c(ab-1)}{ab-1}+\frac{a(bc-1)}{bc-1}+\frac{b(ca-1}{ca-1}$

$=c+a+b$

$=abc.$

Ex. 7. If $x+y=az$, $y+z=bx$ and $z+x=cy$,

find the value of $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$

Since $x+y=az$, $a = \frac{x+y}{z}$.

$\therefore a+1 = \frac{x+y}{z}+1 = \frac{x+y+z}{z}$

$\therefore \frac{1}{a+1}=\frac{z}{x+y+z}$

Similarly $\frac{1}{b+1}=\frac{x}{x+y+z}$

and $\frac{1}{c+1}=\frac{y}{x+y+z}$

$$\therefore \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=\frac{z}{x+y+z}+\frac{x}{x+y+z}+\frac{y}{x+y+z}$$
$$=\frac{x+y+z}{x+y+z}$$
$$=1.$$

Ex. 8. If $a+b+c=1$, $ab+ac+bc=\frac{1}{3}$ and $abc=\frac{1}{27}$.

then $\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}=\frac{27}{4}$.

Since $a+b+c=1$,

$a=1-(b+c)$. $\therefore a+bc=1-(b+c)+bc=(1-b)(1-c)$.

Similarly $b+ca=(1-a)(1-c)$ and $c+ab=(1-a)(1-b)$.

$$\therefore \frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}$$
$$=\frac{1}{(1-b)(1-c)}+\frac{1}{(1-c)(1-a)}+\frac{1}{(1-a)(1-b)}$$
$$=\frac{1-a+1-b+1-c}{(1-a)(1-b)(1-c)}$$
$$=\frac{3-(a+b+c)}{1-(a+b+c)+(ab+ac+bc)-abc}$$
$$=\frac{3-1}{1-1+\frac{1}{3}-\frac{1}{27}}$$
$$=\frac{2}{\frac{8}{27}}$$
$$=\frac{27}{4}.$$

Ex. 9 If $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$, then

(i) $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a-b}=0$;

(ii) $\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}=-(a^2+b^2+c^2)$.

(i) Now $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1.$

$\therefore \frac{a(a+b+c)}{b+c}+\frac{b(a+b+c)}{c+a}+\frac{c(a+b+c)}{a+b}=a+b+c.$

$\therefore a+\frac{a^2}{b+c}+b+\frac{b^2}{c+a}+c+\frac{c^2}{a+b}=a+b+c.$

$\therefore \frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=(a+b+c)-(a+b+c)=0.$

(ii) $\therefore \frac{a^2(a+b+c)}{b+c}+\frac{b^2(a+b+c)}{c+a}+\frac{c^2(a+b+c)}{a+b}=0.$

$\therefore a^2+\frac{a^3}{b+c}+b^2+\frac{b^3}{c+a}+c^2+\frac{c^3}{a+b}=0.$

$\therefore \frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}=-(a^2+b^2+c^2).$

Ex. 10 If $abc=1$, prove that $\frac{1}{1+a+\frac{1}{b}}+\frac{1}{1+b+\frac{1}{c}}+\frac{1}{1+c+\frac{1}{a}}=1.$

Now $1+a+\frac{1}{b}=abc+a+ac=a(bc+1+c).$

$1+b+\frac{1}{c}=\frac{cb+c+1}{c}$

and $1+c+\frac{1}{a}=1+c+bc,$

$\therefore \frac{1}{1+a+\frac{1}{b}}+\frac{1}{1+b+\frac{1}{c}}+\frac{1}{1+c+\frac{1}{a}}$

$=\frac{1}{a(bc+1+c)}+\frac{1}{\frac{c+bc+1}{c}}+\frac{1}{1+c+bc}$

$=\frac{1}{bc+1+c}\left(\frac{1}{a}+c+1\right)$

$=\frac{1}{bc+1+c}(bc+c+1)$

$=1.$

EXERCISE 79.

If $a+b+c=0$, prove the following identities :—

1. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+0$;

$$\frac{a^2+b^2+c^2}{\sqrt{(a^2b^2+b^2c^2+c^2a^2)}}=2.$$

2. $\frac{a+b}{(a+c)(b+c)}+\frac{b+c}{(a+b)(a+c)}+\frac{c+a}{(a+b)(b+c)}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

3. $\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2)+\frac{a+b}{ab}(a^2+b^2-c^2)=0.$

4. $a^3(b-c)+b^3(c-a)+c^3(a-b)=0.$

5. $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc=0.$

6. $\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=9.$

7. $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}=1.$

8. $\frac{a^3}{2a^2+bc}+\frac{b^3}{2b^2+ca}+\frac{c^3}{2c^2+ab}=3$;

and $\frac{a(b^3-c^3)}{b-c}+\frac{b(c^3-a^3)}{c-a}+\frac{c(a^3-b^3)}{a-b}=0.$

9. $\frac{a^4}{2a^2+b^2}+\frac{b^4}{2b^2+ca}+\frac{c^4}{2c^2+ab}=-(ab+bc+ca).$

10. $\frac{(a^2-bc)^2}{(b^2-ca)(c^2-ab)}+\frac{(b^2-ca)^2}{(c^2-ab)(a^2-bc)}+\frac{(c^2-ab)^2}{(a^2-bc)(b^2-ca)}=3.$

11. $\frac{2a}{b^2+c^2-5a^2}+\frac{2b}{a^2+c^2-5b^2}+\frac{2c}{a^2+b^2-5c^2}=0.$

12. $\frac{a^4}{5bc-2(b^2+c^2)}+\frac{b^4}{5ca+2(c^2+a^2)}+\frac{c^4}{5ab+2(a^2+b^2)}$

$$=-(ab+bc+ca).$$

13. $\frac{1}{(a^2+2bc)(b^2+2ca)}+\frac{1}{(b^2+2ca)(c^2+2ab)}+\frac{1}{(c^2+2ab)(a^2+2bc)}=0.$

If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, prove that

14. $\left(\frac{1}{a+b+c}\right)^3=\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}.$

15. $\left(\frac{1}{a+b+c}\right)^{2n+1}=\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}.$

16. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{2}{a+b+c}$, then $(a+b)(b+c)(c+a)=abc$.

17. If $ab+bc+ca=1$, then

$$\left(1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right)^2=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)};$$

and $$\frac{b+c}{1-bc}+\frac{1-ab}{a+b}+\frac{1-ca}{c+a}=a+b+c$$

If $a+b+c+d=0$, prove that

18. $a^2+b^2-c^2-d^2=2(cd-ab)$.

19. $a^3+b^3+c^3+d^3+3(c+d)(ab-cd)=3(a+b)(cd-ab)$.

20. $a^4+b^4+c^4+d^4$
$$=2\{(ab-cd)^2+(ac-bd)^2+(ad+bc)^2\}-4abcd.$$

21. $$\frac{(a^5+b^5+c^5+d^5)}{5}=\frac{a^3+b^3+c^3+d^3}{3}\times\frac{a^2+b^2+c^2+d^2}{2},$$

If $a+b+c=abc$, prove that—

22. $$\frac{a}{1-a^2}+\frac{b}{1-b^2}+\frac{c(1-ab)^2}{(1-a^2)(1-b^2)}=0.$$

23. $$\frac{b}{1-b^2}+\frac{c}{1-c^2}+\frac{a(1-bc)^2}{(1-b^2)(1-c^2)}=0.$$

24. $$\frac{c}{1-c^2}+\frac{a}{1-a^2}+\frac{b(1-ca)^2}{(1-c^2)(1-a^2)}=0.$$

25. $$\frac{2a}{1-a^2}+\frac{2b}{1-b^2}+\frac{2c}{1-c^2}=\frac{2a}{1-a^2}\cdot\frac{2b}{1-b^2}\cdot\frac{2c}{1-c^2}.$$

26. $$\frac{3a-a^3}{1-3a^2}+\frac{3b-b^3}{1-3b^2}+\frac{3c-c^3}{1-3c^2}=\frac{3a-a^3}{1-3a^2}\cdot\frac{3b-b^3}{1-3b^2}\cdot\frac{3c-c^3}{1-3c^2}.$$

27. $$\frac{a+b}{1-ab}+\frac{b+c}{1-bc}+\frac{c+a}{1-ca}=\frac{a+b}{1-ab}\cdot\frac{1+bc}{1-bc}\cdot\frac{c+a}{1-ca}.$$

28. If $x+y+z=0$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$,

then $x^2+y^2+z^2=0$ and $x^4+y^4+z^4=0$.

29. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$, then $\frac{1}{x^2-yz}+\frac{1}{y^2-xz}+\frac{1}{z^2-xy}=0$.

30. If $x=\frac{a-b}{a+b}$, $y=\frac{b-c}{b+c}$ and $z=\frac{c-a}{c+a}$,

then $$\frac{1+x}{1-x}\cdot\frac{1-y}{1-y}\cdot\frac{1+z}{1-z}=1.$$

31. If $x=\frac{b-c}{a}$, $y=\frac{c-a}{b}$ and $z=\frac{a-b}{c}$,

then $x+y+z+xyz=0$.

32. If $x+\frac{1}{y}=1$, $z+\frac{1}{x}=1$, then $y+\frac{1}{z}=1$ and $xyz+1=0$

33. If $\frac{1}{a-b}+\frac{1}{b-c}+\frac{1}{c-a}=0$,

then $(a-b)^2+(b-c)^2+(c-a)^2=0$.

34. If $ax=y$, $by=x$, then $\frac{1}{a+1}+\frac{1}{b+1}=1$

35. If $x+y=az$, $y+z=bx$ and $z+x=cy$,

then $(a-1)(b-1)(c-1)=1$.

36. If $x+y+z=-4$ and $xy+yz+zx=5$,

then $\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=0$.

37. If $x+y=az$, $y+z=bx$ and $z+x=cy$,

then $(a+1)(b+1)(c+1)=0$ if $x+y+z=0$.

38. If $a+b+c=1$, $ab+ac+bc=2$ and $a^3+b^3+c^3=3$

then $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{45}{64}$.

39. If $\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}+\frac{d}{1-d}=1$,

shew that $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1+c}+\frac{1}{1-d}=5$.

40. If $\frac{x}{y}+\frac{y}{x}=1$, then $\frac{x^3}{y}+xy+\frac{y^3}{x}=0$.

41. If $(a+d)(b+c)=(1-ad)(1-bc)$,

then $(a+c)(b+d)=(1-ac)(1-bd)$.

42. If $x+y+z=4$ and $xy+yz+zx=5$,

shew that $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=0$.

43. If $x+y+z=0$, that $\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}=3$.

44. If $a+b+c+2=abc$, then $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1$.

45. If $a+b+c=0$,

then $a^4+b^4+c^4+(ab+ac+bc)(a^2+b^2+c^2)=0$.

46. If $x=\frac{b}{c}-\frac{c}{a}$, $y=\frac{c}{a}-\frac{a}{b}$ and $z=\frac{a}{b}-\frac{b}{c}$

shew that $2(x^2y^2+y^2z^2+z^2x^2)-x^4-y^4-z^4=0$.

47. If $x=\frac{a}{b+c}$, $y=\frac{b}{c+a}$ and $z=\frac{c}{a+b}$,

then $xy+yz+zx+2xyz=1$.

48. If $x+y+z=du$, $y+z+u=ax$, $z+x+u=by$,

and $x+y+u=cz$,

$$\text{then } \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}=1.$$

49. If $X=ax+by+cz$, $Y=bx+cy+az$ and $Z=cx+ay+bz$, shew that $(a^2+b^2+c^2-ab-ac-bc)(x^2+y^2+z^2+xy-yz-zx)=X^2+Y^2+Z^2-XY-XY-YZ$.

50. If $a+b+c=0$,

$$\text{then } \frac{a^7+b^7+c^7}{7}\times\frac{a^3+b^3+c^3}{3}=\left(\frac{a^5+b^5+c^5}{5}\right)^2$$

51. If $2s=a+b+c+d$, then

$$(ad+bc)^2-(a^2-b^2-c^2+d^2)=16(s-a)(s-b)(s-c)(s-d)$$

52. If $a_1+a_2+a_3+\ldots\ldots a_n=\frac{n}{2}s$, then

$$(s-a_1)^2+(s-a_2)^2+(s-a_3)^2+(s-a_4)^2+\ldots\ldots+(s-a_n)^2$$
$$=a_1{}^2+a_2{}^2+a_3{}^2+\ldots\ldots+a^2{}_n.$$

53. Shew that $\frac{1}{a+x}+\frac{2x}{a^2+x^2}+\frac{4x^3}{a^4+x^4}+\frac{8x^7}{a^8+x^8}$

$$=\frac{1}{a-x}-\frac{16x^{15}}{a^{16}-x^{16}}.$$

54. Shew that $(x-y)^4+(y-z)^4+(z-x)^4$
$=2\{(x-y)^2(y-z)^2+(y-z)^2(z-x)^2+(z+x)^2(x-y)^2\}$
$=2(x^2+y^2+z^2-xy-yz-zx)^2$.

55. Shew that $(b+c-a)(c+a-b)(a+b-c)+a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2=4abc$.

56. Shew that $(a-cx)^2+(x^2-1)(b^2-d^2)$

$$=\left(\frac{ab+b^2+cd-d^2}{c-d}\right)^2 \text{ if } x=\frac{a+b}{c-d}.$$

57. If $s=a+b+c$, than $(as+bc)(bs+ca)(cs+ab)$
$=(a+b)^2(b+c)^2(c+a)^2$.

58. If $3s=a+b+c$,
$(s-a)^4+(s-b)^4+(s-c)^4$
$=2\{(s-a)^2(s-b)^2+(s-b)^2(s-c)^2+(s-c)^2(s-a)^2\}$

59. If $\frac{1}{1+l+lm}+\frac{1}{1+m+mn}+\frac{1}{1+n+nl}=1$, prove that—
(i) $lmn=1$, *or* (ii) $(1+l)(1+m)(1+n)=-1$.

60. Show that $\frac{a-b}{n+ab}+\frac{b+c}{n+bc}+\frac{c-a}{n+ca}$
$$=\frac{n(a-b)(b-c)(c-a)}{(n+ab)(n+bc)(n+ca)}$$

61. If $xyz=1$
show that
$$\frac{1-x}{1+x}+\frac{1-y}{1+y}+\frac{1-z}{1-z}=-\frac{1-x}{1+x}\cdot\frac{1-y}{1+y}\cdot\frac{1-z}{1+z}.$$

62. If $xyz=1$
show that
$$\frac{1-x^2}{1+x^2}+\frac{1-y^2}{1+y^2}+\frac{1-z^2}{1+z^2}=-\frac{1-x^2}{1+x^2}\cdot\frac{1-y^2}{1+y^2}\cdot\frac{1-z^2}{1+z^2}.$$

63. If $xyz=1$
show that
$$\frac{1-x^3}{1+x^3}+\frac{1-y^3}{1+y^3}+\frac{1-z^3}{1+z^3}=-\frac{1-x^3}{1+x^3}\cdot\frac{1-y^3}{1+y^3}\cdot\frac{1-z^3}{1+z^3}.$$

64. If $a+b+c=abc$.
show that $(1-ai)(1-bi)(1-ci)$.
$=(1+ai)(1+bi)(1+ci)$ where
$$i=\sqrt{-1}$$

65. If $a+b+c=abc$
show that $\frac{1-ai}{1+ai}\cdot\frac{1-bi}{1+bi}\cdot\frac{1-ci}{1+ci}=1$
from (64) where $i=\sqrt{-1}$

66. If $a+b+c=abc$. show that
$$xyz=1, \text{ where } x=\frac{1-ai}{1+ai},$$
$y=\frac{1-bi}{1+bi}$, and $z=\frac{1-ci}{1+ci}$, from (65) where $i=\sqrt{-1}$

67. If $a+b+c=abc$,
show that $x^2y^2z^2=1$, where $x=\frac{1-ai}{1+ai}$,
$y=\frac{1-bi}{1+bi}$ and $z=\frac{1-ci}{1+ci}$ (i being equal to $\sqrt{-1}$)

Deduce from 66 and hence show that

$$\frac{2a}{1-a^2}+\frac{2b}{1-b^2}+\frac{2c}{1-c^2}=\frac{3a}{1-a^2}\cdot\frac{2b}{1-b^2}\cdot\frac{2c}{1-c^2}$$

from (62) by substituting $\frac{1-ai}{1+ai}\cdot\frac{1-bi}{1+bi}$

$\frac{1-ci}{1+ci}$ for x, y, z in it.

68. If $a+b+c=abc$,

show that $x^2y^2z^2=1$, where $x=\frac{1-ai}{1+ai}, y=\frac{1-bi}{1+bi}$

and $z=\frac{1-ci}{1+ci}$ (i being equal to $\sqrt{-1}$) and hence show that

$$\frac{3a-a^3}{1-3a^2}+\frac{3b-b^3}{1-3b^2}+\frac{3c-c^2}{1-3c^2}=\frac{3a-a^3}{1-3a^2}\cdot\frac{3b-b^2}{1-3b^2}\cdot\frac{3c-c^2}{1-3c^2}$$

from 63 by

substituting $\frac{1-ai}{1+ai}, \frac{1-bi}{1+bi}, \frac{1-ci}{1+ci}$ for x, y and z.

69. Similarly find what identity you will get by substitution of $\frac{1-ai}{1+ai}, \frac{1-bi}{1+bi}, \frac{1-ci}{1+ci}$, for x, y and z

in the result $\frac{1-x^4}{1-x^4}+\frac{1-y^4}{1+y^4}+\frac{1-z^4}{1+z^4}=-\frac{1-x^4}{1+x^4}\cdot\frac{1-y^4}{1+y}\cdot\frac{1-z^4}{1+z^4}$

which is true when $xyz=1$.

70. If $a+b+c=5$, $ab+ac+bc=-4$, $abc=3$.

find the value—

(a) $\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}$

(b) $\frac{c+a}{1+b}+\frac{b+a}{1+c}+\frac{c+b}{1+a}$

(c) $\frac{a^2}{1+bc}+\frac{b^2}{1+ca}+\frac{c^2}{1+ab}$

(d) $\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}$

(e) $\frac{a^3}{(1+b^2)(1+c^2)}+\frac{b^3}{(1+c^2)(1+a^2)}+\frac{c^2}{(1+c^2)(1+c^2)}$

EXAMINATION PAPERS.

Third Series on Chapters XV to XX.

I

(1) If $f(x)=x^3+6x^2-9x-12$, find $f(x+h)-f(x-h)$.

(2) Divide $6x^3-9x^2+8x+27$ by $2x-1$, and find the remainder without actual division.

(3) Resolve into factors :—

(1) $a(b^3-c^3)+bc(c^2-b^2)+a^2(c-b)$

(2) $(a-b+c)^3+a^3-c^3+b^3$

(4) Find the L. C. M. of :—

$8x^3+27y^3$, $8x^3-27y^3$, $16x^4+36x^2y^2+81y^4$

(5) Find the value of $\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}$ if $a+b+c=2$, $ab+ac+bc=1$ and $abc=-1$

(6) Simplify $\frac{x}{x-1}+\frac{x^2}{x^2-1}-\frac{x^4}{x^4-1}$

(7) If $\frac{b}{a+c}=\frac{1}{a+c}-\frac{1}{4b}$, express b in terms of a and c.

(8) If $2s=a+b+c$, show that

$2(s-a)(s+b)(s-c)+a(s-b)(s-c)$

$+b(s-c)(s-a)+c(s-a)(s-b)=abc$.

II

(1) If $3x^2+5x-7\equiv A+B(x-1)+C(x-1)^2+D(x-1)^3$

determine the values of A, B, C, & D,

(2) What conditions must p and q satisfy in order that x^2-px+q may divide x^3-x^2-2x+2 exactly.

(3) Express a^4 as an integral function of $a-1$.

(4) Prove that $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.

Hence show that $(x+y+z)^3+(x+y+z)^3+(y+z-x)^3+(z+x-y)^3=24xyz$.

(5) Find the H. C. F. and L. C. M. of $3x(x^2-2x-35)$ and $2x(x^2-4x-45)$.

(6) $f(y)=6y^3-8y^2+7y-4$

If $y=x-2$, find $f(y)$ as a function of

(7) Simplify :—

$$\frac{x^2+x+1}{(x-y)(x-z)}+\frac{y^2+y+1}{(y-z)(y-x)}+\frac{z^2+z+1}{(z-x)(z-y)}$$

(8) If $x=a(y+z)$, $y=b(z+x)$, $z=c(x+y)$, show that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$$

III

(1) If $f(x)=ax^3+bx-10$, $f(1)=-5$, and $f(-2)+8=0$, determine a and b.

(2) Find the H. C. F. and L. C. M. of :—

$4x^3-8x^2+3x-6$ and $12x^3+4x^2+9x+3$

(3) If $x^3-4x^2+4x-3\equiv A(x-1)(x-2)(x-3)+B(x-2)(x-3)+C(x-3)+D$, determine A, B, C and D.

(4) Simplify :—

$$\left\{2a+1+\frac{a-1}{a+1}-\frac{a+2}{a}\right\}\left\{\frac{1}{2a+1}+\frac{a+1}{a-1}-\frac{a}{a+2}\right\}$$

(5) Resolve into factors :—

$$(a^2+b^2-c^2)(a^2+c^2-b^2)+(a^2+c^2-b^2)(b^2+c^2-a^2)+(b^2+c^2-a^2)(a^2+b^2-c^2)$$

(6) If $3(a^2+b^2+c^2)=(a+b+c)^2$, show that $a=b=c$.

(7) If $x^3+ax^2-(a^2-2ab)x-c^3+3abc$ be divisible by $x+a+b$, show that $a^3+b^3+c^3=3abc$.

(8) If $b=a+\frac{2}{a}$ and $c=a+\frac{1}{a}$, prove that $3bc=1+b^2+2c^2$.

IV

(1) If $F(x)=2x^3-3x+3$ and $f(x)=3x^2-x+1$, find the value of $2F(3)-3f(2)$.

(2) If $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$, determine A, B, C.

(3) Resolve into factors :—

(1) $x(y^2+z^2-x^2)+y(z^2+x^2-y^2)$

(2) $6a^2-ab-b^2+4a+3b-2$

(4) Find the L. C. M. of :—

$12a^3+4a^2-3a-1$ and $8a^3-4a^2-2a+1$.

(5) If $xyz=1$, show that

$$\left(x+\frac{1}{x}\right)^2+\left(y+\frac{1}{y}\right)^2+\left(z+\frac{1}{z}\right)^2=4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)$$

(6) Simplify :—

$$\frac{a+b}{(a^2-bc)(b^2-ac)}+\frac{b+c}{(b^2-ac)(c^2-ab)}+\frac{c+a}{(c^2-ab)(a^2-bc)}$$

(7) Show that $x(x-2a)(x+a)(x-a)+a^4$ is a perfect square.

(8) Show that ax^4+6x^2+c is divisible by x^2+c, if $b=ac+1$.

V

1. Prove that the product of $6a^2+a-2$ and $4a^2-11a-3$ is equal to the product of $8a^2-2a-1$ and $3a^2-7a-6$.

2. Express $x^3-6x^2+12x-8$ in terms of y, if $x-y=2$.

3. Find an integral value of a for which the expression $a^5+10a^4+30a^3+32a^2+12a-4$ shall be exactly divisible by a^3+3a+1.

4. If $y=f(x)=\frac{ax+b}{x-a}$, shew that $f(y)=x$.

5. If $x+y+z=0$, simplify

$$\frac{y+z}{yz}(y^2+z^2-x^2)+\frac{z+x}{zx}(z^2+x^2-y^2)+\frac{(x+y)}{xy}(x^2+y^2-z^2)$$

6. Find the H. C. F. of :—

$4a^3-10a^2+4a+2$ and $3a^4-2a^3-3a+2$.

7. Resolve $x^4-10x^3+35x^2-50x+42$ into factors.

8. If $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2$ show that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

VI

1. Show that $(ax+by+cz)^3-(bx+cy+az)^3$ is divisible by $(a-b)x+(b-c)y+(c-a)z$.

2. Find the H. C. F. of :—

$x^3-6x^2+11x-6$ and $x^3-9x^2+26x-24$.

3. Show that $4+\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)$

$$=\left(\frac{a}{b}+\frac{b}{a}\right)^2+\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2$$

4. Resolve into factors :—

$(a-b)^3+(b-c)+(c-a)^3$

5. If $2s=a+b+c$, show that

$$\frac{s-a}{(s-b)(s-c)}+\frac{s-b}{(s-c)(s-a)}+\frac{s-c}{(s-a)(s-b)}+\frac{s^2}{(s-a)(s-b)(s-b)}=\frac{a^2+b^2+c^2}{(s-a)(s-b)(s-c)}$$

6. If $a+b=cz$, $b+c=ay$, $c+a=bx$, show that

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$$

7. If $ax^2+bxy+cy^2$ and $bx^2-2(a-c)xy-by^2$ have a common factor, then $ax^2+bxy+cy^2$ is an exact square.

8. If $\frac{1}{(x+1)(x+2)(x+3)}\equiv\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}$ find A, B & C.

VII

1. Find the H. C. F. of :—

x^4-6x^2+8x-3, $x^4-2x^3-7x^2+20x-12$ and $x^4-4x^2+12x-9$.

2. If $x^2+z^2=2y^2$, show that $\frac{1}{y+z}+\frac{1}{x+y}=\frac{2}{x+z}$.

3. Find the H. C. F. of two expressions is $a-4$ and their L. C. M. is $a^4-10a^3+35a^2-50a+24$; find the expressions.

4. If $\frac{7x+11}{(x+1)(x+2)(x+3)}\equiv\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}$, find A, B & C.

5. If $A=ax+by+cz$; $B=cx+ay+bz$ and $C=bx+cy+az$, shew that $(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)=A^3+B^3+C^3-3ABC$.

6. Simplify :—

$$\frac{a^4+a^2+1}{(a^2+a+1)(a-b)(a-c)}+\frac{b^4+b^2+1}{(b^2+b+1)(b-c)(b-a)}+\frac{c^4+c^2+1}{(c^2+c+1)(c-a)(c-b)}.$$

7. Show that $(x+y+z)(xy+xz+yz)-xyz=(x+y)(y+z)(z+x)$.

Hence simplify $\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{a}+\frac{a}{b}\right)-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$

8. Express $(x^2+y^2)(a^2+b^2)$ as the sum of two squares in two ways.

VIII

1. Divide $x^4+ax^3+(a^2+4)x^2+(4a+1)x+6a^2$ by x^2+4 and find the value of x for exact divisibility.

2. Resolve into factors :—

$x(y+z)^2+y(z+x)^2+z(x+y)^2-(y+z)(x-y)(x-z)-(z+x)(y-x)(y-z)-(x+y)(z-x)(z-y)$.

3. If $a+b+c=0$, prove that

(i) $a(b^2+c^2-a^2)=b(c^2+a^2-b^2)=c(a^2+b^2-c^2)=-2abc$.

(ii) $(b+c-a)^3+(c+a-b)^3+(a+b-c)^3=-24abc$.

4. Reduce to its lowest terms :—

$$\frac{3x^3-23x^2+43x-8}{x^4-5x^3-6x^2+35x-7}$$

5. Find the H. C. F. of $2x^4+13x^3-4x^2+6x+1$ and $x^4+7x^3-2x^2-21x-3$.

6 Find a homogeneous and symmetrical expression of the second degree in x and y which shall be equal to 3, when x and y are each equal to unity, and shall be equal to 11, when $x=2$, $y=1$

7. If a, b, c be three consecutive numbers, prove that $a^3-2b^3+c^3=6b$.

8. Simplify $\frac{x^3+y^3}{x^4+y^4}-\frac{x+y}{x^2-y^2}-\frac{1}{2}\left\{\frac{x-y}{x^2+y^2}-\frac{1}{x-y}\right\}$.

IX

1. If the expression ax^2+bx+c takes the values $\frac{1}{c}$, $\frac{1}{c+1}$, $\frac{1}{c+2}$ when x takes the values 0, 1, 2 find the values of a, b, c and prove that when $x=c+2$, the expression is equal to $\frac{1}{c+1}$.

2. If $x=\frac{a-b}{m-c}$, $y=\frac{b-c}{m-a}$, $z=\frac{c-a}{m-b}$, find the value of

$$x+y+z+xyz.$$

3. Simplify $\left\{1-\frac{4}{x-1}+\frac{12}{x-3}\right\}\left\{1+\frac{4}{x-1}-\frac{12}{x+3}\right\}$.

4. Find the H. C. F. of a^3-2a^2+1 and $2a^3+a^2+4a-7$.

5. Resolve into factors:—

$$(b-c)(b+c)^3+(c-a)(c+a)^3+(a-b)(a+b)^3.$$

6. Express $8x^3-10x+5$ as a function of $2x-1$.

7. If D is the mid point of the base BC of a triangle ABC, it may be assumed that $AB^2+AC^2=2(AD^2+BD^2)$. If $AB=2p$, $BC=2p+2$, $CA=2p+1$, $AD=p+4$. find the perimeter of the triangle.

8. If $x=\frac{1}{1-y}$, $y=\frac{1}{1-z}$, prove that $z=\frac{1}{1-x}$.

X

1. Resolve fnto factors:—

(*a*) $(x+y-z)^3(x-y)+(y+z-x)^3(y-z)+(z+x-y)^3(z-x)$.

(*b*) $4x^2-7xy-2y^2+10x+7y-6$.

2. Find a symmetrical and homogeneous expression of the second degree in x, y and z which will be equal to -7 when $x=y=1$ and $z=2$, and 23, when $x=1$, $y=-1$ and $z=2$.

3. If $2S=a^2+b^2+c^2$ and $2s=a+b+c$, show that $(S-a^2)(S-b^2)+(S-b^2)(S-c^2)+(S-c^2)(S-a^2)=4s(s-a)(s-b)(s-c)$.

4. If $a+b+c=k$, $ab+ac+bc=l$, and $abc=m$
Find the values of (1) $a^2(b+c)+b^2(c+a)+c^2(a+b)$

(2) $b^2c^2+c^2b^2+a^2b^2$.

5. Simplify $\left\{\dfrac{\dfrac{1}{x}+\dfrac{1}{y+z}}{\dfrac{1}{x}-\dfrac{1}{y+z}}\right\}\left\{1+\dfrac{y^2+z^2+-x^2}{2yz}\right\}$

6. Reduce to its lowest terms:—
$$\frac{x^3+11x+12}{x^5+11x^3-54}$$

7. Find the value of $\dfrac{x^2-y^2+x}{y^2-x^2+y}$ when $x=\dfrac{a-1}{a+1}$ and $y=\dfrac{a+1}{a-1}$.

8. If $b=q\,\dfrac{a+p}{p-a}$, find $a+b$ in terms of a and also in term of b.

CHAPTER XXI.

INVOLUTION AND EVOLUTION.

168. If a quantity be continually multiplied by itself, it is said to be *involved* or raised ; and the power to which it is raised is expressed by the number of times the quantity has been employed in multiplication.

Thus, $a \times a$ or a^2 is the second power of a ; $a \times a \times a$ or a^3 is the third power of a ; and so on.

If the quantity to be *involved* has a negative sign, the sign of the *even* powers will be positive, and the sign of the *odd* powers will be negative.

Thus, $-a \times -a = a^2$; $-a \times -a \times -a = -a^3$;

$-a \times -a \times -a \times -a = a^4$! and so on.

169. A *simple* quantity is raised to any power by multiplying the index of every factor in the quantity by the exponent of that power, and prefixing the proper sign.

Thus, $(am)n = amn$;

$(ab)m = ab \times ab \times ab$ ..... to m factors,

$= (a \times a \times a$ ...to m factors) $\times$ ($b \times b \times b$ .....to m factors)

$= am \times bm$

$(a^2b^4c)^5 = a^{10}b^{20}c^5$.

And $(-an)m = +anm$ if m is *even* ; or $= -amn$ if m is *odd*.

170. To raise a Binomial to any power.

By actual multiplication, the following results can be established.

$$\left.\begin{array}{l}(a+b)^2 = a^2 + 2ab + b^2 \\ (a-b)^2 = a^2 - 2ab + b^2\end{array}\right\} \quad \ldots\ldots\ldots \text{I.}$$

$$\left.\begin{array}{l}(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\ (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\end{array}\right\} \quad \ldots\ldots \text{II.}$$

$$\left.\begin{array}{l}(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ (a-b)^4 = a^4 - 4a^2b + 6a^2b^2 - 4ab^2 + b^4\end{array}\right\} \quad \ldots\ldots\ldots \text{III.}$$

$$\left.\begin{array}{l}(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^2 + 5ab^4 + b^5 \\ (a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5\end{array}\right\} \quad \ldots\ldots \text{IV.}$$

On examining the above cases we observe :—

(1) The number of terms in the resulting expression is one more than the index of the power.

(2) Any power of $a-b$ differs from that of $a+b$ *only* in this, that the signs of the terms of the former are *alternately* + and −, whilst those of the latter are *all* +.

(3) The first term of the expansion is equal to the first term of the binomial raised to the given power. In the succeeding terms, the powers of a decrease by unity, and those of b increase by unity.

(4) The co-efficient of the second term is the same as the index of the power to which the binomial is raised.

(5) If we multiply the co-efficient of any term by the power of a in that term, and divide the product by the number of that term, we get the co-efficient of the following term.

(6) The co-efficient of the terms equidistant from the beginning and the end are equal.

The above laws enable us to expand a binomial raised to any power.

Thus *generally*

$$(x+y)^n = x^n + n.x^{n-1}.y + \frac{n(n-1)}{1.2}x^{n-2}.y^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 + \ldots\ldots\ldots\ldots + \frac{n(n-1)}{1.2}.x^2.y^{n-2} + n.x.y^{n-1} + y^n.$$

$$(x-y)^n = x^n - n.x^{n-1}.y + \frac{n(n-1)}{1.2}.x^{n-2}.y^2 - \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3 + \ldots\ldots\ldots\ldots + \frac{n(n-1)}{1.2}.x^2.(-y)^{n-2} + n.x.(-y)^{n-1} + (-y)^n.$$

Ex. 1. Expand $(a+b)^6$ and $(a-b)^6$.

The *first* term $= a^6$.

The *second* term $= 6a^5b$.

The *third* term $= \frac{6\times5}{2}a^4b^2 = 15a^4b^2$.

The *fourth* term $= \frac{15\times4}{3}a^3b^3 = 23a^3b^3$.

The *fifth* term $= \frac{20\times3}{4}a^2b^4 = 15a^2b^4$.

The *sixth* term $= \frac{15 \times 2}{5} ab^5 = 6ab^5$.

The *seventh* term $= \frac{6 \times 1}{6} b^6 = b^6$.

Hence

$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

and

$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

Ex. 2. Expand $(x-2y)^7$.

The *first* term $= x$.

The *second* term $= -7x^6(2y) = [-14x^6y]$.

The *third* term $= +\frac{7 \times 6}{2} x^5(2y)^2 = 21x^5(2y)^2$

$= [84x^5y^2]$.

The *fourth* term $= -\frac{21 \times 5}{3} x^4(2y)^3 = -35x^4(2y)^3 = [-280x^4y^3]$.

The *fifth* term $= +\frac{35 \times 4}{4} x^3(2y)^4 = 35x^3(2y)^4 = [560x^3y^4]$.

The *sixth* term $= -\frac{35 \times 3}{5} x^2(2y)^5 = -21x^2(2y)^5 = [-672x^2y^5]$.

The *seventh* term $= +\frac{21 \times 2}{6} x(2y)^6 = 7x(2y)^6 = [448xy^6]$.

The *eighth* or last term $= -\frac{7 \times 1}{7}(2y)^7 = -(2y)^7 = [-128y^7]$.

Hence $(x-2y)^7 = x^7 - 14x^6y + 84x^5y^2 - 280x^4y^3 + 560x^3y^4 - 672x^2y^5 + 448xy^6 - 128y^7$.

This result may be verified as in Art. 122.

Thus putting $x=y=1$, we have

$(1-2)^7 = 1 - 14 + 84 - 280 + 560 - 672 + 448 - 128$.

Ex. 3. Find the co-efficient of x^5 in the expansion of $(x+a)^8$.

$(x+a)^8 = x^8 + 8x^7a + \left(\frac{8 \times 7}{2} \text{ or } 28\right)x^6a^2 + \left(\frac{28 \times 6}{3} \text{ or } 56\right)x^5a^3 + \&c.$

Hence the co-efficient of $x^5 = 56a^3$.

Note.—A trinomial raised to any power may be expanded by the repeated application of this rule.

EXERCISE 80.

Evaluate :—

1. $(5a^3b^{\cdot})^3$; $(-2a^2b^3)^5$.

2. $\left(\frac{l^5m^3n}{ab^2c^3}\right)^3$; $\left(-\frac{2x^2yz^3}{3p^3q^2r}\right)^2$

3. $\left(\frac{a^3}{b^{\cdot}}\right)^3 \times \left(\frac{b^3}{a^{\cdot}}\right)^2$) n.

4. $\left(-\frac{3x^2}{4yz}\right)^3 \times \left(\frac{4y^2}{3zx}\right)^2 \div \left(-\frac{3x^{\cdot}}{2z^2}\right)^1$; ^{3}

5. $(a^{2l+3m})^{4n} \div (a^{2m+3n})^{4l}$.

Expand :—

6. $(x-a)^4$. 7. $(a^2+1)^5$. 8. $(a^3-2)^4$. 9. $(x-y)$.

10. $(2-x)^7$ 11. $\left(1+\frac{1}{x}\right)^6$ 12. $(4+a^3)^4$. 13. $(3x-2a)^5$

14. $(ax+by)^5$. 15. $(\frac{1}{2}x-\frac{1}{3}y)^4$.

16. $\left(\frac{m}{n}-\frac{n}{m}\right)^5$ 17. $\left(\frac{2x^2}{yz}+\frac{y^2}{3zx}\right)^4$.

18. $(1+a+a^2)^4$. 19. $(x-1+x^2)^5$.

20. $(ax^2+bx+c)^3$.

Simplify :—

21. $(a+b)^6-(a-b)^6$. 22. $(x+y)^7+(x-y)^7$.

23. $(a+b)^5-a^5-b^5$. 24. $(a+b)^7-a^7-b^7$.

Find the co-efficient of :—

25. x^4 in $(2x+a)^5$. 26. x^5 in $(x-2y)^6$.

27. x^3 in $(x-3a)^7$. 28. x^3 in $(x^2+x+1)^4$.

29. x^4 in $(1-x)^4(1+x)^4$. 30. x^5 in $(1-x)^3(1+x)^3$.

EVOLUTION.

171. Evolution, or the extraction of roots, is the reverse of involution and it is the method of determining a quantity which raised to a proposed power will produce the given quantity.

To find the root of a simple quantity.

We have seen in Art. 42 that $(am)^n = a^{mn}$ or the nth power of am is obtained by raising a to the power of m multiplied by n, *i.e.*, of mn.

Hence, conversely, the quantity whose nth power is a^{mn} is obtained by raising a to a power which is got by *dividing* mn by n

Thus we have the following rule for extracting the root of a simple quantity :—

Rule.—"*Divide the exponent of each factor by the index of the required root and prefix the root of the numerical co-efficient.*"

Thus, $\sqrt[5]{a^{25}} = a^5$; $\sqrt[3]{27a^9} = 3a^3$; $\sqrt[4]{16a^8b^4c^{12}} = 2a^2bc^3$.

Sign of roots.—The root of a positive quantity is positive or negative if the index of the root be an even number.

Thus, $\sqrt{a^2} = \pm a$; $\sqrt[4]{16a^8} = \pm 2a^2$

The root of a negative quantity is negative if the index of the root be an odd number. Thus, $\sqrt[3]{-a^3} = -a$.

The root of a positive quantity is positive if the index of the root be an odd number. Thus, $\sqrt[3]{a^6} = a^2$.

The root of a negative quantity is impossible when the index of the root is an even number. Thus, $\sqrt{-2}$ is an imaginary quantity.

172. The ordinary method of extracting the square root of a compound expression.

Taking the identity $(a+b)^2 = a^2 + 2ab + b^2$,

we find that the square root of a trinomial is a binomial expression, the first term of which is the square root of the first term of the given expression arranged in the descending order of the powers of a given quantity. When the square of the first term, *i.e.*, a^2 is subtracted from the given expression we have the remainder $2ab + b^2$

$$(a+b)^2 - a^2 \qquad = 2ab + b^2 \qquad = (2a+b)b.$$

Examining the remainder we see that it is the product of the second term of the square root and an expression which is the sum of twice the first term and the second term of the square. Thus if we can find $2a+b$, b can be found by dividing the remainder by $2a+b$. Since the first term of the remainder is $2ab$, b can be found more readily from this by dividing it by $2a$, *i.e.*, twice the first term. After finding b thus, $2a+b$ may be obtained ; this, when multiplied by b will give the remainder.

Ex. 1. Extract the square root of $4a^2+b^2+4ab$. Arrange the expression according to descending powers of a.

$$\begin{array}{l|l} & 4a^2+4ab+b^2\ (2a+b \\ & 4a^2 \\ \hline 4a+b & \quad 4ab+b^2 \\ & \quad 4ab+b^2 \end{array}$$

The square root of $4a^2$ is $2a$, which is the first term of the required root. We place it to the right of the given expression and also to the left. We now subtract the square of $2a$ from the given expression, and the remainder is $4ab+b^2$; then we double $2a$, which gives $4a$, by which we divide the first term of the remainder, and this gives b as quotient, which is the second term of the square root. We place it after $2a$ to the right; and we also add it to $4a$. Again we multiply $4a+b$ by b the second term of the root, and subtract the product from the remainder. Since there is no remainder, we have $2a+b$ for the required square root.

Ex. 2. Extract the square root of :—

$$x^6-6ax^5+15a^2x^4-20a^3x^3+15a^3x^2+6a^5x+a^6$$

Arrange the expression according to descending powers of a.

$$\begin{array}{l|l} & a^3-3a^2x+3ax^2-x^3 \\ & a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6 \\ & a^6 \\ \hline 2a^3-3a^2x & \quad -6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6 \\ & \quad -6a^5x+9a^4x^2 \\ \hline 2a^3-6a^2x+3ax^2 & \quad 6a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6 \\ & \quad 6a^4x^2-18a^3x^3+9a^2x^4 \\ \hline 2a^3-6a^2x+6ax^2-x^3 & \quad -2a^3x^3+6a^2x^4-6ax^5+x^6 \\ & \quad -2a^3x^3+6a^2x^4+6ax^5+x^6 \end{array}$$

Hence the required square root is $a^3-3a^2x+3ax^2-x^3$.

We have seen in Art. 122 that the sum of the co-efficients of an expression is equal to the product of the sums of the co-efficients of its factors. Now if the expression is a full square the sum of its co-efficients must be the square of the sum of the co-efficients of the root.

Verification: Put $x=a=1$.

$1-6+15-20+15-6+1=0=(1-3+3-1)^2$.

Ex 3. Extract the square root of $1-\frac{a}{2}+\frac{37a^2}{80}-\frac{a^3}{10}+\frac{a4}{25}$.

The expression is arranged according to ascending powers of a.

$$1-\frac{a}{2}+\frac{37a^3}{80}-\frac{a^3}{10}+\frac{a^4}{25} \quad \Big| \quad 1-\frac{a}{4}+\frac{a^2}{5}$$

$$1$$

$$2-\frac{a}{2} \quad \Big| \quad -\frac{a}{2}+\frac{37a^2}{80}-\frac{a^3}{10}+\frac{a^4}{25}$$

$$-\frac{a}{2}+\frac{a^2}{16}$$

$$2-\frac{a}{2}+\frac{a^2}{5} \quad \Big| \quad \frac{2a^2}{5}-\frac{a^3}{10}+\frac{a^4}{25}$$

$$\frac{2a^2}{5}-\frac{a^3}{10}+\frac{a^4}{25}$$

Hence $1-\frac{a}{4}+\frac{a^2}{5}$ is the required square root.

Ex. 4. Extract the square root of:—

$$x^4+\frac{1}{x^4}+x^2+\frac{1}{x^2}+2 \quad \Big| \quad x^3-\frac{1}{x^3}-\left(x-\frac{1}{x}\right)^2$$

Arrange the expression according to descending powers of x

$$x^4+2x^3+x^2-2x+\frac{2}{x}+\frac{1}{x^2}-\frac{2}{x^3}+\frac{1}{x^4} \quad \Big| \quad x^2+x-\frac{1}{x}+\frac{1}{x^2}$$

$$x^4$$

$$2x^2+x \quad \Big| \quad 2x^3+x^2-2x+\frac{2}{x}+\frac{1}{x^2}-\frac{2}{x^3}+\frac{1}{x^4}$$

$$2x^3+x^2$$

$$2x^2+2x-\frac{1}{x} \quad \Big| \quad -2x+\frac{2}{x}+\frac{1}{x^2}-\frac{2}{x^3}+\frac{1}{x^4}$$

$$-2x-2+\frac{1}{x^2}$$

$$2x^2+2x-\frac{2}{x}+\frac{1}{x^2} \quad \Big| \quad 2+\frac{2}{x}-\frac{2}{x^3}+\frac{1}{x^4}$$

$$2+\frac{2}{x}-\frac{2}{x^3}+\frac{1}{x^4}$$

∴ the square root is $x^2+x-\frac{1}{x}+\frac{1}{x^2}$.

Ex. 5. Extract the square root of :—

$$x^2-2+\frac{4x^3+9x^2+4x+9}{x^4+4x^3+5x^2+4x+4}.$$

In such examples as this, we must first see *whether the fraction is in its lowest terms.*

Now the H. C. F. of :—

$4x^3+9x^2+4x+9$ and $x^4+4x^3+5x^2+4x+4$ is x^2+1 ;

$$\therefore \frac{4x^3+9x^2+4x+9}{x^4+4x^3+5x^2+4x+4}=\frac{(4x+9)(x^2+1)}{(x^2+4x+4)(x^2+1)}=\frac{4x+9}{x^2+4x+4}.$$

The given expression $=x^2-2+\dfrac{4x+9}{x^2+4x+4}=\dfrac{x^4+4x^3+2x^2-4x+1}{x^2+4x+4}$.

The square root of the numerator is x^2+2x-1.

The square root of the denominator is $x+2$.

∴ The square root of the given expression is

$$\frac{x^2+2x-1}{x+2}=x-\frac{1}{x+2}.$$

EXERCISE 81.

Evaluate :—

1. $\sqrt{a^4b^4c^6}$; $\sqrt{16m^{10}n^{12}}$.
2. $\sqrt[3]{27a^6b^9}$; $\sqrt[3]{-8x^{12}y^3}$.
3. $\sqrt[4]{16p^8q^{16}}$; $\sqrt[5]{-243l^5m^{10}n^{25}}$.

Extract the square root of :—

4. $81a^2b^2+18abc^2+c^4$.
5. $a^2x^4+2abx^3+(b^2+2ac)x^2+2bcx+c^2$.
6. $9-6x+13x^2-4x^3+4x^4$.
7. $(2a^2b+b^2)^2+(a^2-2ab^2)(4ab+1)$.
8. $(a^2+b^2)(a^2b^2+1)-2ab(a^2-1)(b^2-1)-4a^2b^2$.
9. $x^6+8x^4-2x^3+16x^2-8x+1$.
10. $\dfrac{x^4}{4}+4x^2+\dfrac{ax^2}{3}+\dfrac{a^2}{9}-2x^3-\dfrac{4ax}{3}$.
11. $x^4-2x^3+\dfrac{3x^2}{2}-\dfrac{x}{2}+\dfrac{1}{16}$.
12. $\dfrac{x^4}{4y^4}+\dfrac{4y^4}{x^4}+\dfrac{x^2}{y^2}+\dfrac{4y^2}{x^2}+3$.
13. $\dfrac{2x^2}{3}+2+\dfrac{8}{3x}+\dfrac{4x^2}{9}+\dfrac{4}{x^2}+\dfrac{x^4}{4}$.
14. $\dfrac{x^4}{y^4}+\dfrac{y^4}{x^4}-2\left(\dfrac{x^3}{y^3}+\dfrac{y^3}{x^3}\right)+3\left(\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}\right)-4\left(\dfrac{x}{y}+\dfrac{y}{x}\right)+5$.
15. $x^2+\dfrac{1}{x^2}-2\left(x+\dfrac{1}{x}\right)+3$.
16. $\left(x^2+\dfrac{1}{x^2}\right)^2-4\left(x+\dfrac{1}{x}\right)^2+12$.

17. $\frac{1}{a^3}-\frac{1}{a^2}+\frac{2}{a^3}+3-2a+a^2$.

18. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$.

19. $a^4+b^4+c^4+d^4-2a^2(b^2+d^2)-2b^2(c^2-d^2)+2c^2(a^2-d^2)$.

20. $a^4+2(2b-c)a^3+(4b^2-4bc+3c^2)a^2+2c^2(2b-c)a+c^4$.

21. $a^2+3+\frac{6(a^3-1)}{4a^3-8a^2+5a-1}$.

22. $x^2+2x+\frac{8-16x+6x^2}{2-9x+12x^2-4x^3}$.

23. $a^2+2-\frac{28a^2+45a-7}{7a^3+27a^2+24a-4}$.

24. Find the value of
$(a-b)^2+(b-c)^2+(c-a)^2+$
$2(a-b)(b-c)+2(b-c)(c-a)+2(c-a)(a-b)$.

25. Find the value of
$(3x-2y)^2+(y-2x)^2+(y-x)^2$
$+2(3x-2y)(y-2x)+2(y-2x)(y-x)$
$+2(y-x)(3x-2y)$.

26. Find the square root of
$(a-b)^2+2(a^2-b^2)+(a+b)^2$.

27. Find the square root of
$9x^4+6x^3+7x^2+2x+1$; hence deduce the square root of 96721.

28. If $a=x^2-yz$, $b=y^2-zx$, $c=z^2-xy$ show that $x^3+y^3+z^3-3xyz$ is the square root of $a^2+b^3+c^3-3abc$.

173. *When $n+1$ figures of a square root have been found by the ordinary method, n more may be obtained by division only, supposing $2n+1$ to be the whole number of figures in the root.*

Let N represent the number whose square root is required ; a the part of the root already found ; and x the part to be found, then, $\sqrt{N} = a+x$.

$\therefore N = a^2+x^2+2ax$.

$\therefore N-a^2 = 2ax+x^2$.

$\therefore \frac{N-a^2}{2a} = x+\frac{x^2}{2a}$.

If we show that $\frac{x^2}{2a}$ is a *proper fraction* that x = the quotient of $N - a^2$ divided by $2a$.

Since x contains n digits, x^2 cannot contain more than $2n$ digits; but a contains $2n+1$ digits. Therefore $\frac{x^2}{2a}$ is a proper fraction.

Ex. Find the square root of

$$a^8 - 6a^7 + 13a^6 - 14a^5 + 12a^4 - 10a^3 + 5a^2 - 2a + 1.$$

As the given expression is of the 8th degree the square root must be of the 4th degree and consist of five terms. We shall find the first three terms by the ordinary method and the next two by simple division.

$$\begin{array}{r|l}
 & a^4 - 3a^3 + 2a^2 - a + 1 \\
\hline
a^8 & a^8 - 6a^7 + 13a^6 - 14a^5 + 12a^4 - 10a^3 + 5a^2 - 2a + 1 \\
2a^4 - 3a^3 & \quad - 6a^7 + 13a^6 - 14a^5 + 12a^4 - 10a^3 + 5a^2 - 2a + 1 \\
 & \quad - 6a^7 + 9a^6 \\
\hline
2a^4 - 6a^3 + 2a^2 & \quad 4a^6 - 14a^5 + 12a^4 - 10a^3 + 5a^2 - 2a + 1 \\
 & \quad 4a^6 - 12a^5 + 4a^4 \\
\hline
 & \quad - 2a^5 + 8a^4 - 10a^3 + 5a^2 - 2a + 1 \\
 & \quad - 2a^5 + 6a^4 - 2a^3 \\
\hline
 & \quad 2a^4 - 8a^3 + 5a^2 - 2a + 1 \\
 & \quad 2a^4 - 6a^3 + 2a^2 \\
\hline
\end{array}$$

Hence the square root is $a^4 - 3a^3 + 2a^2 - a + 1$.

Verification :—

$1 - 6 + 13 - 14 + 12 - 10 + 5 - 2 + 1 \equiv 0 \equiv (1 - 3 + 2 - 1 + 1)^2$.

Note —This article is useful in finding the square root to a certain number of terms of an expression which is not a full square.

174 Application of the Principle of Indeterminate Coefficients.

To find the square root of $x^4 + 4x^3 + 10x^2 + 12x + 9$.

Ex. 1. Since the given expression is one of the fourth degree in x the square root will be one of the second degree and contains 3 terms. As the first term of the given expression is x^4 the first term of the square root corresponding to this is x^2 and of the other 2 terms one contains x and the other is independent of x.

∴ The square root is of the form $x^2 + px + q$ where p and q are to be determined.

Then $x^4 + 4x^3 + 10x^2 + 12x + 9 \equiv (x^2 + px + q)^2$

$\therefore\ x^4+4x^3+10x^2+12x+9 \equiv x^4+2px^3+(p^2+2q)x^2+2pqx+q^2.$

Since this is true for all values of x, the co-efficients of the like powers of x are equal.

Hence, $2p=4$. $\therefore\ p=2$;

$p^2+2q=10$. $\therefore\ 2q=10-4=6\ \therefore\ q=3$.

$\therefore$ The square root of the given expression is x^2+2x+3.

Ex. 2 Find the value of p so that

$4x^4-12x^3+px^2-6x+1$ may be a perfect square.

The square root of $4x^4=2x^2$.

The square root is an expression of the second degree in x, the first term being $2x^2$.

Let it be $2x^2+ax+b$ where a and b are to be determined.

Then $4x^4-12x^3+px^2-6x+1 \equiv (2x^2+ax+b)^2$

$\therefore\ 4x^4-12x^3-px^2-6x+1 \equiv 4x^4+4ax^3+x^2(a^2+4b)+2abx+b^2.$

Equating the co-efficients of like powers of x

$4a=-12$ $\therefore\ a=-3$

$a^2+4b=p$.

$2ab=-6$ $\therefore\ 2-3\times b=-6$

$\therefore\ b=1$

But $p=a^2+4b$

$\therefore\ p=-3\times-3+4=13$

Ex 3. To find the square root of:—

$64x^6-192x^5y+240x^4y^2-160x^3y^3+60x^2y^4-12xy^5+y^6.$

As the given expression is of the 6th degree the square root must be of the 3rd degree.

Let the expression $\equiv (px^3+qx^2y+rxy^2+sy^3)^2$.

Now obviously $p^2 = 64$ $\therefore\ p=8$.

and $s^2 = 1$ $\therefore\ s=\pm 1$.

Also $2pq = -192$ $\therefore\ q=-12$.

and $2rs = -12$ $\therefore\ r=\mp 6$.

The value of p is taken to be positive as it is the first co-efficient The ambiguity in the values of r and s can easily be removed as the sum of the co-efficients of the given expression must be a full square of the sum of the co-efficients of the root.

The sum of the co-efficients of the given expression = 1.

$\therefore p+q+r+s$ must be equal to $\sqrt{1}$ *i.e.*, ± 1.
$\therefore r$ must be equal to $+6$.
and s must be equal to -1.
Hence the square root is $8x^3-12x^2y+6xy^2-y^3$.

173. Miscellaneous Examples.

Ex. 1. To find the square root of:—
$4a^2+b^2+9c^2+6bc-12ac-4ab$ by reducing it to the form $(A\pm B)^2$.
The given expression
$=(4a^2+b^2-4ab)+9c^2+6bc-12ac$
$=(2a-b)^2+9c^2-6c(2a-b)$
$=(2a-b)^2+(-3c)^2+2(-3c)(2a-b)$
$=(2a-b-3c)^2$.
Hence, the square root is $2a-b-3c$.

Ex. 2. What value of x will make $x^4+4x^3+10x^2+14x+3$ a *perfect* square?

Rule.—"*Extract the square root in the ordinary way and put the remainder* $=0$."

$$\begin{array}{l|l} x^4+4x^3+10x^2+14x+3 & x^2+2x+3 \\ x^4 & \\ \hline 2x^2+2x \;\big|\; 4x^3+10x^2+14x+3 & \\ \quad\big|\; 4x^3+4x^2 & \\ \hline 2x^2+4x+3 \;\big|\; 6x^2+14x+3 & \\ \quad 6x^2+12x+9 & \\ \hline \quad 2x-6 & \end{array}$$

Putting $2x-6=0$, we have $2x=6$. $\therefore x=3$.

Ex. 3. The remainder after finding the first two terms of a square root of the form ax^2+bx+c is $-6x^2+4x+1$; determine the root.

$(ax^2+bx+c)^2$ is the whole expression of which the square root is ax^2+bx+c and $(ax^2+bx)^2$ is that portion of the expression which is necessary for finding the first two terms of the root.

$\therefore (ax^2+bx+c)^2-(ax^2+bx)^2=-6x^2+4x+1.$
$\therefore (2ax^2+2bx+c)c \quad =-6x^2+4x+1$
$\therefore (2acx^2+2bcx+c^2) \quad =-6x^2+4x+1.$

Since this is true for all values of x, the co-efficients of the like powers of x are equal.

$\therefore \quad c^2=1 \quad \therefore c=+1$;
$\quad 2bc=4 \quad \therefore b=+2$;
$2ac=-6 \quad \therefore a=-3$.

$\therefore$ the root is $-3x^2+2x+1$ or $3x^2-2x-1$,

or thus: The form of the trial divisor after finding the first two terms must be $2a^2+2bx$

$\therefore$ the form of the remainder must be $(2ax^2+2bx+c)c$

$\therefore$ $(2ax^2+2bx+c)c=-6x^2+4x+1$.

Hence a, b and c can be found.

Ex. 4. Find the relation between a, b and c in order that ax^2+bx+c may be an *exact* square.

If ax^2+bx+c be an exact square, it must be the square of a binomial of the form $x\sqrt{a}+\sqrt{c}$

$\therefore$ $ax^2+bx+c=(x\sqrt{a}+\sqrt{c})^2=ax^2+2x\sqrt{ac}+c$.

$\therefore$ $b=2\sqrt{ac}$.

$\therefore$ $b^2=4ac$.

Ex. 5. Find the relation between a, b, c and d, in order that $x^4+ax^3+bx^2+cx+d$ may be a *perfect* square.

If the given expression be an exact square, it must be the square of a trinomial of the form $x^2+px+\sqrt{d}$.

$\therefore$ $x^4+ax^3+bx^2+cx+d$.

$=(x^2+px+\sqrt{d})^2=x^4+2px^3+(p^2+2\sqrt{d})x^2+2px\sqrt{d}+d$.

Equating the co-efficients of the like powers of x,

$$a=2p;\ b=p^2+2\sqrt{d};\ c=2p\sqrt{d}.$$

$\therefore$ $p=\frac{a}{2}$; $b=\frac{a^2}{4}+2\sqrt{d}$ and $c=a\sqrt{d}$.

$\therefore$ $4b-a^2=8\sqrt{d}$ and $c^2=a^2d$,

or thus: Extract the square root in the ordinary way thus:

$$\begin{array}{r|l}
x^4+ax^3+bx^2+cx+d & x^2+\frac{a}{2}x+\frac{b-\frac{a^2}{4}}{2} \\
\hline
x^4 & \\
\hline
2x^2+\frac{a}{2}x \;\Big|\; ax^3+bx^2+cx+d & \\
ax^3+\frac{a^2}{4}x^2 & \\
\hline
2x^2+ax+\frac{b-\frac{a^2}{4}}{2} \;\Big|\; x^2\left(b-\frac{a^2}{4}\right)+cx+d & \\
x^2\left(b-\frac{a^2}{4}\right)+\frac{a}{2}\left(b-\frac{a^2}{4}\right)x+\frac{1}{4}\left(b-\frac{a^2}{4}\right)^2 & \\
\hline
 & \left(c-\frac{a}{2}\left(b-\frac{a^2}{4}\right)\right)x+d-\frac{1}{4}\left(b-\frac{a^2}{4}\right)^2
\end{array}$$

Now this remainder must vanish identically if the given expression is a perfect square. It will vanish if

$$c-\frac{a}{2}\left(b-\frac{a^2}{4}\right)=0 \text{ and, } d-\frac{1}{4}\left(b-\frac{a^2}{4}\right)^2=0.$$

$$\therefore\ d=\frac{1}{4}\left[b-\frac{a^2}{4}\right]^2. \qquad \therefore\ \sqrt{d}=\frac{1}{2}\left[b-\frac{a^2}{4}\right] \qquad \therefore\ 8\sqrt{d}=4b-a^2.$$

$$\therefore\ c-a(\sqrt{d})=0. \qquad \therefore\ c=a\sqrt{d}. \qquad \therefore\ c^2=a^2d.$$

Ex. 6. Extract the square root of $1-x$ to 4 terms

$$\begin{array}{l|l}
1-x & 1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16} \\
\hline
2-\frac{x}{2}-x & 1 \\
\quad -x+\frac{x^2}{4} & \\
\hline
2-x-\frac{x^2}{8} & -\frac{x^2}{4} \\
\quad -\frac{x^2}{4}+\frac{x^3}{8}+\frac{x^4}{64} & \\
\hline
2-x-\frac{x^2}{4}-\frac{x^3}{16} & -\frac{x^3}{8}-\frac{x^4}{64} \\
\quad -\frac{x^3}{8}+\frac{x^4}{16}+\frac{x^5}{64}+\frac{x^6}{256} & \\
\hline
 & -\frac{5x^4}{64}-\frac{x^5}{64}-\frac{x^6}{256}\ \text{(Remdr.)}
\end{array}$$

Hence $\sqrt{(1-x)}=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-$ &c.

$$\therefore 1-x=\left[1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}\right]^2-\frac{5x^4}{64}-\frac{x^5}{64}-\frac{x^6}{256}$$

Changing the sign of x, we have $\sqrt{(1+x)}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-$ &c.

Ex. 7. The product of any four quantities which increase or decrease by a common difference together with the fourth power of that difference is a perfect square

Let $a+x$ be the 1st term and x the common difference

Then $(a+x)(a+2x)(a+3x)(a+4x)+x^4$ shall be a perfect square.

The expression $= \{(a+x)(x+4x)\}\{(a+2x)(a+3x)\} + x^4$

$= (x^2+5ax+4x^2)(a^2+5ax+6x^2)+x^4$

$= p(p+2x^2)+x^4$ .....putting p for $a^2+5ax+4x^2$

$= p^2+2px^2+x^4$

$= (p+x^2)^2$

$= (a^2+5ax+4x^2+x^2)^2$

$= (a^2+5ax+5x^2)^2$.

Hence $x(x+1)(x+2)(x+3)+1$ and $(x-2)(x+2)(x+4)+16$ are perfect squares.

I. E —The prodact of any four consecutive numbers together with unity is a perfect square ; and the product of any four consecutive odd or even numbers together with 16 *is a perfect square.*

Ex. 8. If three quantities increase or decrease by a common difference, then the product of the first and the third together with the square of the common difference = square of the second.

Let $a+x$, $a+2x$, $a+3x$ be the three quantities x being the common difference

$(a+x)(a+3x)+x^2=(a+2x)^2$.

Now $(a+x)(x+3x)+x^2 = a^2+4ax+3x^2+x^2$

$= a^2+4ax+4x^2$

$= (a+2x)^2$.

Hence $(a+1)(a+3)+1=(a+2)^2$;

and $(a-2)(a-6)+4=(a-4)^2$;

and $(a-b)(a-5b)+4b^2=(a-3b)^2$.

EXERCISE 82.

Find, *by the Method of Indeterminate Co-efficients*, the square roots of—

1. $9x^4-12x^3+10x^2-4x+1$.
2. $x^4-\frac{1}{2}x+\frac{3}{2}x^2+\frac{1}{16}-2x^3$.
3. $x^2+\frac{2ax}{3}-bx+\frac{a^2}{9}+\frac{b^2}{4}-\frac{ab}{3}$
4. $x^4-6x^3y+13x^2y^2-12xy^3+4y^4$.
5. $x^6+6x^5y+15y^2x^4+6xy^5+20x^3y^3+y^6+15x^2y^4$
6. $x^6-12x^5+60x^4-160x^3+240x^2-182x+49$.
7. $x^8-2a^2x^6-a^4x^4+2a^6x^2+a^8$.
8. $729-2916x^2+4860x^4-4320x^6+2160x^8-576x^{10}+64x^{12}$.
9. $16a^4+16a^3-4a^2-28a-11+\frac{6}{a}+\frac{9}{a^2}$.

10. $9\frac{a^4}{b^4}+24\frac{a^3}{b^3}+34\frac{a^2}{b^2}+12\frac{a}{b}-7\ -12\frac{b}{a}+4\ \frac{b^2}{a^2}$.

11. $4(x-2)^4+20(x-2)^3+(x-2)^2-60(x-2)+36$.

12. $(a-1)^4+6(a-1)^3+5(a-1)^2-12(a-1)+4$.

Find the square roots of the following reducing each to the form $(A+B)^2$:—

13. $a^2+b^2+c^2+2ab-2ac-2bc$.

14. $a^4+4b^4+c^4-2a^2b^2+2a^2c^2-4b^2c^2$.

15. $\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)$. 16. $a^4+\frac{1}{a^4}+2\left(a^2+\frac{1}{a^2}\right)+3$.

17. $\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)+6$. 18. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{2x}{y}+\frac{2y}{x}+3$.

For what value of x will each of the following expressions be an *exact* square ?

19. $x^4+6x^3+33x^2+3x+31$. 23. $x^4+4x^3+2x^2-3x-1$.

21. $x^4+4bx^3+2b^2x^2-3b^3x-a^4$.

22. $4x^4+24x^3+44x^2+12x+124$.

23. $25x^4-30ax^3+49a^2x^2-22a^3x+4a^4$.

24. $x^4+6x^3+10x^2+x+20$.

25. The remainder after finding the first two terms of a square root of the form ax^2+bx+c is $4x^2-2x+1$. Find the whole root.

26. The remainder after finding the first two terms of a square root of the form px^2+qx+r is $6x^2+18x+9$; determine the whole root.

27. The remainder after finding the first two terms of a square root of the form ax^2+bx+c is $12x^2+12x+9$. Find the whole root.

28. The remainder after finding the first three terms of a square root of the form pa^3+qa^2+ra+s is $6a^3+6a^2+18a+9$. Find the whole root.

29. Find the relation between a, b, c and d in order that $x^4+4ax^3+6bx^2+4cx+d$ may be a perfect square.

30. Find the relation between p and q in order that x^2+px+q may be a perfect square.

31. Find the relation between p, q, r and s, in order that $x^4+px^3+qx^2+rx+s$ may be a perfect square.

32. Find the relation between p and q, in order that $x^4+6p^2q^2x^2+(p^2+q^2)x^2+p^2x+p^2q^2$ may be a perfect square.

33. If $3x^2+ax+b$ be the square root of $9x^4+6x^3-11x^2-4x+4$, determine $a+b$.

34. Determine k in order that $4x^4-16x^3+36x^2-40x+k$ may be a perfect square.

35. Determine m so that $x^4-8x^3+22x^2+mx+9$ may be a perfect square.

36. Determine k and l so that $9x^4+6x^3-17x^2+kx+l$ may be a perfect square.

37. Shew by inspection that the square root of the expression $4x^6-4x^5+x^4+12x^3-6x^2+6$ does not contain a term involving x.

38. If $4a^4+12a^3b+pa^2b^2+6ab^3+b^4$ is a perfect square find p.

39. Find what term is wanting to make the following expression a *complete* square;

$$(a^2x^4+64b^2)-4(ax^2+8b)(a-b)x.$$

Find the square root of each of the following to 5 terms :—

40. $1-a^2$; $1+a^2$.

41. a^2-x^2 ; a^2+x^2.

42. $1+x+x^2$; $1-x+x^2$.

43. $a+b$; $a-b$.

Shew that each of the following is a perfect square :—

44. $4(x+1)(x+2)(2x+1)(2x+3)+1$;

45. $(x^2+x+1)(x^2+3x+3)+(x+1)^2$.

46. $(x+2)(2x+3)(3x+4)(4x+5)+(x+3)^4$;

47. $(x-a-b)(x-3a-5b)+(a+2b)^2$.

48. $(a^2+2a+2)(a^2-2a+2)(a^4+3)(a^4+2)(a^4+1)+1$.

49. $(a^2-2ab+2b^2)(a^2+2ab+2b^2)(a^4+b^4)(a^4+2b^4)(a^4+3b^4+b^8)$.

50. $(x+a+b)(x+a+3b)(x+a-b)(x+a-3b)+16b^4$.

51. $(a-b)^2(b-c)^2+(b-c)^2(c-a)^2+(c-a)^2(a-b)^2$.

52. $2\{(x+y)^4+(2y-x)^4+(2x-y)^4\}$.

176. The fourth root of an expression is the square root of the square root of the expression.

Thus, $\sqrt{\sqrt{16}}=\sqrt{(\sqrt{16}=\sqrt{4}=2)}$.

The sixth root of an expression is the square root of the cube root of the expression ; or the cube root of the square root of the expression.

Thus $\sqrt{\sqrt[3]{64}}=\sqrt{\sqrt[3]{64}}=\sqrt{4}=2$; or $=\sqrt[3]{\sqrt{64}}=\sqrt[3]{8}=2$.

177. Miscellaneous Examples.

Ex. 1. Extract the cube root of

$x^6+6x^5+15x^4+20x^3+15x^2+6x+1$,

by *the Method of Indeterminate Co-efficients*.

If the given expression be a perfect cube, it must be the cube of a trinomial of the form x^2+px+1.

$\therefore x^6+6x^5+15x^4+20x^3+15x^2+6x+1$

$=(x^2+px+1)^3$

$=x^6+3px^5+$(terms containing x^4, x^3, x^2 and x)$+1$.

Since this is true for all values of x, the co-efficients of the like powers of x are equal.

$\therefore 6=3p \therefore p=2$.

This value must be verified by equating the co-efficients of the other powers of x.

Hence the cube root of the given expression is x^2+2x+1.

Ex. 2. Find the relation between a, b, c and d.

When ax^3+bx^2+cx+d is a *complete* cube.

Let $ax^3+bx^2+cx+d=(px+q)^3=p^3x^3+3p^2qx^2+3pq^2x+q^3$.

The co-efficients of the *like* powers of x are equal.

$\therefore a=p^3$; $b=3p^2q$; $c=3pq^2$ and $d=q^3$.

$\therefore \frac{a}{d}=\frac{p^3}{q^3}=\left(\frac{p}{q}\right)^3$;

$\frac{b}{c}=\frac{3p^2q}{3pq^2}=\frac{p}{q} \therefore \frac{b^3}{c^3}=\left(\frac{p}{q}\right)^3$.

$\therefore \frac{a}{d}=\frac{b^3}{c^3} \therefore ac^3=b^3d$ ..... (1).

Again, $ad=p^3q^3$ and $bc=3p^3q^3$. $\therefore 3ad=bc$ ..... (2).

Ex. 3. The product of any three consecutive quantities together with the middle number is the cube of the middle one.

i. e., $(a+1)(a+2)(a+3)+(a+2)=(a+2)^3$.

$(a+1)(a+2)(a+3)+(a+2)=(a+2)\{(a+1)(a+3)+1\}$

$=(a+2)(a+2)^2$

$=(a+2)^3$.

Ex. 4. The product of any three quantities which increase or decrease by a common difference together with the product of the square of that difference and the middle number is the cube of the middle one.

Let the three quantities be $a+b$, $a+3b$, $a+5b$, the common difference being $2b$.

Then $(a+b)(a+3b)(a+5b)+(2b)^2(a+3b)=(a+3b)^3$.

Left side expression $=(a+3b)\{(a+b)(a+5b)+(2b)^2\}$

$=(a+3b)\{a^2+6ab+5b^2+4b^2\}$

$=(a+3b)(a+3b)^2=(a+3b)^3$.

EXERCISE 83.

Extract the cube root of each of the following expressions, by the *Method of Indeterminate Coefficients*:—

1. $x^6-9x^5+33x^4-63x^3+66x^2-36x+8$
2. $8x^6+48cx^5+60c^2x^4-80c^3x^3-90c^4x^2+108c^5x-27c^6$.
3. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
4. $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.
5. $x^6+15x^2+\frac{15}{x^2}+20+\frac{6}{x^4}+\frac{1}{x}+6x^4$.
6. $27x^6-6x-54x^5-44x^3+63x^4+21x^2+1$.
7. (*a*) Find the *fourth* root of $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12$.

   (*b*) Find the *sixth* root of:—

   $x^6-12x^5+60x^4-160x^3-240x^2-192x+64$.

8. Find the conditions that $x^3+3ax^2+3bx+c$ should be a *perfect* cube.

9. Find the relation between p, q and r, in order that x^3+px^2+qx+r shall be a *perfect* cube.

10. Find the conditions that $ax^4+bx^3+cx^2+dx+e$ should be a *complete fourth power*.

Shew that each of the following is a complete cube:—

11. $\left(\frac{a+b}{a-b}\right)^3+\left(\frac{a-b}{a+b}\right)^3+6\,\frac{a^2+b^2}{a^2-b^2}$.
12. $(x-b)(x-4b)(x-7b)x+9b^2(x-4b)$.
13. $(x+a-b)(x+2a-3b)(x+3a-5b)+(a-2b)^2(x+2a-3b)$.
14. Show that

    $(a+b)^3+3(a^2-b^2)(a+b)+3(a^2-b^2)(a-b)+(a-b)^3=8a^3$

15. Show that $b^2=3ac$ and $c^2=3bd$,

    if the expression

    ax^3+bx^2+cx+d is a perfect cube

CHAPTER XXII.

INDICES.

178. Definition of a^m. The product of m factors each equal to a is denoted by a^m.

The Index Law is $a^m \times a^n = a^{m+n}$. We have already proved this in Art. 42 when m and n are *positive* integers. We shall assume that the law is true for *all* values of m and n and proceed to find the meanings of a_0, a^{-n} and $a^{\frac{m}{n}}$.

179. (i) *To find the meaning of* a^0.

Now $a^m + a^n = a^{m+n}$.

Let $m = 0$; then $a^0 \times a^n = a^{0+n} = a^n$.

$\therefore a^0 = \dfrac{a^n}{a^n} = 1$.

Thus any quantity raised to the power zero is equal to 1.

(ii) *to find the meaning of* a^{-n}.

Now $a^m \times a^n = a^{m+n}$.

Let $m = -n$; then $a^{-n} \times a^n = a^{-n+n} = a^0 = 1$.

$\therefore a^{-n} = \dfrac{1}{a^n}$ and $a^n = \dfrac{1}{a^{-n}}$.

Thus, $a^{-3} = \dfrac{1}{a^3}$; $a^{-\frac{1}{2}} = \dfrac{1}{a^{\frac{1}{2}}}$; $a^{m-n} = \dfrac{1}{a^{n-m}}$.

a^{-n} **is the** *reciprocal* **of** a^n.

(iii) *To find the meaning of* $a^{\frac{m}{n}}$, *when* m *and* n *are positive integers.*

$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}}$ . . to n factors $= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}}$, . . to n terms. $= a^{\frac{m}{n} \times n} = a^m$.

Hence $a^{\frac{m}{n}}$ is the n^{th} root of a^m; i. e., $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

In a fractional index, the numerator denotes a power and the denominator a root.

Thus, $a^{\frac{1}{n}} = \sqrt[n]{a^1} = \sqrt[n]{a}$; $a^{\frac{2}{3}} = \sqrt[3]{a^2}$; $a^{\frac{1}{4}} = \sqrt[4]{a}$.

$a^{\frac{p}{q}} = \left(\dfrac{1}{a^q}\right)^p = \left(a^p\right)^{\frac{1}{q}}$.

180. In like manner we can prove that—

(i) To prove that $(a^m)^n = a^{mn}$.

$(a^m)^n = a^m \times a^m \times a^m \times$ . . . . . n factors.

$= (a \times a \times a$ . . . . m factors$) \times (a \times a \times a$ . . . . .

m factors$) \times (a \times a \times a$ . . . m factors$) \times ($

. . . . to n groups.

$\because$ $(a^m = a \times a \times a$ . . . . m factors)

$= a \times a \times a$ . . . . . mn factors.

$\because$ there are n groups each containing m factors

$= a^{mn}$.

(ii) $(ab)^n = a^n b^n$.

$(ab)^n = ab \times ab \times ab$ . . . . . n factors.

$= (a \times b) \times (a \times b) \times (a \times b)$ . . . . . n groups.

$= a \times a \times a$ . . . n factors $\times b \times b \times$ . . . n factors.

$= a^n b^n$.

(iii) Similarly $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Ex. 1. Express, with fractional indices, $\sqrt[6]{a^7 b^{10} c^9 d^{-15}}$.

$\sqrt[6]{a^7 b^{10} c^9 d^{-15}} = (a^7 b^{10} c^9 d^{-15})^{\frac{1}{6}} = a^{\frac{7}{6}} b^{\frac{10}{6}} c^{\frac{9}{6}} d^{-\frac{15}{6}} = a^{\frac{7}{6}} b^{\frac{5}{3}} c^{\frac{3}{2}} d^{-\frac{5}{2}}$

Ex. 2. Simplify x^{2a-b-c}, x^{2b-c-a}, x^{2c-a-b}.

The expression $= x^{2a-b-c+2b-c-a+2c-a-b} = x^0 = 1$.

Ex. 3. Simplify $\{(a+b)^{m-n}(a+b)^{m+n}\}^{\frac{1}{4mn}}$.

The expression $= \{(a+b)^{m-n+m+n}\}^{\frac{1}{4mn}} = \{(a+b)^{2m}\}^{\frac{1}{4mn}}$

$= (a+b)^{\frac{2m}{4mn}}$

$= (a+b)^{\frac{1}{2n}}$

Ex. 4. Multiply $x^{-\frac{1}{2}} - x^{-\frac{1}{4}} y^{-\frac{1}{4}} + y^{-\frac{1}{2}}$ by $x^{-\frac{1}{4}} + y^{-\frac{1}{4}}$.

Putting a for $x^{-\frac{1}{4}}$, b for $y^{-\frac{1}{4}}$.

Multiplicand $= a^2 - ab - b^2$ and multiplier $= a + b$

Hence the product

$$=(a^2-ab+b^2)(a+b)=a^3+b^3$$
$$=\left(x^{-\frac{1}{4}}\right)^3+\left(y^{\frac{1}{4}}\right)^3$$
$$=x^{-\frac{3}{4}}+y^{\frac{3}{4}}.$$

Ex. 5 Divide $a-2a^{\frac{1}{2}}+1$ by $a^{\frac{1}{3}}-2a^{\frac{1}{6}}+1$.

Putting x for $a^{\frac{1}{6}}$,

the dividend $=x^6-2x^3+1$; and the divisor $=x^2-2x+1$.

$$\because a=\left(a^{\frac{1}{6}}\right)^6;\ a^{\frac{1}{2}}=\left(a^{\frac{1}{6}}\right)^3;\ a^{\frac{1}{3}}=\left(a^{\frac{1}{6}}\right)^2.$$

Hence $\dfrac{a-2a^{\frac{1}{2}}+1}{a^{\frac{1}{3}}-2a^{\frac{1}{6}}+1}=\dfrac{x^6-2x^3+1}{x^2-2x+1}$

$$=\frac{(x^3-1)^2}{(x-1)^2}$$
$$=\left(\frac{x^3-1}{x-1}\right)^2$$
$$=(x^2+x+1)^2$$
$$=x^4+2x^3+3x^2+2x+1$$
$$=\left(a^{\frac{1}{6}}\right)^4+2\left(a^{\frac{1}{6}}\right)^3+3\left(a^{\frac{1}{6}}\right)^2+2a^{\frac{1}{6}}+1$$
$$=a^{\frac{2}{3}}+2a^{\frac{1}{2}}+3a^{\frac{1}{3}}+2a^{\frac{1}{6}}+1.$$

Ex. 6. Extract the square root of:—

$$a+b^{\frac{2}{3}}-2a^{\frac{1}{2}}b^{\frac{1}{3}}+c^{\frac{1}{3}}+2a^{\frac{1}{2}}c^{\frac{1}{4}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}.$$

Arrange the expression according to descending powers of a.

$$a-2a^{\frac{1}{2}}b^{\frac{1}{3}}+2a^{\frac{1}{2}}c^{\frac{1}{4}}+b^{\frac{2}{3}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}}\ \Big|\ a^{\frac{1}{2}}-b^{\frac{1}{3}}+c^{\frac{1}{4}}$$

a

$2a^{\frac{1}{2}}-b^{\frac{1}{3}}$ | $-2a^{\frac{1}{2}}b^{\frac{1}{3}}+2a^{\frac{1}{2}}c^{\frac{1}{4}}+b^{\frac{2}{3}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}}$

$-2a^{\frac{1}{2}}b^{\frac{1}{3}}\qquad +b^{\frac{2}{3}}$

$2a^{\frac{1}{2}}-2b^{\frac{1}{3}}+c^{\frac{1}{4}}$ | $2a^{\frac{1}{2}}c^{\frac{1}{4}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}}$

$2a^{\frac{1}{2}}c^{\frac{1}{4}}-2b^{\frac{1}{3}}c^{\frac{1}{4}}+c^{\frac{1}{2}}$

Hence, the square root is $a^{\frac{1}{2}}-b^{\frac{1}{3}}-c^{\frac{1}{4}}$.

Ex. 7. Simplify :—

$$\frac{1}{1+x^{m-n}+x^{m-p}}+\frac{1}{1+x^{n-m}+x^{n-p}}+\frac{1}{1+x^{p-m}+x^{p-n}}.$$

The first term $=\dfrac{x^{-m}}{x^{-m}(1+x^{m-n}+x^{m-p})}=\dfrac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}$.

The second term $=\dfrac{x^{-n}}{x^{-n}(1+x^{n-m}+x^{n-p})}=\dfrac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}}$.

The third term $=\dfrac{x^{-p}}{x^{-p}(1+x^{p-m}+x^{p-n})}=\dfrac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}$.

Hence, the given expression

$$=\frac{x^{-m}}{x^{-m}+x^{-n}+x^{-p}}+\frac{x^{-n}}{x^{-n}+x^{-m}+x^{-p}}+\frac{x^{-p}}{x^{-p}+x^{-m}+x^{-n}}$$

$$=\frac{x^{-m}+x^{-n}+x^{-p}}{x^{-m}+x^{-n}+x^{-p}}$$

$$=1.$$

Ex. 8. If $a^b=b^a$, shew that $\left(\frac{a}{b}\right)^{\frac{a}{b}}=a^{\frac{a}{b}-1}$

and if $a=2b$, shew that $b=2$.

Since $a^b=b^a$, $\therefore a=b^{\frac{a}{b}}$. Hence $\left(\frac{a}{b}\right)^{\frac{a}{b}}=\frac{a^{\frac{a}{b}}}{b^{\frac{a}{b}}}=a^{\frac{a}{b}}\div b^{\frac{a}{b}}=b^{\frac{a}{b}\cdot 1}$.

Since $a^b=b^a$ and $a=2b$. $\therefore (2b)^b=b^{2b}=(b^2)^b$. $\therefore 2b=b^2$ $\therefore b=2$

Ex. 9. If $x=(a+\sqrt{a^2+b^3})^{\frac{1}{3}}+(a-\sqrt{a^2+b^3})^{\frac{1}{3}}$,

shew that $x^3+3bx-2a=0$.

Putting p for $a+\sqrt{a^2+b^3}$ and q for $a-\sqrt{a^2+b^3}$,

we have $x=p^{\frac{1}{3}}+q^{\frac{1}{3}}$.

$$\therefore x^3=(p^{\frac{1}{3}}+q^{\frac{1}{3}})^3=p+q+3p^{\frac{1}{3}}q^{\frac{1}{3}}(p^{\frac{1}{3}}+q^{\frac{1}{3}})$$

$$=p+q+3p^{\frac{1}{3}}q^{\frac{1}{3}}(x)$$

But $p+q=a+\sqrt{a^2+b^3}+a-\sqrt{a^2+b^3}=2a$

and $pq=(a+\sqrt{a^2+b^3})(a-\sqrt{a^2+b^3})=a^2-(a^2+b^3)=-b^3$

$\therefore p^{\frac{1}{3}}q^{\frac{1}{3}}=(-b^3)^{\frac{1}{3}}=-b$

$\therefore x^3=2a+3(-b)(x)=2a-3bx$.

$\therefore x^3+3bx-2a=0$.

EXERCISE 84.

1. Multiply $a^{-2}-1+a^2$ by $a^{-1}+a$.

2. Multiply $a^{\frac{3}{4}}+a^{\frac{1}{2}}+a^{\frac{1}{4}}+1$ by $a^{\frac{1}{4}}-1$.

3. Multiply $x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{\frac{1}{4}}y$ by $x^{\frac{1}{4}}-y^{\frac{1}{2}}$.

4. Divide $a-b^{\frac{1}{2}}+c^{\frac{1}{4}}+3a^{\frac{1}{4}}b^{\frac{1}{2}}c^{\frac{1}{4}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{2}}+c^{\frac{1}{4}}$.

5. Divide $x^{\frac{3p}{2}}-y^{\frac{3p}{2}}$ by $x^{\frac{p}{2}}-y^{\frac{p}{2}}$.

6. Simplify $\left(\sqrt[bc]{x}\right)^{b-c}+\left(\sqrt[ca]{x}\right)^{c-a}+\left(\sqrt[ab]{x}\right)^{a-b}$.

7. Simplify :—

$$\frac{a^{\frac{1}{2}}}{(a^{\frac{1}{2}}-b^{\frac{1}{2}})(a^{\frac{1}{2}}-c^{\frac{1}{2}})}+\frac{b^{\frac{1}{2}}}{(b^{\frac{1}{2}}-a^{\frac{1}{2}})(b^{\frac{1}{2}}-c^{\frac{1}{2}})}+\frac{c^{\frac{1}{2}}}{(c^{\frac{1}{2}}-a^{\frac{1}{2}})(c^{\frac{1}{2}}-b^{\frac{1}{2}})}.$$

8. Resolve into factors $x^{-2}-7x^{-1}+12$; $x^{\frac{2}{3}}-y^{\frac{2}{3}}$; $a^{\frac{1}{4}}+b^{\frac{1}{4}}$; $x^3+x^{-3}-2$; and $(a^{-3}+a^{-2}+a^{-3})(a^{-1}+a^{-2}+a^{-4})-a^{-6}$.

9. Find the H. C. F. and L. C. M. of—

$$x(x+\sqrt{xy})-y(y+\sqrt{xy}) \text{ and } x^2+y^2+3(x+y)\sqrt{xy}+4xy.$$

10. Express $\left\{\left(\frac{a-m}{b-n}\right)^{-\frac{1}{p}}\times\left(\frac{-m}{\frac{b}{-n}}\right)\right\}^{\frac{q}{m+n}}$ in the simplest form. What must be the relation between, p and q so that the expression may be equal to $\frac{a}{b}$?

11. Extract the square root of $y^2x^{-1}+\frac{1}{4}x^2y^{-1}+\frac{2y^{\frac{3}{2}}-x}{x^{\frac{1}{2}}y^{\frac{1}{2}}}$

12. Extract the square root of :—

$$4x-12x^{\frac{1}{2}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}+16x^{\frac{1}{2}}z^{\frac{1}{4}}-24y^{\frac{1}{3}}z^{\frac{1}{4}}+16z^{\frac{1}{2}}.$$

13. Find the square of $x^{\frac{1}{3}}-2x^{\frac{1}{4}}+x^{\frac{1}{2}}$.

14. Simplify

$$(x^{a+b})^{a-b}\times(x^{a-b})^{a+b}\times(x^{b-a})^{a+b}.$$

15. Simplify

$$(a+b)^m+(a-b)^m\times(a^2+b^2)^m$$

16. Simplify $\left(\frac{x^a}{x^b}\right)^{a+b}\div\left(\frac{x^{a+b}}{x^{a-b}}\right)^{a}$

17. Simplify

$$\left[\sqrt{\frac{x^2}{y^4}}+\sqrt{\frac{y^3}{x}}\right]^{12}\times x^{22}$$

18. Simplify

$$\left(\frac{x}{x^n}\right)^{m+n}\times\left(\frac{x^n}{x^l}\right)^{n+l}\times\left(\frac{x^l}{x^m}\right)^{l+m}$$

19. Simplify

$$\left(\frac{x^p}{x^q}\right)^{p+q}\div\left(\frac{x^p+q}{x^p-q}\right)^{\frac{p}{q}^2}$$

20. Simplify

$$\left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2}\times\left(\frac{x^m}{x^n}\right)^{m^2+mn+n^2}\times\left(\frac{x^n}{x^l}\right)^{n^2+ln+l^2}$$

21. Simplify

$$\left\{\left(a^x\right)^{x-\frac{1}{x}}\right\}^{\frac{1}{x+1}}$$

22. Simplify

$$\left\{(a^{\frac{3}{4}}b^{\frac{2}{3}})^{-\frac{1}{8}}\times(a^{-\frac{1}{4}}b^{-\frac{1}{8}})^{12}\right\}^{-20}$$

23. If $a=2^{\frac{1}{3}}-2^{\frac{1}{3}}$, prove that $2a^3+6a=3$.

CHAPTER XXIII

RATIO AND PROPORTION.

181. Definitions.

Ratio is the relation that one quantity bears to another of the *same* kind, the comparison being made by considering what multiple, part or parts, one quantity is of the other.

The ratio of a to b is usually written $a : b$. a and b are called its terms. Of these a is called the **antecedent** and b the **Consequent**.

The ratio $a : b$ is said to be one of **greater inequality**, one of **equality** or one of **less inequality**, according as a is greater than, equal to or less than b.

Proportion. If a, b are quantities of the same kind and c, d are quantities of the same kind (but not necessarily of the same kind as a and b) a, b, c, d are said to be in **proportion** if the ratio of a to b is equal to the ratio of c to d; that is, when $a : b = c : d$. a, b, c, d are called **proportionals**; a and d are called **extremes** and b and c are called **means**, and d is said to be a **fourth proportional** to a, b, c.

If a, b, c are magnitudes of the same kind such that $a : b = b : c$ then b is called a **mean proportional** to a and c, and c a **third proportional to** a and b.

If a, b, c, d are quantities of the same kind such that $a : b = b : c = c : d = d : e = \ldots\ldots\ldots$ then $a, b, c, d, e \ldots$ are said to be in **continued proportion**.

182. Theorems on Ratio and Proportion.

(1) If $a > b$, then $\frac{a}{b} > \frac{a+x}{b+x}$.

Since $a > b$, $\therefore ax > bx$.

$\therefore ax + ab > bx + ab$.

$\therefore a(x+b) > b(x+a)$

$\therefore \frac{a}{b} > \frac{x+a}{x+b}$, i. e., $\frac{a+x}{b+x}$.

(2) If $a < b$, than $\frac{a}{b} < \frac{a+x}{b+x}$.

Since $a<b$, $\therefore\ ax<bx$.

$\therefore\ ax+ab<bx+ab$,

$\therefore\ a(x+b)<b(x+a)$.

$\therefore\ \frac{a}{b}<\frac{x+a}{x+b}$, i. e., $\frac{a+x}{b+x}$.

From (1) and (2) we infer that *a ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.*

Similarly it can be proved that if $a>b$, $\frac{a}{b}<\frac{a-x}{b-x}$;

and that if $a<b$, $\frac{a}{b}>\frac{a-a}{b-x}$.

That is, *a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by subtracting the same quantity from both its terms.*

183. I. If $\frac{a}{b}=\frac{c}{d}$ then $ad=bc$

That is, *when four quantities are proportionals, the product of the extremes is equal to the product of the means.*

II. If $\frac{a}{b}=\frac{c}{d}$ then

adding 1 to both sides

$$\frac{a+b}{b}=\frac{c+d}{d} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

subtracting 1 from both sides

$$\frac{a-b}{b}=\frac{c-d}{d} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and dividing (i) and (ii)

$$\frac{a+b}{a-b}=\frac{c+d}{c-d} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

184. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

then each of the ratios $=\left(\frac{pa^n+qc^n+re^n}{pb^n+qd^n+rf^n}\right)^{\frac{1}{n}}$

where p, q, r, n are any quantities whatever.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=bk$; $c=kd$ and $e=kf$.

$\therefore p.an = p(kb)^n$; $qcn = q(kd)^n$ and $ren \equiv r(kf)^n$.

$\therefore p(kb)^n + q(kd)^n + r(kf)^n = pan + qcn + ren$

$\therefore k^n = \dfrac{pan + qcn + ren}{pbn + pdn + rfn}$

$\therefore k = \left(\dfrac{pan + qcn + ren}{pbn + pdn + rfn}\right)^{\frac{1}{n}}$

Note – By giving to p, q, r, n different values, many particular cases may be deduced ; or they may be proved by the same method.

If $p=q=r=n=1$, then each of the ratios $\dfrac{a}{b}$, $\dfrac{c}{d}$ and $\dfrac{e}{f} = \dfrac{a+c+e}{b+d+f}$.

That is, *when a number of fractions are equal each of them is equal to the sum of all the numerators divided by the sum of all the denominators.*

The same may be proved independently thus :—

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k$; then $kb = a$; $kd = c$ and $kf = e$.

$\therefore kb + kd + kf = a + c + e$.

$\therefore k(b+d+f) = a+c+e$.

$\therefore k = \dfrac{a+c+e}{b+d+f}$

Ex 1. If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$, shew that $a : b = c : d$.

From the given relation we have $\dfrac{a+b+c+d}{a+b-c-d} = \dfrac{a-b+c-d}{a-b-c+d}$

$\therefore \dfrac{a+b+c+d+a+b-c-d}{a+b+c+d-a-b+c+d} = \dfrac{a-b+c-d+a-b-c+d}{a-b+c-d-a+b+c-d}$ .....

[II of Art 183]

$\therefore \dfrac{a+b}{a-b} = \dfrac{a-b}{c-d}$

$\therefore \dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$

$\therefore \dfrac{a+b+a-b}{a+b-a+b} = \dfrac{c+d+c-d}{c+d-c+d}$ .....[II of Art 183]

$\therefore \dfrac{a}{b} = \dfrac{c}{d}$ i.e., $a : b = c : d$.

Ex. 2. If $x = \dfrac{2ab}{b^2+1}$, find the value of $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$

Since $x=\dfrac{2ab}{b^2+1}$, $\dfrac{x}{a}=\dfrac{2b}{b^2+1}$, $\therefore \dfrac{a}{x}=\dfrac{b^2+1}{2b}$.

$\therefore \dfrac{a+x}{a-x}=\dfrac{b^2+1+2b}{b^2+1-2b}=\dfrac{(b+1)^2}{(b-1)^2}$. [II of Art. 183.]

$\therefore \dfrac{\sqrt{a+x}}{\sqrt{a-x}}=\dfrac{b+1}{b-1}$.

$\therefore \dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{b+1+b-1}{b+1-b+1}=b$.

Ex. 3. If $x=\dfrac{\sqrt[3]{p+1}+\sqrt[3]{p-1}}{\sqrt[3]{p+1}-\sqrt[3]{p-1}}$,

shew that $x^3-3px^2+3x-p=0$.

Since $x=\dfrac{\sqrt[3]{p+1}+\sqrt[3]{p-1}}{\sqrt[3]{p+1}-\sqrt[3]{p-1}}$, $\therefore \dfrac{x+1}{x-1}=\dfrac{\sqrt[3]{p+1}}{\sqrt[3]{p-1}}$ .....

[II of Art. 183.]

$\therefore \dfrac{(x-1)^3}{(x-1)^3}=\dfrac{p+1}{p-1}$.

$\therefore \dfrac{(x+1)^3+(x-1)^3}{(x+1)^3-(x-1)^3}=\dfrac{p+1+p-1}{p+1-p+1}$ .....[II of Art. 183.]

$\therefore \dfrac{x^3+3x}{3x^2+1}=p$.

$\therefore x^3+3x=3px^2+p$.

$\therefore x^3-3px^2+3x-p=0$.

Ex. 4. If $\dfrac{a}{b}-\dfrac{c}{d}=\dfrac{e}{f}$, shew that $\dfrac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}=\dfrac{ace}{bdf}$.

Let $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=k$, then $a=bk$, $c=dk$, $e=fk$.

$\therefore a^3b=b(b^3k^3) = b^4k^3$,

$2c^2e=2.(dk)^2.fk = 2d^2fk^3$,

$-3ae^2f=-3(bk)(fk)^2f = -3bf^3k^3$.

$\therefore$ by adding, $k^3(b^4+2d^2f-3bf^3)=a^3b+2c^2e-3ae^2f$.

$\therefore k^3=\dfrac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}$ and $k^3=\dfrac{a}{b}\times\dfrac{c}{d}\times\dfrac{e}{f}=\dfrac{ace}{bdf}$.

$\therefore \dfrac{a^3b+2c^2e-3ae^2f}{b^4+2d^2f-3bf^3}=\dfrac{ace}{bdf}$.

Ex. 5. If $\dfrac{ay-bx}{c}=\dfrac{cx-az}{b}=\dfrac{bz-cy}{a}$, shew that $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$.

Multiplying the terms of the given ratios by c, b, a respectively, we have $\frac{acy-bcx}{c^2}=\frac{bcx-abz}{b^2}=\frac{abz-acy}{a^2}$.

$\therefore$ each $=\frac{acy-bcx+bcx-abz-abz-acy}{c^2+b^2+a^2}$

.....[Art. 184.]

$=0$.

$\therefore \frac{acy-bcx}{c^2}=0$. $\therefore \frac{ay-bx}{c}=0$. $\therefore ay=bx$. $\therefore \frac{x}{a}=\frac{y}{b}$.

Again $\frac{bcx-abz}{b^2}=0$. $\therefore \frac{cx-az}{b}=0$. $\therefore cx=a^2$. $\therefore \frac{x}{a}=\frac{z}{c}$.

$\therefore \frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Ex. 6 If $\frac{a}{b}=\frac{c}{d}$, shew that $\frac{ma+nc}{mb+nd}=\sqrt{\frac{a^2+c^2}{b^2+d^2}}$.

Since $\frac{a}{b}=\frac{c}{d}$, $\therefore \frac{ma}{mb}=\frac{nc}{nd}$.

$\therefore$ each $=\frac{ma+nc}{mb+nd}$.....(1) [Art. 184.]

Again since $\frac{a}{b}=\frac{c}{d}$ $\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2}$.

$\therefore$ each of these $=\frac{a^2+c^2}{b^2+d^2}$.....(2) [Art. 184.]

We have $\frac{ma+nc}{mb+nd}=\frac{a}{b}$ and $\frac{a^2+c^2}{b^2+d^2}=\frac{a^2}{b^2}$.

$\therefore \frac{ma+nc}{mb+nd}=\sqrt{\frac{a^2+c^2}{b^2+d^2}}$.

EXERCISE 85.

1. If $x=\frac{4ab}{a+b}$, find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$.

2. If $x=\frac{\sqrt[3]{a+1}+\sqrt[3]{a-1}}{\sqrt[3]{a+1}-\sqrt[3]{a-1}}$, find the value of $\frac{x^3+3x}{3x^2+1}$.

3. If $x=\frac{\sqrt{m+1}+\sqrt{m-1}}{\sqrt{m+1}-\sqrt{m-1}}$, shew that $x^2-2mx+1=0$.

4. If $x=\frac{\sqrt{3p+3q}+\sqrt{2p-3q}}{\sqrt{2p+3q}-\sqrt{2p-2q}}$,

shew that $3qx^2 - 4px + 3q = 0$.

5. If $\dfrac{\sqrt{x^2+y^2} - \sqrt{x^2-y^2}}{\sqrt{x^2+y^2} + \sqrt{x^2-y^2}} = m$, shew that $\dfrac{1+m^2}{2m} = \dfrac{x^2}{y^2}$.

6. If $x = \dfrac{\sqrt[4]{a+1} + \sqrt[4]{a-1}}{\sqrt[4]{a+1} + \sqrt{a-1}}$,

shew that $x^4 - 4ax^3 + 6x^2 - 4ax + 1 = 0$.

7. If $(a+3b+2x+6y)(a-3b-2x+6y)$
$= (a-3b+2x-6y)(a+3b-2x-6y)$,

shew that $\dfrac{a}{b} = \dfrac{x}{y}$.

8. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, shew that :—(i) $\dfrac{a}{b} = \left(\dfrac{a^2 - ac + e^2}{b^2 - bd + f^2}\right)^{\frac{1}{2}}$;

(ii) $\dfrac{a}{b} = \left(\dfrac{a^3 - 3ace + e^3}{b^3 - 3bdf + f^3}\right)^{\frac{1}{3}}$;

and (iii) $\dfrac{a}{b} = \dfrac{3a - 4c + 5e}{3b - 4d + 5f}$.

9. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$,

shew that $\sqrt{(a+c+e)(b+d+f)} = \sqrt{ab} + \sqrt{cd} + \sqrt{ef}$

10. If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$,

shew that $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$.

11. If $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$, shew that :—(i) $\dfrac{a}{b+d} = \dfrac{c^3}{c^2d + d^3}$;

and (ii) $\dfrac{2a+3d}{3a-4d} = \dfrac{2a^3+3b^3}{3a^3-4b^3}$.

12. If $\dfrac{a}{b} = \dfrac{c}{d}$, shew that $\dfrac{a-c}{b-d} = \dfrac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$

13. If $\dfrac{a}{b} = \dfrac{b}{c}$, shew that $b^4 = \dfrac{a^2 - b^2 + c^2}{a^{-2} - b^{-2} + c^{-2}}$.

14. If $\dfrac{a}{b} = \dfrac{c}{d}$, shew that $\dfrac{a^2+b^2}{c^2+d^2} = \dfrac{a^2(c+d)}{c^2(a+b)}$.

If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, shew that $\dfrac{ay - bx}{c} = \dfrac{cx - az}{b} = \dfrac{bz - cy}{a}$.

16. If $ad=bc$, prove the following identities :—

(i) $\left(\frac{a+c}{b+d}\right)^2=\frac{a(a-c)^2}{b(b-d)^2}$; (ii) $\frac{a^2+b^2}{a^2-b^2}=\frac{ac+bd}{ac-bd}$;

(iii) $\frac{a^2+ab+b^2}{a^2+ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$; (iv) $\frac{a^2+ab}{c^2+cd}=\frac{b^2-2ab}{d^2-2cd}$.

17. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$;

shew that $(b-c)x+(c-a)y+(a-b)z=0$.

18. If $\frac{a}{y+z}=\frac{b}{z+x}=\frac{c}{x+y}$,

prove that $\frac{a(b-c}{y^2-z^2}=\frac{b(c-a)}{z^2-x^2}=\frac{c(a-b)}{x^2-y^2}$.

19. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove that

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}.$$

20. If $\frac{a}{q+r-p}=\frac{b}{r+p-q}=\frac{c}{p+q-r}$,

shew that $(q-r)a+(r-p)b+(p-q)c=0$.

21. If $\frac{x}{ax+by+cz}=\frac{y}{bx+cy+az}=\frac{z}{cx+ay+bx}$,

then each $=\frac{1}{a+b+c}$.

22. If $\frac{a+b}{a-b}=\frac{b+c}{2(b-c)}=\frac{c+a}{3(c-a)}$, then $8a+9b+5c=0$.

23. If $\frac{mx-a-b}{nx-c-d}=\frac{mx-a-c}{nx-b-d}$, then each $=-1$.

24. If $x(a+x-b)=y(b+y-c)=z(c+z-a)$,

then each $=\frac{x+y+z}{x^{-1}+y^{-1}+z^{-1}}$.

25. If $\frac{ad-bc}{a-b-c+d}=\frac{ac-bd}{a-b-d+c}$, then each $=\frac{1}{4}(a+b+c+d)$.

26. If b and c are unequal and $\frac{b^2-ca}{a+c-2b}=\frac{c^2-bd}{b+d-2c}$,

then each $=\frac{bc-ad}{a+d-b-c}$.

CHAPTER XXIV.

HARDER SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

We shall deal in this chapter with equations of harder type.

Ex. 1. Solve

$$(x+a)(x+b)=(x+c)(x+d)$$

Multiplying out we have

$$x^2+ax+bx+ab=x^2+cx+dx+cd$$

cancelling x^2 on both sides, and transposing

$$ax+bx-cx-dx = cd-ab$$

$$\therefore\ x(a+b-c-d) = cd-ab$$

$$\therefore\ x = \frac{cd-ab}{a+b-c-d}$$

Ex. 2. Solve $\dfrac{3}{x-1}+\dfrac{21}{x+4}=\dfrac{7}{2x-2}+\dfrac{4}{3x+12}$.

The L. C. M. of the denominators is $6(x-1)(x+4)$.

Multiplying each side by $6(x-1)(x+4)$, we have

$$18(x+4)+12(x-1) = 21(x+4)+8(x-1).$$

$$\therefore 18x+72+12x-12 = 21x+84+8x-8.$$

$$\therefore 18x+12x-21x-8x = 84-8+12-72.$$

$$\therefore x=16.$$

Ex. 3. Solve $\dfrac{1}{x}+\dfrac{b}{x+a}=\dfrac{1+b}{x+b}$.

Multiplying each side by $x(x+a)(x+b)$, the L. C. M. of the denominators, we have

$$(x+a)(x+b)+bx(x+b)=x(x+a)(1+b).$$

$$\therefore x^2+x(a+b)+ab+bx^2+b^2x=x^2+bx^2+ax+abx.$$

Cancelling like terms on both sides, we get

$$bx+b^2x-abx=-ab.$$

Dividing both sides by b, $x+bx-ax=-a$.

$$\therefore\ x(1+b-a)=-a.$$

$$\therefore x=\frac{-a}{1+b-a}=\frac{a}{a-b-1}.$$

In the following examples, the solution is facilitated by suitable transposition and combination of terms.

Ex. 4. Solve $\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}$.

We have $\frac{1}{x-1}+\frac{2}{x-2}=\frac{1}{x-3}+\frac{2}{x-3}$.

By transposition, $\frac{1}{x-1}-\frac{1}{x-3}=\frac{2}{x-3}-\frac{2}{x-2}$.

$$\therefore \frac{x-3-x+1}{(x-1)(x-3)}=\frac{2x-4-2x+6}{(x-3)(x-2)}.$$

$$\therefore \frac{-2}{(x-1)(x-3)}=\frac{2}{(x-3)(x-2)}.$$

$$\therefore \frac{-1}{x-1}=\frac{1}{x-2}.$$

Multiply both sides by $(x-1)(x-2)$.

We have $-1(x-2)=x-1$.

$$\therefore -x+2=x-1.$$

$$\therefore -2x=-3.$$

$$\therefore x=\tfrac{3}{2}=1\tfrac{1}{2}.$$

Ex. 5. Solve $\frac{b-c}{x+a}+\frac{a-b}{x+b}=\frac{a-c}{x+c}$.

Transposing, $(b-c)\left(\frac{1}{x+a}-\frac{1}{x+c}\right)=(a-b)\left(\frac{1}{x+c}-\frac{1}{x+b}\right)$

$$\therefore (b-c)\ \frac{c-a}{(x+a)(x+c)}=(a-b)\ \frac{b-c}{(x+c)(x+b)}.$$

Multiply both sides by $(x+a)(x+b)$.

$$\frac{c-a}{x+a}=\frac{a-b}{x+b}.$$

We have $(x+b)(c-a)=(a-b)(x+a)$.

$$\therefore x(c-a)+b(c-a)=x(a-b)+a(a-b).$$

$$\therefore x(c-a-a+b)=a(a-b)-b(c-a)=a^2-bc.$$

$$\therefore x=\frac{a^2-bc}{b+c-2a}.$$

Ex. 6. Solve $\frac{x-a^3}{a^2+a}+\frac{x-a^3}{a^3+1}+\frac{x-a}{a+1}=1+a+a^2$.

We have by transposition,

$$\frac{x-a^3}{a^2+a}-1+\frac{x-a^3}{a^3+1}-a+\frac{x-a}{a+1}-a^2=0.$$

Simplifying,

$$\frac{x-a^3-a^2-a}{a^3+a}+\frac{x-a^2-a^3-a}{a^2+1}+\frac{x-a-a^3-a^2}{a+1}=0.$$

$$\therefore (x-a-a^2-a^3)\left(\frac{1}{x^3+a}+\frac{1}{a^3+1}+\frac{1}{a+1}\right)=0.$$

Dividing both sides by $\left(\frac{1}{a^2+a}+\frac{1}{a^3+1}+\frac{1}{a+1}\right)$,

we have $x-a-a^2-a^3=0$.

$$\therefore x=a+a^2+a^3.$$

EXERCISE 86.

Solve the following equations :—

1. $x-a=(b-a)x$.
2. $(x-a)(x-b)=x(x-c)$.
3. $(2+x)(a-3)=4-2ax$.
4. $x(x+a)+x(x+b)-2(x+a)(x+b)=0$.
5. $(3a-x)(a-b)-4b(a+x)+2ax=0$.
6. $(x+a)^2=5a^2+(x-a)^2$.
7. $(a+2x)(b+2x)=4(x+2a)(x+2b)$.
8. $(x-a)(x-b)+a^2+b^2=(x-a-b)^2$.
9. $(6x+a)^2+(8x-b)^2=(10x+a)^2-10a^2$.
10. $(x-1)(x-2)(x-3)+(x+1)(x+2)(x+3)=x(2x^2+21)+1$.
11. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.
12. $(x+a)^3+(x+b)^3+(x+c)^3=3(x+a)(x+b)(x+c)$.
13. $(mx-a-b)(nx-x-d)=(mx-c-a)(nx-d-c)$.
14. $(x+a)(x+b)(x+c+d)-(x+c)(x+d)(x+a+b)=0$.
15. $(x+a)(2x+b+c)^2=(x+b)(2x+c+a)^2$.
16. $(b-c)(x-a)^3+(c-a)(x-b)^3+(a-b)(x-c)^3=0$.
17. $(b-c)^3(x-a)+(c-a)^3(x-b)+(a-b)^3(x-c)=0$.
18. $(b-c)^3(2x+b+c)+(c-a)^3(2x+c+a)+(a-b)^3(2x+a+b)=0$.
19. $\frac{\cdot 01x}{\cdot 02}-\frac{x}{30}=\frac{\cdot 01x}{\cdot 5}+1\cdot 34$.
20. $(1-2x)(\cdot 01-\cdot 03x)-\cdot 23=(\cdot 6x+\cdot 1)(\cdot 1x-\cdot 1)-\cdot 03x$.
21. $3\cdot 75x+\cdot 5=2\cdot 26x+8$.
22. $\frac{4\cdot 05}{9x}-\frac{\cdot 3}{\cdot 8-2x}=\frac{1\cdot 8}{x}-\frac{3\cdot 6}{2\cdot 4-6x}$.
23. $\cdot 65x+\frac{\cdot 585x-\cdot 975}{\cdot 6}=\frac{1\cdot 56}{2}-\frac{\cdot 39x-\cdot 78}{\cdot 9}$.

24. $(\cdot1x+\cdot2)^2+\cdot7(\cdot3x-\cdot1)=\cdot06(2x+4)+(\cdot1x-\cdot2)^2-\cdot65.$

25. $\frac{x}{8}-\frac{x-1}{2\frac{1}{2}}=\frac{3x-4}{15}+\frac{x}{12}.$

26. $\frac{11x-13}{25}+\frac{19x+3}{7}-\frac{5x-25\frac{1}{3}}{4}=28\frac{1}{7}-\frac{17x+4}{21}.$

27. $\frac{x-1}{3}+\frac{1\frac{1}{2}-2x}{4\frac{1}{2}-5x}=\frac{\frac{1}{3}(2x-1)}{2}+2-\frac{7}{18x+6}-\frac{7x}{6x+2}.$

28. $\frac{6-5x}{5}-\frac{3}{14}.\frac{7-2x^2}{x-1}-1\frac{1}{10}=\frac{1+3x}{7}-x+\frac{1}{35}.$

29. $\frac{1}{x-1}-\frac{2}{x-7}=\frac{1}{7(x-1)}.$

30. $\frac{1}{x-3}+\frac{1}{x-6}=\frac{1}{x-9}.$

31. $\frac{2}{x-4}-\frac{5}{x-2}+\frac{3}{x-6}=0.$

32. $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}.$

33. $\frac{1}{x-3}-\frac{1}{x-4}=\frac{3}{3x-13}-\frac{3}{3x-16}.$

34. $\frac{8}{x-3}-\frac{4}{x+9}-\frac{5}{x-27}+\frac{6}{x-15}=0.$

35. $\frac{3}{x-2}+\frac{5}{x-6}=\frac{8}{x+3}.$

36. $\frac{2}{2x-5}+\frac{1}{x-3}=\frac{6}{3x-1}.$

37. $\frac{3}{4x+1}+\frac{4}{4x+5}=\frac{7}{4x+3}.$

38. $\frac{1}{x+a}+\frac{1}{x+c}=\frac{2}{x+b}.$

39. $\frac{1}{1-3x}+\frac{5}{1-5x}+\frac{4}{2x-1}=0.$

40. $\frac{a-c}{x+2b}+\frac{b-c}{x+2a}=\frac{a+b-2c}{a+b+x}.$

41. $\frac{x-a^2}{b^2+c^2}+\frac{x-b^2}{c^2+a^2}+\frac{x-c^2}{a^2+b^2}=3.$

42. $\frac{x+a}{b+c+d}+\frac{x-b}{c+d+a}+\frac{x+c}{d+a+b}+\frac{x+d}{a+b+c}+4=0.$

43. $\frac{x-ab}{a+b}+\frac{x-a}{a+1}+\frac{x-b}{1+b}=1+b+a.$

44. $\frac{x-ab}{a+b}+\frac{x-bc}{b+c}+\frac{x-ca}{c+a}=a+b+c.$

45. $\frac{x-6ab}{3a+2b}+\frac{x-3ac}{3a+c}+\frac{x-2bc}{2b+c}=3a+2b+c.$

46. $\frac{1}{x+6a}+\frac{2}{x-3a}+\frac{3}{x+2a}=\frac{6}{x+a}.$

47. $\frac{1}{x-6a}+\frac{2}{x+3a}+\frac{3}{x-2a}=\frac{6}{x-a}.$

48. $\dfrac{x-4}{(x-1)(x-3)}+\dfrac{x-7}{(x-1)(x-6)}+\dfrac{x-9}{(x-3)(x-6)}=\dfrac{3}{x}$.

49. $\dfrac{1}{x-7}+\dfrac{1}{x-13}=\dfrac{1}{x-9}+\dfrac{1}{x-11}$.

50. $\dfrac{b+c}{x-a}=\dfrac{b}{x-c}+\dfrac{c}{x-b}$.

51. $\dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}$.

52. $\dfrac{a+b}{x-c}=\dfrac{a}{x-b}+\dfrac{b}{x-a}$.

53. $\dfrac{1}{x-2}-\dfrac{1}{x-4}=\dfrac{1}{x-6}-\dfrac{1}{x-8}$.

54. $\dfrac{1}{x-9}-\dfrac{1}{x-11}=\dfrac{1}{x-15}-\dfrac{1}{x-17}$.

55. $\dfrac{1}{x-a}-\dfrac{1}{x-a+c}=\dfrac{1}{x-b-c}-\dfrac{1}{x-b}$.

56. $\dfrac{x-a}{a-b}-\dfrac{x+a}{a+b}=\dfrac{2ax}{a^2-b^2}$.

57. $\dfrac{m(x+a)}{x+b}+\dfrac{n(x+b)}{x+a}=m+n$.

58. $\dfrac{a+c}{x-2b}-\dfrac{b+c}{x+2b}=\dfrac{a-c}{x+2a}-\dfrac{b-c}{x+2a}$.

185. Equations involving Fractions—Special Methods.

A. *Write the fractions in mixed form, by dividing the numerator by the denominator.*

Ex. 1. Solve $\dfrac{x-1}{x-2}=\dfrac{x-2}{x-5}$.

Dividing out each fraction,

we have $1+\dfrac{1}{x-2}=1+\dfrac{3}{x-5}$.

$\therefore \quad \dfrac{1}{x-2}=\dfrac{3}{x-5}$

$\therefore \quad x-5=3(x-2)=3x-6$.

$\therefore \quad -2x=-1$.

$\therefore \quad x=\frac{1}{2}$.

Ex. 2. Solve $\dfrac{x^3+7x^2+24x+30}{x^2+5x+13}=\dfrac{2x^3+11x^2+37x+45}{2x^2+7x+20}$.

Dividing out each fraction,

we have $x+2+\dfrac{x+4}{x^2+5x+13}=x+2+\dfrac{2x+5}{2x^2+7x+20}$.

$\therefore \dfrac{x+4}{x^2+5x+13}=\dfrac{2x+5}{2x^2+7x+20}$.

$\therefore \dfrac{x^2+5x+13}{x+4}=\dfrac{2x^2+7x+20}{2x+5}$.

Dividing the numerator by the denominator,

we have $x+1+\frac{9}{x+4}=x+1+\frac{15}{2x+5}$,

$\therefore \quad \frac{9}{x+4}=\frac{15}{2x+5}$,

$\therefore \quad \frac{3}{x+4}=\frac{5}{2x+5}$,

$\therefore 6x+15=5x+20$,

$\therefore \quad x=5$.

Ex 3. Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$.

Dividing out each fraction,

we have $1+\frac{1}{x-2}-\left(1+\frac{1}{x-3}\right)=\frac{1}{x+6}-\left(1+\frac{1}{x-7}\right)$

$\therefore \frac{1}{x-2}-\frac{1}{x+3}=\frac{1}{x-6}-\frac{1}{x-7}$.

$\therefore \frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}$.

$\therefore \frac{-1}{(x-2)(x+3)}=\frac{-1}{(x-6)(x-7)}$.

$\therefore \frac{x-6}{x-2}=\frac{x-3}{x-7}$.

Dividing out each fraction, we have,

$1-\frac{4}{x-2}=1+\frac{4}{x-7}$

$\therefore \quad \frac{-1}{x-2}=\frac{1}{x-7}$.

$\therefore -(x-7)=x-2$.

$\therefore -2x=-9$.

$\therefore x=4\frac{1}{2}$

EXERCISE 87.

Solve the following equations :—

1. $\frac{x+4}{x+8}=\frac{x+8}{x+14}$.
2. $\frac{x-7}{x-5}=\frac{2x-7}{2x-5}$.
3. $\frac{7x-4}{x-1}=\frac{7x-26}{x-3}$.
4. $\frac{2x-7}{2x-3}=\frac{x-2}{x+2}$.
5. $\frac{x-4}{x-1}+\frac{x-2}{x-4}=2$.

6. $\frac{x+15}{x+10}+\frac{x+3}{x+6}=2.$

7. $\frac{x-1}{x-3}-\frac{x-5}{x-1}=2.$

8. $\frac{3(x+5)}{x+5}+\frac{4(x+5)}{x+6}=7.$

9. $\frac{2x+15}{2x+9}+\frac{2x-1}{2x-3}=2.$

10. $\frac{5x^2+x-3}{5x-4}=\frac{7x^2-3x-9}{7x-10}.$

11. $\frac{3x}{x+1}+\frac{3x-2}{3x-5}=4.$

12. $\frac{x+a}{x-a}=\frac{x+b}{x-b}$

13. $\frac{3x+5}{3x-5}+\frac{2x+4}{x-2}=8.$

14. $\frac{x^2+15x+56}{x+8}=\frac{2x^2+23x+63.}{x+7}$

15. $\frac{2x^3+5x^2+7x+5}{2x^2+3x^2}=\frac{x^3+9x^2+11x+10.}{x^3+8x+2}$

16 $\frac{4x^3+6x^2+11x+7}{4x^2+2x+5}=\frac{2x^3+6x^2+11x+9.}{2x^2+4x+5}$

17. $\frac{2x^3+15x^2+18x-34}{2x^3+9x-11}=\frac{x^2+4x^2-20x-76.}{x^3+x-24}$

18. $\frac{x^2+2x-2}{x-1}+\frac{x^2-2x+2}{x+1}=\frac{2x^2-6x+2.}{x-3}$

19. $\frac{x^4+10x^3+38x^2+65x+43}{x^3+6x^2+13x+10}=\frac{5x^2+46x^2-147x+163.}{5x^3+26x+38}$

20. $\frac{x+4a+b}{x+a+b}=\frac{4x+a+2b}{x+a-b}=5.$

21. $\frac{x+2a}{x+2b}=\left(\frac{x+a}{x-b}\right)^2.$

22. $\frac{2x-3}{x+1}=\frac{4x+5}{4x+4}+\frac{3x+3.}{4x+1}$

23. $\frac{x-7}{x-3}+\frac{x-2}{x-9}+\frac{x-4}{x-1}=3.$

24. $\frac{x-5}{x-4}-\frac{x-6}{x-5}=\frac{x-3}{x-4}-\frac{x-2}{x-3}.$

25. $\frac{x-4}{x-5}+\frac{x-6}{x-7}=\frac{x-2}{x-3}+\frac{x-8}{x-9}.$

26. $\frac{x+3}{x+1}+\frac{x-6}{x-4}=\frac{x-5}{x-3}+\frac{x+4}{x+2}.$

27. $\frac{x+6}{x+7}+\frac{x+11}{x+12}=\frac{x+7}{x+8}+\frac{x+10}{x+11}.$

28. $\frac{x+a+b}{x-a-b}=\frac{x-a-b.}{x+a+b}$

29. $\frac{x+4a-2b}{x+a-2b}+\frac{2x-a-2b}{x+a-4b}=3.$

30. $\frac{(x-a)(x-b)}{(x-c)(x-d)}=\frac{x-a-b}{x-c-d}$

31 $\frac{x+a}{x+b}=\left(\frac{2x+a+c}{2x+b+c}\right)^2.$

32. $\frac{x-a}{x-a-1}-\frac{x-a-1}{x-a-2}=\frac{x-b}{x-b-1}-\frac{x-b-1.}{x-b-2}$

33. $\dfrac{x}{x-2}+\dfrac{x-9}{x-7}=\dfrac{x+1}{x-1}+\dfrac{x-8}{x-6}$.

34. $\dfrac{3+x}{3-x}-\dfrac{2+x}{2-x}-\dfrac{1+x}{1-x}=1$.

35. $\left(\dfrac{x-a}{x-b}\right)^2=\dfrac{x-2a+b}{x+a-2b}$.

36. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+17}{x+4}=\dfrac{x^2+4x+5}{x+2}+\dfrac{x^2+6x+10}{x+3}$.

37. $\dfrac{x^2-2x+2}{x-1}+\dfrac{x^2-8x+20}{x-4}=\dfrac{x^2-4x+6}{x-2}+\dfrac{x^2-6x+12}{x-3}$.

38. $\dfrac{x-4}{x-5}+\dfrac{x-8}{x-9}=\dfrac{x-7}{x-8}+\dfrac{x-5}{x-6}$.

39. $(x+2a)(x+2b)(x-a-b)=(x-a)(x-b)(x+2a+2b)$.

40. $\dfrac{x-7}{x+7}=\dfrac{2x-15}{2x-6}-\dfrac{1}{2(x+7)}$.

41. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{11x+18}{11x-18}$.

42. $\dfrac{x-5}{x+5}-\dfrac{x+5}{x-5}+\dfrac{x^2}{x^2-25}=1$.

43. $\dfrac{2x-1}{2x-3}+\dfrac{2x-7}{2x-5}+\dfrac{3x-13}{3x-4}+\dfrac{3x+4}{3x-5}=4$.

44. $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x+1}$.

45. $\dfrac{x+a}{x-a}+\dfrac{x+4a}{x-4a}=\dfrac{x+3a}{x-3a}+\dfrac{x+2a}{x-2a}$.

46. $\dfrac{4x-3}{4x-5}-\dfrac{4x-7}{4x-9}=\dfrac{4x-11}{4x-13}-\dfrac{4x-15}{4x-17}$.

47. $\dfrac{7x-55}{x-8}+\dfrac{2x-17}{x-9}=\dfrac{6x-71}{x-12}+\dfrac{3x-14}{x-5}$.

48. $\dfrac{x^2+3x+3}{x+2}+\dfrac{x^2-15}{x-4}=\dfrac{x^2+7x+11}{x+5}+\dfrac{x^2-4x-20}{x-7}$

49. $\dfrac{x+7a}{x+6a}+\dfrac{x-a}{x-3a}+\dfrac{a-x}{x+2a}=\dfrac{x+7a}{x+a}$.

50. $\dfrac{x^2-(a-b)^2+c^2}{x-a-b}-\dfrac{(x+a-b)^2+c^2}{x+a-b}=2\left(\dfrac{c^2}{x+b}-\dfrac{c^2}{x+a}\right)$.

B. *When an equation consists of two fractions only, its solution may be simplified by the application of one or other of the following results :—*

If $\dfrac{a}{b}=\dfrac{c}{d}$, then

I. $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}=\frac{a-c}{b-d}$ (Arts 183 and 184)

II. $\frac{a+b}{d}=\frac{c+d}{d}$; $\frac{a-b}{b}=\frac{c-d}{d}$.

III. $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Ex. 1. Solve $\frac{mx-a-b}{nx-c-d}=\frac{mx-a-b}{nx-b-d}$.

$\therefore \frac{mx-a-b}{nx-c-d}=\frac{mx-a-b-mx+a+c}{nx-c-d-nx+b+d}$ . . . by 1.

$=\frac{c-b}{b-c}$

$=-1$.

$\therefore mx-a-b=-nx+c+d$.

$\therefore x(m-n)=a+b+c+d$.

$\therefore x=\frac{a+b+c+d}{m+n}$

Ex. 2. Solve $x^3\frac{x^3+3a^2}{c^3+3x^2}=ac\frac{a^3+3x^2}{x^3+3c^2}$.

$\therefore \frac{x(x^3+3a^2)}{c(c^3+3x^2)}=\frac{a(a^3+3x^2)}{x(x^2+3c^2)}$.

$\therefore \frac{x^3+3a^2x}{a^3+3ax^2}=\frac{c^3+3cx^2}{x^3+3xc^2}$.

$\therefore \frac{x^3+3a^2x+a^3+3ax^2}{x^3+3a^2x-a^3-3ax^2}=\frac{c^3+3cx^2+x^3+3xc^2}{c^3+3cx^2-x^3-3xc^2}$. . . by III.

$\therefore \left(\frac{x+a}{x-a}\right)^3=\left(\frac{c+x}{c-x}\right)^3$.

$\therefore \frac{x+a}{x-a}=\frac{c+x}{c-x}$.

Again by III, $\frac{x+a+x-a}{x+a-x+a}=\frac{c+x+c-x}{c+x-c+x}$.

$\therefore \frac{x}{a}=\frac{c}{x}$.

$\therefore x^2=ac$.

$\therefore x=\pm\sqrt{ac}$.

Ex. 3. Solve :—

$(x-1)(x-5)(x-7)(x-9)=(x-2)(x-4)(x-6)(x-10)$.

We may write it thus $\dfrac{(x-1)(x-5}{(x-6)(x-10)} = \dfrac{(x-2)(x-4)}{(x-7)(x-9)}$.

$$\therefore \frac{x^2-6x+5}{x^2-16x+60} = \frac{x^2-6x+8}{x^2-16x+63}.$$

$\therefore$ by I. $\dfrac{x^2-6x+5}{x^2-16x+60} = \dfrac{x^2-6x+5-x^2+6x-8}{x^2-16x+60-x^2+16x-63}$

$$= \frac{-3}{-3}$$

$$= 1.$$

$$\therefore x^2-6x+5 = x^2-16x+60.$$

$$\therefore 10x = 55.$$

$$\therefore x = 5\tfrac{1}{2},$$

Ex. 4. Solve $(x-a)^3(x+a-2b) = (x-b)^3(x-2a+b)$.

$$\therefore \frac{(x-a)^3}{(x-b)^3} = \frac{x-2a+b}{x+a-2b}.$$

$\therefore$ by III. $\dfrac{(x-a)^3+(x-b)^3}{(x-a)^3-(x-b)^3} = \dfrac{x-2a+b+x+a-2b}{x-2a+b-x-a+2b}$.

$$\therefore \frac{(2x-a-b)\{x^2-(a+b)x+a^2-ab+b^2\}}{(b-a)\{3x^2-3(a+b)x+a^2+ab+b^2\}} = \frac{2x-a-b}{3(b-a)}.$$

$$\therefore 5(2x-a-b)\{x^2-(a+b)x+a^2-ab+b^2\}$$
$$= (2x-a-b)\{3x^2-3(a+b)x+a^2+b^2+ab\}.$$

$$\therefore (2x-a-b)\{2a^2-4ab+2b^2\} = 0.$$

$$\therefore 2x-a-b = 0.$$

$$\therefore 2x = a+b$$

$$\therefore x = \frac{a+b}{2}.$$

EXERCISE 88.

Solve the following equations :—

1. $\dfrac{x-1}{x+1} = \dfrac{1-a}{1+a}$.

2. $\dfrac{x-a}{x-b} = \dfrac{x-m+n}{x+m-n}$.

3. $\dfrac{ax-b-c}{px-q-r} = \dfrac{ax-b-q}{px-c-r}$.

4. $\dfrac{mx+n}{mx-n} = \dfrac{b+c-a}{c+a-b}$.

5. $\dfrac{ax^2+bx+c}{px^2+qx+r} = \dfrac{ax+b}{px+q}$.

6. $\dfrac{x+a}{x-a} = \dfrac{(a-c)(b+d)}{(a+c)(b-d)}$.

7. $(x-1)(2x-3)^2 = (x-3)(2x-1)^2$.

8. $\dfrac{x-b-a}{x+2a+2b} = \dfrac{(x-a)(x-b)}{(x+2a)(x+2b)}$.

9. $\dfrac{x^2x^2+3a^4}{c^4+3x^2}=a^2c^2\dfrac{a^4+3x^2}{x^2+3c^4}$.

10. $\dfrac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+5)(a^4+10a^2+5)}=ac$.

11. $(x+1)(2x+5)^2=4(x+2)^2$.

12. $(x-a)(2x-b)^2=(x-b)(2x-a)^2$.

13. $(x+1)(x+2)(x+3)=(x+4)(x+5)(x-3)$.

14. $(x+5)(x+6)^2(x+9)=(x+4)(x+7)^2(x-8)$.

15. $(x+1)(x+6)^2=(x+2)(x+4)(x+7)$.

16. $(x+2a+b)(x+a+2b)=(x+a+b)^2$.

17. $(x-1)(x-2)(x+6)=(x+2)(x+3)(x+4)$.

18. $(x-1)(x-2)^2(x-5)=x(x-3)^2(x-4)$.

19. $\dfrac{(x+1)(x+9)}{(x+2)(x+4)}=\dfrac{(x+6)(x+10)}{(x+5)(x+7)}$.

20. $\dfrac{(x+2)(x+5)}{(x+4)(x+6)}=\dfrac{(x+1)(x+7)}{(x+3)(x+8)}$.

186. Equations involving Radicals.

Ex. 1. Solve $\sqrt{x+4}+\sqrt{x+11}=7$

Transposing, we have, $\sqrt{x+4}=7-\sqrt{x+11}$

Squaring, $x+4=49+x+11-14\sqrt{x+11}$.

$\therefore 14\sqrt{x+11}=56$.

$\therefore \sqrt{x+11}=4$.

Squaring, $x+11=16$.

$\therefore x=5$.

Or thus $\sqrt{x+4}+\sqrt{x+11}=7$ . . . . . .(A)

Identically, $(x+4)-(x+11)=-7$ . . . . . .(B)

Dividing (B) by (A), $\sqrt{x+4}-\sqrt{x+11}=-1$ . . . . . .(C)

Adding (A) and (C), $2\sqrt{x+4}=6$.

$\therefore \sqrt{x+4}=3$.

Squaring, $x+4=9$.

$\therefore x=5$.

Ex. 2. Solve $\sqrt{cx-a}-\sqrt{cx+b}+\sqrt{dx+a}-\sqrt{dx-b}=0$.

Writing it thus; $\sqrt{cx-a}-\sqrt{dx+b}+\sqrt{cx-b}-\sqrt{dx+a}$. . . .(A)

Identically, $(cx-a)-(cx+b)=(dx-b)-(dx+a)$ . . . .(B)

Dividing (B) by (A), $\sqrt{cx-a}+\sqrt{cx+b}=\sqrt{dx-b}+\sqrt{dx-a}$. . . .(C)

Adding (A) and (C), $2\sqrt{cx-a}=2\sqrt{dx-b}$.

$\therefore \sqrt{cx-a}=\sqrt{dx-b}$.

Squaring, $cx-a=dx-b$.

$\therefore x(c-d)=a-b$.

$\therefore x=\dfrac{a-b}{c-d}$.

Ex. 3. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}=b$.

Applying III of Art. 185 B.

$$\frac{\sqrt{a+x}+\sqrt{a-x}+\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}-\sqrt{a+x}+\sqrt{a-x}}=\frac{b+1}{b-1}$$

$$\therefore \frac{\sqrt{a+x}}{\sqrt{a-x}}=\frac{b+1}{b-1}.$$

Squaring, $\dfrac{a+x}{a-x}=\dfrac{(b+1)^2}{(b-1)^2}$.

Applying the same formula, $\dfrac{a+x+a-x}{a+x-a+x}=\dfrac{(b+1)^2+(b-1)^2}{(b+1)^2-(b-1)^2}$.

$$\therefore \frac{a}{x}=\frac{b^2+1}{2b}.$$

$$\therefore x=\frac{2ab}{b^2+1}.$$

Ex. 4. Solve $\sqrt{\dfrac{1+x}{1+x}}+\sqrt{\dfrac{1-x}{1+x}}=2$.

Putting a for $\sqrt[4]{\dfrac{1+x}{1-x}}$ and $\dfrac{1}{a}$ for $\sqrt[4]{\dfrac{1-x}{1+x}}$,

we have $a^2+\dfrac{1}{a^2}=2$.

$$\therefore a^2-2+\frac{1}{a^2}=0$$

$$\therefore \left(a-\frac{1}{a}\right)^2=0$$

$$\therefore a-\frac{1}{a}=0.$$

$$\therefore a=\frac{1}{a}.$$

$$\therefore a^2=1$$

$$\therefore \sqrt[4]{\left(\frac{1+x}{1-x}\right)^2}=1.$$

$$\therefore \sqrt{\frac{1+x}{1-x}}=1.$$

$$\therefore \frac{1+x}{1-x}=1.$$

$$\therefore 1+x=1-x.$$

$$\therefore 2x=0. \quad \therefore \quad x=0.$$

Ex. 5. Solve $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b$.

Cubing both sides,

$$a+x+a-x+3\sqrt[3]{a+x}.\sqrt[3]{a-x}(\sqrt[3]{a+x}+\sqrt[3]{a-x})=b^3.$$

$$\therefore 2a-3.\sqrt[3]{a^2-x^2}\ (\sqrt[3]{a+x}+\sqrt[3]{a-x})=b^3.$$

But $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b$.

$$\therefore \quad 2a+3.\sqrt[3]{a^2-x^2}.\ (b)=b^3.$$

$$\therefore \quad 3b\sqrt{a^2-x^2}=b^3-2a.$$

$$\therefore \quad \sqrt[3]{a^2-x^2}=\frac{b^3-2a}{3b}.$$

Cubing both sides, $a^2-x^2=\left(\frac{b^3-2a}{3b}\right)^3$.

$$\therefore x^2=a^2-\left(\frac{b^3-2a}{3b}\right)^3.$$

$$\therefore x=\pm\sqrt{\left\{a^2-\left(\frac{b^3-2a}{3b}\right)^3\right\}}.$$

Ex. 6. Solve $\dfrac{x^2+16a^2}{x+4a+\sqrt{8ax}}-\dfrac{x^2+64a^2}{x+8a+\sqrt{16ax}}=-2a\sqrt{2}.$

$$x^2+16a^2=(x+4a)^2-8ax=(x+4a\sqrt{8ax}(x+4a-\sqrt{8ax})$$

and $x^2+64a^2\equiv(x+8a)^2-16ax$

$$=(x+8a+\sqrt{16ax})\ (x+8a-\sqrt{16ax}).$$

$\therefore$ the equation becomes

$$x+4a-\sqrt{8ax}-(x+8a-\sqrt{16ax})=-2a\sqrt{2}.$$

$$\therefore \sqrt{16ax}-\sqrt{8ax}=4a-2a\sqrt{2}.$$

$$\therefore \sqrt{ax}(4-2\sqrt{2})=a(4-2\sqrt{2}).$$

$$\therefore \sqrt{ax}=a.$$

$$\therefore ax=a^2.$$

$$\therefore \quad x=a.$$

Ex. 7. Solve $\sqrt[2pq]{x^{p+q}} = \frac{1}{2c}\left(\sqrt[p]{x} + \sqrt[q]{x}\right)$

$$\therefore \frac{p+q}{x^{2pq}} = \frac{1}{2c}\left(\frac{1}{x^p} + \frac{1}{x^q}\right)$$

$$\therefore \frac{x^{\frac{1}{p}} + x^{\frac{1}{q}}}{\frac{p+q}{2pq}} = c.$$

$$\therefore \frac{x^{\frac{1}{p}} + x^{\frac{1}{q}} + 2x^{\frac{p+q}{2pq}}}{x^{\frac{1}{p}} + x^{\frac{1}{q}} - 2x^{\frac{p+q}{2pq}}} = \frac{c+1}{c-1}.$$

$$\therefore \left(\frac{x^{\frac{1}{2p}} + x^{\frac{1}{2q}}}{x^{\frac{1}{2p}} + x^{\frac{1}{2q}}}\right)^2 = \frac{c+1}{c-1}$$

$$\therefore \frac{x^{\frac{1}{2p}} + x^{\frac{1}{2q}}}{x^{\frac{1}{2p}} + x^{\frac{1}{2q}}} = \frac{\sqrt{c+1}}{\sqrt{c-1}}.$$

$$\therefore \frac{x^{\frac{1}{2p}}}{x^{\frac{1}{2q}}} = \frac{\sqrt{c+1} + \sqrt{c-1}}{\sqrt{c+1} - \sqrt{c-1}}.$$

$$\therefore x^{\frac{1}{2p} - \frac{1}{2q}} \text{ or } x^{\frac{q-p}{2pq}} = \frac{\sqrt{c+1} + \sqrt{c+1}}{\sqrt{c+1} - \sqrt{c-1}}.$$

$$\therefore x = \left(\frac{\sqrt{c+1} + \sqrt{c-1}}{\sqrt{c+1} - \sqrt{c-1}}\right)^{\frac{2pq}{q-p}}.$$

Ex. 8. Solve $\sqrt{x+a} + \sqrt{x+b} + \sqrt{x+c} = 0$.

If $p+q+r=0$, then $p^3+q^3+r^3 = 3pqr$.

$\therefore x+a+x+b+x+c = 3\sqrt[3]{(x+a)(x+b)(x+c)}$

$\therefore (3x+a+b+c)^3 = 27(x+a)(x+b)(x+c)$.

$$\therefore 27x^3+27x^2(a+b+c)+9x(a+b+c)^2+(a+b+c)^3$$
$$=27x^3+27x^2(a+b+c)+27x(ab+bc+ca)+27abc.$$
$$\therefore 9x\{(a+b+c)^2+3(ab+bc+ca)\}=27abc-(a+b+c)^3$$
$$\therefore x=\frac{27abc-(a+b+c)^3}{9(a+b+c)^2-27(ab+bc+ca)}$$
$$=\frac{27abc-(a+b+c)^3}{9(a^2+b^2+c^2-ab-ac-bc)}$$

Ex. 9. Solve $\sqrt{a+x}+\sqrt{a-x}=\dfrac{2x}{\sqrt{a+\sqrt{a^2+x^2}}}$.

Writing it thus: $\sqrt{a+x}+\sqrt{a-x}=\dfrac{a+x-(a-x)}{\sqrt{a+\sqrt{a^2+x^2}}}$.

Dividing each side by the common factor $(\sqrt{a+x}+\sqrt{a-x})$,

we have $1=\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+\sqrt{a^2+x^2}}}$

$$\therefore \sqrt{a+\sqrt{a^2+x^2}}=\sqrt{a+x}-\sqrt{a-x}.$$

Squaring, $a+\sqrt{a^2+x^2}=a+x+a-x-2\sqrt{a^2-x^2}$.

$$\therefore \sqrt{a^2+x^2}-a=-2\sqrt{a^2-x^2}.$$

Squaring, $a^2+x^2+a^2-2a\sqrt{a^2+x^2}=4(a^2-x^2)$.

$$\therefore 5x^2-2a^2=2a\sqrt{a^2+x^2}.$$
$$\therefore 25x^4+a^4-20a^2x^2=4a^4+4a^2x^2.$$
$$\therefore 25x^4-24a^2x^2=0.$$
$$\therefore 25x^2=24a^2.$$
$$\therefore x^2=\frac{24}{25}a^2.$$
$$\therefore x=\pm\frac{a}{5}\sqrt{24}=\pm\frac{2a}{5}\sqrt{6}.$$

EXERCISE 89.

Solve the following equations:—

1. $\sqrt{4x}+\sqrt{4x-7}=7$.
2. $\sqrt{(x+14)}+\sqrt{(x-14)}=14$.
3. $\sqrt{(9x+4)}+\sqrt{(9x-1)}=3$.
4. $\sqrt{x-a}+\sqrt{x-b}=\sqrt{a-b}$.
5. $\sqrt{(x+3a)}-\sqrt{(x-2a)}=\sqrt{a}$.
6. $\sqrt{x+4a}-\sqrt{x+2a}=2\sqrt{x-a}$.
7. $\sqrt{x-4}+\sqrt{x+5}=\sqrt{4x+1}$.

8. $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$.

9. $x+\sqrt{2ax+x^2}=a$.

10. $\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}$.

11. $a+x+\sqrt{2ax+x^2}=b$.

12. $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$

13. $\dfrac{ax-b^2}{\sqrt{ax+b}}+c=\dfrac{\sqrt{ax-b}}{n}$.

14. $\sqrt{5x-14}-\sqrt{5x-21}=1$.

15. $\sqrt{x^2+3}+\sqrt{x^2+8}=5$.

16. $\sqrt{x+p^2}+\sqrt{x+q^2}=\sqrt{2r-p^2}+\sqrt{2r-q^2}$.

17. $\sqrt{x+a-c}+\sqrt{x+b-c}=\sqrt{c-a}+\sqrt{c-b}$.

18. $\sqrt{x+7}-\sqrt{x-3}=\sqrt{3x-14}-\sqrt{3x-24}$.

19. $\sqrt{x-2}+\sqrt{x-7}=\sqrt{3x-10}-\sqrt{3x-15}$.

20. $\sqrt{5x-4}-\sqrt{4x-3}=-\sqrt{2x-1}+\sqrt{3x-2}$.

21. $\sqrt{ax+b}+\sqrt{ax+c}=\sqrt{dx+b}+\sqrt{dx+c}$.

22. $\sqrt{a-x}+\sqrt{b-x}-\sqrt{c+x}=\sqrt{a-b+c+x}$.

23. $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=4$.

24. $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}}=1$.

25. $\dfrac{a+\sqrt{2ax-a^2}}{a-\sqrt{2ax-a^2}}=a$

26. $\dfrac{\sqrt{36x+1}+\sqrt{36x}}{\sqrt{36x+1}-\sqrt{36x}}=9$.

27. $\dfrac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\dfrac{a}{x-a}$.

28. $\dfrac{\sqrt{4x+5}+\sqrt{x}}{\sqrt{4x+5}-\sqrt{x}}=2$.

29. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\dfrac{m+n}{m-n}$.

30. $\dfrac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=b$.

31. $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2$.

32. $\dfrac{x^4+6x^2+1}{4x(x^2+1)}=a$.

33. $\dfrac{\sqrt{a+b}+\sqrt{ax-b}}{\sqrt{ax+b}-\sqrt{ax-b}}=\left(\dfrac{p+1}{p-1}\right)^2$.

34. $\dfrac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}}=a$.

35. $\dfrac{1+x+x^2}{1-x+x^2}=\dfrac{62.1+x}{63.1-x}$.

36. $\dfrac{1}{x+\sqrt{x^2-a^2}}+\dfrac{1}{x-\sqrt{x^2-a^2}}=4.$

37. $\sqrt[3]{x+a}-\sqrt[3]{x-a}=2\sqrt[3]{a}.$

38. $b+b^2+3b\sqrt[3]{x}=x.$

39. $\dfrac{\sqrt{1+x}-1}{\sqrt{1+x}+1}+\dfrac{\sqrt{1+x}+1}{\sqrt{1+x}-1}=b.$

40. $\sqrt[3]{x+1}+\sqrt[3]{x+2}+\sqrt[3]{x-3}=0.$

41. $\sqrt[3]{x+3a}+\sqrt[3]{x+3b}+\sqrt[3]{x+3c}=0.$

42. $\sqrt[m]{2a+x}=\sqrt[2m]{x^2-4a^2-4ax}.$

43. $(1-b)^{\frac{1}{2}}\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{4}}+(1+b)^{\frac{1}{2}}\left(\dfrac{a-x}{a+x}\right)^{\frac{1}{4}}=2(1-b^2)^{\frac{1}{4}}.$

44. $\dfrac{x^2+4}{x^3-2x+2}-\dfrac{x^4+9-3x^2}{x^2+3x+3}=24.$

45. $p\sqrt{\dfrac{1+x}{1-x}}+(p+2)\sqrt{\dfrac{1-x}{1+x}}=2\sqrt{p(p+2)}.$

46. $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}.$

47. $\dfrac{2}{19}\left(\sqrt{x^2+39x+374}-\sqrt{x^2+20x+51}\right)=\sqrt{\dfrac{x+22}{x+17}}$

186. A simple Equation cannot have more than one root.

Every simple equation can be reduced to the form $ax+b=0$.

If possible, let this equation have two different roots a and B.

Then we have $\left.\begin{aligned} a+b&=0\\ B+b&=5\end{aligned}\right\}$

$\therefore$ by subtraction $a(a-B)=0$.

$\therefore$ either $a=0$, or $a-B=0$.

Now if $a=0$, there will be no equation and if $a-B=0$, then $a=B$, which is contrary to our supposition.

$\therefore$ *a simple equation cannot have more than one root.*

CHAPTER XXV.

HARDER PROBLEMS LEADING TO SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

We shall work out some examples of harder type below.

Ex 1. There is a number of two digits whose sum is 13, and if 27 be subtracted from the number, the remainder will form a number with the digits *inverted*; find the number.

Let x = the digit in the units' place;

then $13 - x$ = the digits in the tens' place.

The number $= 10(13 - x) + x$.

The number when the digits are *reversed* $= 10x + 13 - x$.

By the question, $10(13 - x) + x - 27 = 10x + 13 - x$.

$\therefore -10x + x - 10x + x = 13 + 27 - 130$.

$\therefore -18x = -90$.

$\therefore \quad x = 5$ = the digit in the units' place,

and $13 - x = 8$ = the digit in the tens' place.

Hence 85 is the number.

Ex 2. A workman was employed for 60 days, on condition that for every day he worked he should receive 1*s.* 3*d.*, and for every day he was idle he should forfeit 5*d.*; at the end of the time he had 20 shillings to receive; find the number of days he worked.

Let x = the number of days he worked;

then $60 - x$ = the number of days he was idle.

His pay for x days in pence $= 15x$.

The sum forfeited in pence $= 5(60 - x) = 300 - 5x$.

By the question, $15x - (300 - 5x) = 20 \times 12 = 240$.

$\therefore 20x = 540$.

$\therefore x = 27$.

and $60 - x = 33$,

$\therefore$ the number of days he worked is 27

and the number of days he was idle is 33.

Ex. 3. A hare is 50 leaps before a greyhound and takes 4 leaps to the greyhound's 3 leaps; but 2 of the greyhound's leaps are equal to

3 of the hare's; how many leaps must the greyhound take to catch the hare?

Let $3x$ = number of the greyhound's leaps;

then $4x$ = number of hare's leaps in the same time;

Since 2 of the greyhound's leaps = 3 of the hare's;

therefore $3x$ number of the greyhound's $= \frac{9x}{2}$ leaps;

By the question $\frac{9x}{2} - 4x = 50$.

$\therefore \frac{1}{2}x = 50$.

$\therefore x = 100$.

$\therefore 3x = 300$.

$\therefore$ the number of leaps the greyhound has to take to catch the hare is 300.

Ex. 4. A can do a piece of work in a days; B can do the same in b days; find in what time A and B together can do it.

Let x be the number of days in which A and B do the work together.

Then if W be the work,

$\frac{W}{x}$ = the work done by A and B together in 1 day.

But $\frac{W}{a}$ = the work done by A in 1 day,

and $\frac{W}{b}$ = the work done by B in 1 day.

$\therefore \frac{W}{a} + \frac{W}{b}$ = the total work done by A and B in 1 day.

By the question,

$$\frac{W}{a} + \frac{W}{b} = \frac{W}{x}$$

$$\therefore \frac{1}{x} = \frac{1}{a} + \frac{1}{b}.$$

$$\therefore x = \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a+b}.$$

$\therefore$ A and B together can do the work in $\frac{ab}{a+b}$ days.

Ex. 5 At what time between 5 and 6 o'clock are the hands of a clock exactly *at right angles* to each other?

When the hour-hand travels over 5 minute-divisions the minute-hand travels over 60 minute-divisions.

∴ the minute-hand travels 12 times as fast as the hour hand.

When the hands are at right angles there must be 15 minute-divisions between them. They will be at right angles *twice* before 6 o'clock, once before and again after they pass each other.

To begin with, by the question the hour-hand points to V and the minute-hand points to XII.

Let the hour-hand travel over x minute-divisions before they are at right angles *first*;

then the minute-hand travels over $12x$ minute-divisions;

the hour hand then points to $12x+15$ minutes measured from XII as it is in advance of the minute-hand by 15 minutes, and also to $25+x$ minutes measured from XII, for it pointed to V originally, and then travelled x minute-divisions.

Hence

$$12x+15=25+x.$$

$$\therefore 12x=10.$$

$$\therefore \quad x=\tfrac{10}{11}.$$

$$\therefore 12x=\frac{120}{11}=10\tfrac{10}{11}.$$

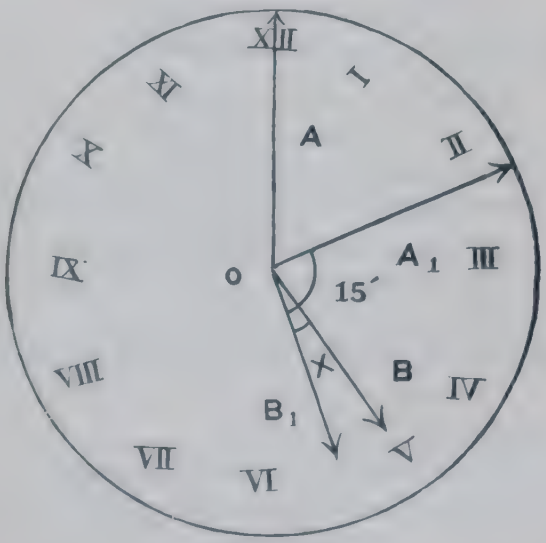


FIG 1.

∴ the two hands will be exactly at right angles to each other for the first time at $10\frac{10}{11}$ minutes past 5 o'clock . . . . . (1)

Again suppose the minute-hand has passed the hour-hand; then we see at once that $12x = 25 + x + 15$.

$\therefore 11x = 40.$

$\therefore 1x = 3\frac{7}{11}$

$\therefore 12x = \frac{483}{11} = 43\frac{10}{11}$

FIG. 2.

$\therefore$ the two hands will be exactly at right angles to each other for the second time at $43\frac{7}{11}$ minutes past 5 o'clock. . . . . (ii)

Note.—In the above figures OA, OB represent the original positions of the two hands; OA_1, OB_1 represent their positions when they are at right angles for the first time and OA_2, OB_2 represent their positions when they are at right angles for the second time.

Ex. 6. A person buys two sorts of ghee, one at Re. 1 a viss and the other at Re. 1-8 per viss. He wishes to mix them, so that by selling the mixture at Re. 1-3 a viss he may gain $12\frac{1}{2}$ per cent. on the whole. What is the proportion of the mixture?

Let x = the fraction of a viss of the inferior sort in *one* viss of the mixture;

then $1 - x$ = the fraction of a viss of the superior sort.

The price of x viss of the inferior ghee = Rs. x.

" $1 - x$ " superior " = Rs. $\frac{3}{2}(1 - x)$.

$\therefore$ the price of 1 viss of the mixture $= x + \frac{3}{2}(1 - x)$.

The gain $= 12\frac{1}{2}$ p. c. of the cost price $= \frac{1}{8}\{x + \frac{3}{2}(1 - x)\}$.

The selling price of 1 viss of the mixture $= \frac{9}{8}\{x + \frac{3}{2}(1 - x)\}$.

By the question, $\frac{8}{9}\{x+\frac{3}{2}(1-x)\}=1\frac{3}{16}$.

$\therefore 18\{x+\frac{3}{2}(1-x)=19.$

$\therefore 18x+27-27x+19.$

$\therefore \quad -9x=-8.$

$\therefore \quad x=\frac{8}{9}.$

$\therefore$ in 1 viss of the mixture there is

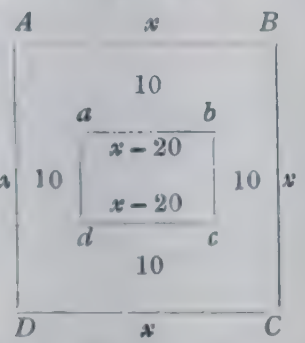
$\frac{8}{9}$ viss of the inferior sort and $(1-\frac{8}{9})$ or $\frac{1}{9}$ viss of the superior sort.

$\therefore$ the proportion of the mixture is superior ghee: inferior ghee $=\frac{1}{9}:\frac{8}{9}=1:8$

Ex. 7. An officer can form his men into a *hollow* square 10 deep. The number of men in the regiment is 2,800. Find the number of men in the front of the hollow square.

Let x = the number of men in the front of the hollow square;

then the number of men in the square $=x^2-(x-20)^2$.



By the question,

$x^2-(x-20)^2=2800$

$\therefore 40x-400=2800$

$\therefore 40x=3200$

$\therefore \quad x=80$

$\therefore$ the number of men in the front of the hollow square is 80.

Note.—$ABCD$ is a *solid* square having x men in the front; $abcd$ is another having x.—20 men in the front, for the difference in the number of men between AB and ab is twice the depth of the hollow square, *i.e.*, 2×10 or 20.

And $ABCD$—$abcd$ is a *hollow* square having x men in the front but only 10 deep.

Ex. 8. A man rides one-third of the distance from A to B at the rate of a miles per hour and the remainder at the rate of $2b$ miles per hour. If he had travelled at a uniform rate of $3c$ miles per hour he could have ridden from A to B and back again in the same time. Prove that $\frac{1}{a}+\frac{1}{b}=\frac{2}{c}$.

Let x = the distance from A to B in miles;

then $\frac{\frac{1}{2}x}{a}$ = time in hours taken to ride $\frac{1}{2}$ of the distance

$\frac{\frac{1}{2}x}{2b}$ " " " $\frac{1}{2}$ " "

and $\frac{2x}{3c}$ = " " " from A to B and back again.

By the question,

$$\frac{\frac{1}{2}x}{a}+\frac{\frac{1}{2}x}{2b}=\frac{2x}{3c}.$$

$$\therefore \frac{x}{3x}+\frac{2x}{6b}=\frac{2x}{3c}$$

$$\therefore \frac{1}{a}+\frac{1}{b}=\frac{2}{c}.$$

EXERCISE 89.

1. Find two numbers the difference of which is 14, and the greater number being divided by the less is 8.

2. Divide the number A into 4 parts, such that if to the first you add B, from the second subtract B, multiply the third by B, and divide the fourth by B, the results will be all equal. If $A=90$, $B=2$, what will the results be?

3. From a certain sum of money I took away one-third part and put in its stead Rs. 50; from the sum thus increased I took away one-fourth part and put in its stead Rs. 70. I then found I had Rs. 120. What was the original sum?

4. A certain fraction is equal to $\frac{2}{3}$; when its numerator is increased by 5, and its denominator by 9 it becomes $\frac{5}{8}$. Find the fraction.

5. Find a number of three digits, each greater by unity than that which follows it, so that its excess above $\frac{1}{4}$th of the number formed by inverting the digits shall be 36 times the sum of the digits.

6. A person bought a certain number of eggs at 2 a penny and as many at 3 a penny, and sold them at the rate of 5 for 2d. losing 4d. by the bargain. How many eggs did he buy?

7. A person bought equal number of two kinds of sheep, one at £3 each, the other at £4 each. If he had expended his money equally on the two kinds, he would have had 2 sheep more than he did. How many did he buy?

8. At a contested election, 1 793 votes are polled, and the defeated candidate is left in a minority of 313. Find the number of votes for each candidate.

9. A landlord let his farm for £10 a year in money, and a corn-rent. When corn sold at 10*s.* a bushel he received at the rate of 10*s.* an acre for his land, but when it sold at 13*s.* 6*d.* a bushel, 13*s.* an acre. Of how many bushels did the corn-rent consist?

10. A person buys some tea at 3*s.* a lb. and some at 5*s.* a lb.; he wishes to mix them so that by selling the mixture at 3*s.* 8 *d.* a lb. he may gain 10 per cent. on his outlay. In what proportion should he mix them?

11. A ship sails with a supply of biscuit for 60 days at a daily allowance of 1 lb. a head; after being at sea 20 days, she encounters a storm in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ ths of a lb. Find the original number of the crew.

12. On a certain morning mangoes were sold at a certain price per score; the next morning as many mangoes could be bought for one rupee as scores for Rs. 30 the day before; the whole price of 30 mangoes, 15 bought one day and 15 the other, was 12 as. 6 p. Find the price of a mango on each day.

13. A person bought a picture at a certain price and paid the same price for the frame; if the frame had cost £ 1 less and the picture 15*s.* more the price of the frame would have been only half that of the picture. Find the cost of the picture.

14. How many bundles of hay at Rs. 5 per thousand must a person mix with 5,600 bundles at Rs. 6 per thousand in order that he may gain 20 per cent. by selling the whole at 11 annas per hundred?

15. A garrison of 1,500 men have provisions for 36 days; but after 16 days it was reinforced; and the provisions were exhausted in 12 days. Find the number of men in the reinforcement.

16. How much tea at 4*s.* 6*d.* per lb. must be mixed with 50 lbs. at 6*s.* per lb. so that the mixture may be worth 5*s* 6*d.* per lb. ?

17. From two bags containing the same number of coins, sums are taken in the ratio of 4 : 5. And if 12 more coins were taken from the bag which now has the fewer, the number of coins taken from it would be double that taken from the other. How many coins were taken from each ?

18. At what time between 7 and 8 o'clock will the hour and minute hands of a clock be

(1) together,

(2) at right angles,

and (3) in a straight line.

19. A man hires a labourer on this condition, that for every day he worked he should receive 2*s.* ; but that for every day he was absent he was to forfeit 1*s.* 4*d.* ; when 390 days were past neither of them was indebted to the other. How many days did the man work ?

20. A labourer is engaged for n days, on condition that he receives p pence for every day he works, and pays q pence for every day he is idle. At the end of the time he receives a pence. How many days did he work and how many days was he idle ?

21. A greyhound spying a hare at the distance of 60 of his own leaps from him pursues her, making 4 leaps for every 5 leaps of the hare, but passing over as much ground in 3 leaps as the hare does in 4. How many leaps did each make during the whole course ?

22. A can do a piece of work in 50 days, B in 60 days and C in 75 days. In what time will they do it, all working together ?

23 A can do a piece of work in 4 hours, C in $3\frac{3}{5}$ hours and B in $5\frac{1}{7}$ hours ; in what time will they do the work together ?

24. To complete a certain work A requires m times as many days as B and C together ; B requires n times as many days as A and C together and C requires p times as many days as A and B together. Prove that $\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1$.

25. Two passengers are charged for excess of luggage 2*s.* 10*d.* and 7*s.* 6*d.* respectively ; had the luggage all belonged to one of them

he would have been charged for excess 14*s.* 6*d.* How much would they have been charged if none had been allowed free?

26. Two persons *A* and *B* were engaged in counting a number of sovereigns, and *A* counted three for every two counted by *B*; when *B* had counted 22 he forgot his reckoning and was obliged to recommence; and when he had counted 64 there were no more left to count. Find the number of sovereigns.

27. At the review of an army the troops were drawn up into a solid mass 40 deep, when there were just $\frac{1}{4}$th as many men in front as there were spectators. Had the depth however been increased by 5, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before. Find the number of men in the army.

28. An officer on attempting to draw up his regiment in the form of a solid square finds that he has 31 men over and that he would require 24 men more in order to increase the side of the square by 1 man. How many men were there in the regiment?

29. An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former. Find the number of men.

30. A person has a number of rupees which he tries to arrange in the form of a square. On the first attempt he has 116 over. When he increases the side of the square by three rupees he wants 25 to complete the square. How many rupees has he?

31. A rower who can pull at the rate of 6 miles an hour can pull 10 miles down the river in half the time that he will take to pull 10 miles up it. Find the rate at which the river flows.

32. A person rows from Cambridge to Ely, a distance of 20 miles, and back again in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

33. A person has just 2 hours at his disposal. How far may he ride in a coach which travels 12 miles an hour, so as to return home in time, walking back at the rate of 4 miles an hour?

34. The number of soldiers present at a review is such that they could be formed into a solid square and also could be formed into four hollow squares, each four deep and each containing twenty-four more men in the front rank than when formed into a solid square. Find the whole number of soldiers. (S.S.L.C. 1918)

35. If a train, which travels at the rate of 35 miles an hour, start one-quarter of an hour after a luggage train, and overtake it in 10 minutes, find the speed of the luggage train.

36. Two persons walk at the rate of 5 and 6 miles an hour respectively; they set out to meet each other from two places 22 miles apart. Having passed each other once, find the place of their second meeting, supposing them to continue their journey between the two places. Also find the time when the second meeting takes place.

37. A, B, C, travel from the same place at the rate of 4, 5, 6 miles an hour respectively and B starts 2 hours after A. How long after B must C start in order that they may both overtake A at the same instant?

38. A train which travels at the uniform rate of 40 miles an hour meets a person walking along the line in the opposite direction at the rate of 4 miles an hour, and passes him in $5\frac{1}{2}$ seconds. Find the length of the train.

39. Two trains running at the rates of 25 and 20 miles respectively on parallel rails in opposite directions are observed to pass each other in 8 seconds; and when they are running in the same direction at the same rates as before, a person sitting in the faster train observes that he passes the other in $31\frac{1}{2}$ seconds. Find the lengths of the trains.

40. Two coaches start at the same time from York and London, a distance of 200 miles; the one from London travels at $9\frac{1}{2}$ miles an hour, and that from York at $10\frac{1}{2}$ miles. Where and when will they meet?

41. Two men set out on a journey, walking at the rate of 4 miles an hour; after walking for six hours, one of the two lessens his rate to 3 miles an hour; the other continues on at the same rate, and arrives at the end of his journey an hour before his companion. Find the length of their journey.

42. A railroad runs from A to C; a goods train starts from A at 12 o'clock and a passenger train at 1 o'clock: after going two-thirds of the distance the goods train breaks down, and can only travel at three-fourths of its former speed. At 20 minutes before 3, a collision occurs, 10 miles from C. The rate of the passenger train being double of the diminished speed of the goods train, find the distance from A to C, and the rates of the two trains in miles per hour.

43. A luggage train going at the rate of 10 miles an hour is some distance in advance of an express engine, which starts to overtake it, and just comes up with it in 5 hours; had the express engine travelled 10 miles an hour less, it would have been $7\frac{1}{2}$ hours in coming up with the luggage train. Find the distance between the train and the express engine at first, and the rate at which the latter travelled.

44. AB is a railway 220 miles long; and the trains, P, Q and R travel upon it at 25, 20 and 30 miles per hour respectively; P and Q leave A at 7 A.M., and 8-15 A.M., respectively, and R leaves B at 10-30 A.M., when and where will P be equidistant from Q and R?

CHAPTER XXVI.

HARDER SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

187. We shall work out some examples of harder type below:—

Ex. 1. Solve the equations—

$$\frac{6}{x}-\frac{7}{y}=2. \tag{1}$$

$$\frac{2}{x}+\frac{14}{y}=3. \tag{2}$$

Here the unknown quantities are $\frac{1}{x}$ and $\frac{1}{y}$.

Multiplying (2) by 3, we get

$$\frac{6}{x}+\frac{42}{y}=9 \tag{3}$$

Subtracting (1) from (3), we get

$$\frac{49}{y}=7 \therefore y=7$$

Putting $y=7$ in (1) we get

$$\frac{6}{x}-1=2 \therefore \frac{6}{x}=3.$$

$$\therefore x=2.$$

$$\left.\begin{matrix}\therefore x=2\\ y=7\end{matrix}\right\}$$

Ex. 2 Solve the equations:—

$$\frac{p}{x}+\frac{q}{y}=a \tag{1}$$

$$\frac{q}{x}+\frac{p}{y}=b \tag{2}$$

Here the unknown quantities are $\frac{1}{x}$ and $\frac{1}{y}$.

From (1) $\times q$, we have $\frac{pq}{x}+\frac{q^2}{y}=aq$ (3)

From (2) $\times p$, we have $\frac{pq}{x}+\frac{p^2}{y}=bp$ (4)

Subtracting (4) from (3), get

$$\frac{1}{y}(q^2-p^2)=aq-bp$$

$$\therefore \frac{1}{y}=\frac{aq-bp}{q^2-p^2}$$

$$\therefore y=\frac{q^2-p^2}{aq-bp}$$

Similarly, the value of $\frac{1}{x}$ can be got, hence the value of x.

Ex. 3. Solve the equations :—

$$(a+c)x-(b+d)y=c \quad (1)$$

$$(a+d)x-(b+c)y=d \quad (2)$$

Subtracting (2) from (1), we get

$$x(a-d)+y(c-d)=c-d$$

$$\therefore x+y=1 \quad (3)$$

$$\therefore (b+d)x+(b+d)y=b+d \quad (4)$$

Adding (1) and (4), we get

$$x(a+b+c+d)=b+c+d$$

$$\therefore x=\frac{b+c+d}{a+b+c+d}$$

From (3), $y=1-x=1-\frac{b+c+d}{a+b+c+d}=\frac{a}{a+b+c+d}$.

EXERCISE 90.

Solve the following equations :—

(1) $\left.\begin{aligned}\frac{3}{y}-\frac{1}{x}&=1\\ \frac{2}{5x}+\frac{3}{2y}&=7\end{aligned}\right\}$

(2) $\left.\begin{aligned}\frac{3}{x}+\frac{1}{y}&=2\\ \frac{4}{x}-\frac{3}{y}&=\frac{7}{12}\end{aligned}\right\}$

(3) $\left.\begin{aligned}\frac{x}{4}+\frac{2}{y}&=2\\ \frac{2x}{5}+\frac{3}{2y}&=\frac{47}{20}\end{aligned}\right\}$

(4) $\left.\begin{aligned}\frac{2}{x}+\frac{3}{y}&=1\\ \frac{7}{x}+\frac{4}{y}&=\frac{15}{8}\end{aligned}\right\}$

(5) $\left.\begin{aligned}2x+3y&=13xy\\ 8x+9y&=42xy\end{aligned}\right\}$

(6) $\left.\begin{aligned}\frac{2a}{x}-\frac{b}{y}&=1\\ \frac{2b}{y}-\frac{a}{x}&=1\end{aligned}\right\}$

(7) $axy = c(bx + ay)$, $bxy = c(ax - by)$

(8) $\frac{x}{a+b} + \frac{y}{a-b} = 2$, $x + y = 2a$

(9) $ax + by = c$, $px + qy = r$

(10) $\frac{x}{a} - \frac{y}{b} = m$, $\frac{x}{c} + \frac{y}{d} = n$

(11) $\frac{x}{a} + \frac{y}{b} = 1 - \frac{x}{c}$, $\frac{y}{a} + \frac{x}{b} = 1 + \frac{y}{c}$

(12) $a(x+y) + b(x-y) = 1$, $c(x+y) + d(x-y) = 1$

(13) $a^2x + bcy = b^2$, $ax + by = c$

(14) $a^2x + b^2y = n$, $b^2x + a^2y = -n$

(15) $\frac{a+x}{ax} - \frac{b+y}{by} = \frac{a+b}{ab}$, $\frac{1}{x} + \frac{1}{y} = \frac{2}{a}$

(16) $\frac{a^2}{x} + \frac{b^2}{y} = \frac{(a+b)^2}{xy}$, $x + y = \frac{4b}{a}$

(17) $(a+b)x + (a-b)y = 2a$, $(a-b)x + (a+b)y = 2a$

(18) $ax + by = c^2$, $\frac{a}{b+y} - \frac{b}{a+x} = 0$

(19) $(a-d)x + (b-d)y = (b-a)(c-d)$, $ax + by = (b-a)c$

(20) $x + y + 1 = 0$; $a^2x + b^2y + c(a+b) = ab$

(21) $(b-a)x - (c-b)y = c - a$, $(a-c)x - (b-a)y = b - c$

(22) $(b+c-a)x + (c+a-b)y = 2c$, $(c+a-b)x + (a+b-c)y = 2a$

(23) $ax + by = a^3 + b^3$, $(b-c)x + (c-a)y = (b-c)a^3 + (c-a)b^3$

24. $(a+b)x - (a-b)y = \frac{a+5b}{5}(x^2 - y^2)$; $(a-b)x + (a+b)y = \frac{5a-b}{5}(x^2 - y^2)$.

25. $\sqrt{y-x} - \sqrt{a-x} = \sqrt{y-x}$, $2\sqrt{y-x} + 9\sqrt{a-x} = 5\sqrt{a-x}$.

26. $\sqrt{x} - \sqrt{a+x} = \sqrt{x+y}$, $3\sqrt{a+x} + 2\sqrt{x+y} = 9\sqrt{a+x}$.

27. $\frac{a+b}{x} + \frac{a-b}{y} = \frac{2a}{a^2 - b^2}$, $\frac{1}{x} + \frac{1}{y} = \frac{2(a^2+b^2)}{(a^2-b^2)^2}$.

28. $(6a+b)x + (a+6b)y = \frac{xy(a+b)}{49}$; $(9a-2b)x - (2a-9b)y = \frac{xy(a+b)}{49}$

29. $\frac{x}{a+b} + \frac{y}{a-b} = 2$; $b\left(\frac{1}{x} + \frac{1}{y}\right) = a\left(\frac{1}{y} - \frac{1}{x}\right)$.

30. $x+y=a+b$, $(a+2b)^3x-(2a+b)^3y=(a-b)^3(a+b)$.

31. $\dfrac{x}{a^2+1}+\dfrac{y}{b^2+1}=\dfrac{1}{a^2-b^2}$; $\dfrac{x}{a^2-1}-\dfrac{y}{b^2-1}=\dfrac{1}{a^2+b^2}$.

32. $\dfrac{x}{a}+\dfrac{y}{b}=1$, $\dfrac{b}{x}+\dfrac{a}{y}=\dfrac{a^2}{by}$.

33. $(pa+qb)x+y(pa^2+qb^2)=pa^2+qb^2$; $(pa^2+qb^2)x+y)(pa^3+qb^3)=pa^4+qb^4$.

34. Find a linear integral function of x which will vanish when $x=\frac{3}{5}$ and will become -1 when $x=-\frac{1}{3}$.

35. If $x+3y=4$ and $2x+5y=7$ find the value of $7x-y$ and $5x+7y$.

36. If the equation $y=ax+b$ is satisfied by the values $(2,-3)$, $(-3,2)$ of x and y respectively find $a+b$.

37. Find the equation to the straight line passing through the points $(1,-2)$, $(-\frac{1}{2},\frac{1}{4})$.

38. Find the equation to the straight line joining the points $(-\frac{1}{3},\frac{1}{4})$ and $(\frac{2}{5},-1)$.

39. Construct a homogeneous symmetric function of x and y of the second degree which shall have values 2, 28 corresponding to values (1,0) and (2,2) of x and y respectively.

40. If P be a homogeneous symmetric function of x and y of the second degree and has the values $15, -5$ when x and y are equal to (i) 1, 2 (ii) $2, -1$ respectively, find the value of P when $x=0$, and $y=1$.

41. Find algebraically the equation to the straight line that joins the origin with the point of intersection of the lines represented by $x+y=4$ and $2x-y+1=0$.

42. Find the equation to the straight line that passes through the points of intersection of the straight lines represented by the following pairs of equations :—

(i) $y=2x$, $x+y=3$.

(ii) $2y=x$, $3x-2y=2$.

43. Show algebraically that the straight lines represented by the equations $4x+3y=5$ $2x-3y=8$.

cut on the line $10x-6y=29$.

44. Determine k so that

$2x+y=7$, $x+y=5$, $2x-y=k$ may be consistent.

45. Find the value of p so that

$$4x - y - 1 = 0$$
$$3x + 2y - 13 = 0$$
and $$px - 2y + 7 = 0$$

may have a common solution.

188 Simultaneous Equations of the first degree involving three or more unknown quantities.

We have seen that when two unknown quantities are involved, one equation of them is eliminated and a relation in only one unknown is obtained and solved. So if more than two are involved, *e.g.*, for example, three, one of them must be eliminated and an equation in only two unknowns should be obtained, and since we must have two equations to enable us to determine the values of two unknowns, one more equation in the same two unknowns should be obtained and the two equations in two unknowns should be solved. The value of the third may be obtained by substitution in one of the three equations. If, however, 4 equations involving 4 unknowns be given we must get 3 equations in only 3 unknowns out of the 4 by taking two and two of the given equations and eliminating the same unknown quantity.

By a similar process, 2 equations in only 2 unknowns should be got, and finally one equation in one unknown solved.

If we have three independent equations, involving three unknown quantities, for example,

$$ax + by + cz = m \quad \ldots\ldots(1)$$
$$a_1x + b_1y + c_1z = n \quad \ldots\ldots(2)$$
$$a_2x + b_2y + c_2z = p \quad \ldots\ldots(3)$$

the values of x, y and z may be determined thus :—

x may be eliminated by

Multiplying (1) by a_1, $aa_1x + a_1by + a_1cz = a_1m$

Multiplying (2) by a, $aa_1x + ab_1y + ac_2z = an$.

$$\therefore y(a_1b - ab_1) + z(a_1c - ac_1) = a_1m - an \quad \ldots\ldots(4)$$

Again we must get another equation in y and z.

For this purpose take any pair, *i. e.* 1 and 3 and eliminate x

By multiplying (1) by a_2, $aa_2x + ab_2y + a_2cz = a_2m$.

Multiplying (3) by a, $aa_2x + ab_2y + ac_2z = ap$.

$$\therefore y(a_2b - ab_2) + z(a_2c - ac_2) = a^2m - ap \quad \ldots\ldots(5)$$

The values of y and z may be determined from the equations (4) and (5); and by substituting their values so obtained in any of the given equations the value of x can be determined.

Note.—*This method may be extended to equations containing four or more unknown quantities.*

Ex. 1. Solve the equations—

$$4x-2y+5z=18 \quad \ldots\ldots(1)$$
$$2x+4y-3z=22 \quad \ldots\ldots(2)$$
$$6x+7y-z=63 \quad \ldots\ldots(3)$$

From (1), $4x-2y+5z=18$.

From (2) $\times 2$, $4x+8y-6z=44$.

Subtracting, $10y-11z=26$ ..... (4)

From (2) $\times 3$, $6x+12y-9z=66$.

From (3), $6x+7y-z=63$.

Subtracting, $5y-8z=3$ ..... (5)

From (4), $10y-11z=26$.

From (5) $\times 2$, $10y-16z=6$.

$\therefore 5z=20$.

$\therefore z=4$.

From (4), $10y=26+11z=26+44=70$.

$\therefore y=7$.

From (1), $4x=18+2y-5z=18+14-20=12$.

$\therefore x=3$.

Ex. 2. Solve the equations—

$$x+y=c \quad \ldots\ldots(1)$$
$$y+z=a \quad \ldots\ldots(2)$$
$$z+x=b \quad \ldots\ldots(3)$$

Adding (1), (2) and (3), $2(x+y+z)=a+b+c$.

$$\therefore x+y+z=\frac{a+b+c}{2} \quad \ldots\ldots(4)$$

Subtracting (1) from (4), $z=\frac{a+b+c}{2}-c=\frac{a+b-c}{2}$

..... (2) ..... $x=\frac{a+b+c}{2}-b=\frac{b+c-a}{2}$

..... (3) ..... $y=\frac{a+b+c}{2}-b=\frac{a+c-b}{2}$

EXERCISE 91.

Solve the following equations :—

1. $x+y+z=6$
$x+2y+3z=14$
$x+3y+4z=19.$

2. $x+y-z=0$
$2x+3y-4z=-4$
$3x-2y-3z=-10$

3. $x+y+z=90$
$2x-3y=-20$
$2x-4z=-30.$

4. $2x+3y+4z=29$
$3x+2y+5z=32$
$4x+3y+2z=25.$

5. $5x+3y=65.$
$2y-z=11$
$3x+4z=57.$

6. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62$
$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47$
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38$

7. $x+y=3$
$y+z=5$
$z+x=4.$

8. $x+y-z=6.$
$z+x-y=10$
$y+z-x=14.$

9. $y+\frac{z}{3}=\frac{x}{5}+5$
$\frac{x-1}{4}-\frac{y-2}{5}=\frac{z+3}{10}$
$x-\frac{2y-5}{3}=1\frac{3}{4}-\frac{z}{12}.$

10. $\frac{x}{2}-\frac{y}{3}+\frac{z}{4}=22$
$\frac{x}{3}+\frac{y}{4}-\frac{z}{5}=-1$
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38$

11. $\frac{1}{x}-\frac{5}{y}+\frac{1}{z}=-3.$
$\frac{3}{x}+\frac{4}{y}-\frac{2}{z}=5$
$\frac{5}{x}+\frac{7}{y}+\frac{6}{z}=18.$

12. $\frac{a}{x}+\frac{b}{y}=\frac{1}{m}$
$\frac{b}{y}+\frac{c}{z}=\frac{1}{n}$
$\frac{c}{z}+\frac{a}{x}=\frac{1}{p}.$

13. $\frac{1}{x}+\frac{1}{y}$ or $\frac{x+y}{xy}=70$
$\frac{1}{y}+\frac{1}{z}$ or $\frac{y+z}{yz}=140.$
$\frac{1}{z}+\frac{1}{x}$ or $\frac{z+x}{zx}=84.$

14. $\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=\frac{38}{5}$
$\frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=\frac{61}{6}$
$\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=\frac{161}{10}.$

15. $\frac{2}{x}-\frac{1}{y}=\frac{3}{z};\ \frac{3}{z}+\frac{2}{y}=2;\frac{1}{x}+\frac{1}{z}=\frac{4}{3}.$

16. $x+\frac{1}{2}(y+z)=y+\frac{2}{3}(z+x)=z+\frac{3}{4}(x+y)=x+y+z-1.$

17. $\dfrac{6y-4x}{3z-7}=\dfrac{5z-x}{2y-3z}=\dfrac{y-4z}{3y-2x}=1.$

18. $x+y+z=0.$
$(a+b)x+(a+c)y+(b+c)z=0$
$abx+acy+bcz=1.$

19. $xy=a(x+y)$
$yz=c(y+z).$
$zx=b(z+x).$

20. $x-ay+a^2z=a^3$
$x-by+b^2z=b^3$
$x-cy+c^2z=c^3.$

21. $bx+ay=c.$
$cx+az=b$
$cy+bz=a.$

22. $\dfrac{a}{y}+\dfrac{b}{z}=\dfrac{a}{z}+\dfrac{b}{x}=\dfrac{a}{x}+\dfrac{b}{y}=c.$

23 $3(x-z)=by;\ 2(x-y)=z;\ (y-z)=1.$

24. $xyz=a(yz-zx=xy)=b(zx-yz-xy)=c(xy-yz-zx).$

25. $ax+by+cz=a+b+c;\ x+y+z=3;\ ay+bx=a+b.$

26. $\dfrac{ay+bx}{c}=\dfrac{bz+cy}{a}=\dfrac{az+cx}{b}=1.$

27. $x+y+z+u=10$
$x-2y+3z=6$
$y+z-2u=-3$
$3x-2z+4u=13.$

28. $x+y+z+u=0$
$15u+13x=-(11y+9z)$
$71u+31y=-(47x+23z)$
$1+105u=-(45x+21y).$

29. $x+y+z+u=10$
$x+y+z-u=2$
$y+z+u-x=8$
$z+x+u-y=6.$

30. $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=6$
$\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{u}=7$
$\dfrac{1}{x}+\dfrac{1}{z}+\dfrac{1}{u}=8$
$\dfrac{1}{y}+\dfrac{1}{z}+\dfrac{1}{u}=9$

31. $ax+by+cz=mxyz$
$bx+cy+az=nxyz$
$cx+ay+bz=bxyz$

32. $xy+a(x+y)=yz+a(y+z$
$=zx+a(z+x)$
$=3a^2$

33. The expression ax^2+bx+c takes the values 0, 1, 2 according as $x=-1, \frac{1}{2}, 1$ respectively; find a, b, c.

34. Construct an expression of the second degree in x which will have the values $-1, -3, 1$ corresponding to the values $0, 1, 2$ of x respectively.

35. If y, be an integral function of x of the second degree and its values be 6, 15, 28 when $x=1, 2, 3$ respectively, find its values where $x=-1$

36. The expression $ax+by+c$ has the values $-1, 1, -2$ corresponding to the values $(0, 1), (1, -1), (0, 0)$ of x, y respectively: determine a, b, c.

37. Construct a linear integral function of x, y which will have the values 4, 9, 14 corresponding to the values $(0, 1), (1, 2), (2, 3)$ of x, y respectively.

38. Find an expression of the second degree in x such that it may be equal to $-\frac{2}{3}, \frac{1}{4}, \frac{5}{6}$ when $x=\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively.

39. Construct a symmetric homogeneous expression of the third degree in x, y, z which will have the values 1, 4, 90 corresponding to the values $(0, 0, 1), (1, 1, 0), (1, 2, 3)$ of x, y, z respectively.

40. Construct an equation to a straight line which cuts an intercept of 3 units on the y axis and passes through the point $(1, 5)$.

189. Method of Cross Multiplication.

If $a_1x+b_1y+c_1z=0$

and $a_2x+b_2y+c_2z=0$,

shew that $\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}$.

Multiply (1) by a_2; $a_1a_2x+a_2b_1y+a_2c_1z=0$.

Multiply (2) by a_1; $a_1a_2x+a_1b_2y+a_1c_2z=0$.

Subtracting, $y(a_2b_1-a_1b_2)+z(a_2c_1-a_1c_2)=0$.

$\therefore\ y(a_2b_1-a_1b_2)=-z(a_2c_1-a_1c_2)$.

$\therefore \frac{y}{c_1a_2-a_1c_2}=\frac{z}{a_1b_2-a_2b_1}$ ..... (1)

Again, multiply (1) by b_2; $a_1b_2x+b_1b_2y+b_2c_1z=0$.

Multiply (2) by b_1; $a_2b_1x+b_1b_2y+b_1c_2z=0$.

Subtracting, $x(a_1b_2-a_2b_1)+z(b_2c_1-b_1c_2)=0$.

$\therefore x(a_1b_2-a_2b_1=-z(b_2c_1-b_1c_2)$.

$$\therefore \frac{x}{b_1c_2-b_2c_1}=\frac{z}{a_1b_2-a_2b_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence from (1) and (2),

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}.$$

These results can be easily remembered by writing down the co-efficients thus :—

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 \\ a_2 & b_2 & c_2 & a_2 \end{matrix}$$

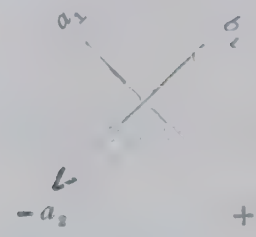
The quantity under x is obtained by writing the co-efficient of y and z and multiplying across thus :—

$$\begin{matrix} b_1 & & c_1 \\ & \times & \\ -b_2 & & +c_2 \end{matrix} \qquad =b_1c_2-c_1b_2$$

The quantity under y is obtained in the same way from the co-efficients of z and x.

$$\begin{matrix} c_1 & & a_1 \\ & \times & \\ -c_2 & & +a_2 \end{matrix} \qquad =c_1a_2-a_1c_2.$$

The quantity under z is obtained in the same way from the coefficients of x and y.



$$= a_1b_2 - b_1a_2.$$

If $z=1$ in the above equations,

we have $\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{y}{c_1a_2 - c^2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$

which gives the solution of the equations,

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

If the above values are committed to memory, it will enable us to solve readily simple equations involving two unknown quantities.

Ex. 1. Solve $5y - 3x = 9$; $5x + 2y = 16$.

Writing the equations thus: $3x - 5y + 9 = 0 \quad . \quad . \quad , (1)$

$5x + 2y - 16 = 0 \quad . \quad . \quad . (2)$

Here $a_1 = 3$, $b_1 = -5$, $c_1 = 9$

and $a_2 = 5$, $b_2 = 2$, $c_2 = -16$.

$$\therefore \frac{x}{(1-5)(-16) - 9 \times 2} = \frac{y}{9 \times 5 - 3(-16)} = \frac{1}{3 \times 2 - (-5)5}$$

$$\text{or } \frac{x}{80 - 18} = \frac{y}{45 + 48} = \frac{1}{6 + 25}.$$

$$\therefore \frac{x}{62} = \frac{y}{93} = \frac{1}{31}.$$

$$\therefore x = \frac{62}{31} = 2.$$

$$\text{and } y = \frac{93}{31} = 3$$

Ex. 2. Solve $\dfrac{3}{x} + \dfrac{4}{y} = 2$; $\dfrac{2}{x} + \dfrac{3}{y} = \dfrac{17}{12}$.

Write the equations thus $3\left(\frac{1}{x}\right) + 4\left(\frac{1}{y}\right) - 2 = 0 \quad . \quad . \quad (1)$

$$2\left(\frac{1}{x}\right)+3\left(\frac{1}{y}\right)-\frac{17}{12}=0 \ldots (2)$$

Hence

$$\frac{\frac{1}{x}}{4\left(-\frac{17}{12}\right)-(-2\times3)} = \frac{\frac{1}{y}}{(-2\times2)-3\left(-\frac{17}{12}\right)} = \frac{1}{3\times3-2\times4}$$

$$\text{or } \frac{\frac{1}{x}}{-\frac{17}{3}+6} = \frac{\frac{1}{y}}{-4+\frac{17}{4}} = \frac{1}{9-8},$$

$$\text{or } \frac{\frac{1}{x}}{\frac{1}{3}} = \frac{\frac{1}{y}}{\frac{1}{4}} = 1,$$

$$\text{or } \frac{1}{x} = \frac{1}{3} \text{ or } x=3,$$

$$\text{and } \frac{1}{y} = \frac{1}{4} \text{ or } y=4.$$

Ex. 3. Solve $x-2y+z=0$ ..... (1)

$$9x-8y+32=0 \ldots\ldots (2)$$

$$2x+3y+5z=36 \ldots\ldots (3)$$

From (1) and (2),

$$\frac{x}{(-2)\times32-(-8)\times(-8)} = \frac{y}{1\times32-1\times9} = \frac{z}{1\times(-8)-9\times(-2)}$$

$$\text{or } \frac{x}{-64-64} = \frac{y}{32-9} = \frac{z}{-8+18},$$

$$\text{or } \frac{x}{-128} = \frac{y}{23} = \frac{z}{10} = k \text{ (suppose)};$$

then $x=-128k$, $y=23k$ and $z=10k$.

Substituting in (3), we get $-256k+69k+50k=36$,

or $-137k=36$, or $k=-\frac{36}{137}$,

$\therefore x=-128\times-\frac{36}{137}=\frac{4608}{137}$,

$y=23\times-\frac{36}{137}=-\frac{828}{137}$,

and $z=10\times-\frac{36}{137}=-\frac{360}{137}$.

Ex. 4. Solve $x+y+z=a+b+c$ ..... (1)

$$ax+by+cz=0 \ldots\ldots (2)$$

$$bx+cy+az=a^2+b^2+c^2-ab-ac-bc \ldots\ldots (3)$$

From (1) and (2), $\frac{x}{c-b}=\frac{y}{a-c}=\frac{z}{b-a}=k$ (suppose).

$\therefore x=k(c-b)$, $y=k(a-c)$ and $z=k(b-a)$.

Substituting these values in (3), we have—

$bk(c-b)+ck(a-c)+ak(b-a)=a^2+b^2+c^2-ab-ac-bc$.

$\therefore k\{bc-b^2+ca+c^2+ab-a^2\}=a^2+b^2+c^2-ab-ac-bc$.

$\therefore k=-1$.

$\therefore x=-(c-b)=(b-c)$,

$y=-(a-c)=(c-a)$,

and $z=-(b-a)=(a-b)$.

Ex. 5. Solve $x+y+z=a+b+c$.....(1)

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3 \text{ .....(2)}$$

$$ax+by+cz=a^2+b^2+c^2 \text{ .....(3)}$$

The questions (1) and (2) may be written thus :—

$$(x-a)+(y-b)+(z-c)=0.$$

$$\frac{1}{a}(x-a)+\frac{1}{b}(y-b)-\frac{1}{c}(z-c)=0.$$

Hence, $\frac{x-a}{\frac{1}{c}-\frac{1}{b}}=\frac{y-b}{\frac{1}{a}-\frac{1}{c}}=\frac{z-c}{\frac{1}{b}-\frac{1}{a}}$

or $\frac{bc(x-a)}{b-c}=\frac{ac(y-b)}{c-a}=\frac{ab(z-c)}{a-b}=k$ (suppose)

Then $x-a=\frac{k(b-c)}{bc}$ or $x=a+\frac{k(b-c)}{bc}$.

Similarly $y=b+\frac{k(c-a)}{ac}$.

and $z=c+\frac{k(a-c)}{ab}$.

Substituting the values of x, y, z in (3), we have—

$$a^2+\frac{ak(b-c)}{bc}+b^2+\frac{bk(c-a)}{ac}+c^2+\frac{ck(a-b)}{ab}=a^2+b^2+c^2$$

$$\therefore k\left\{\frac{a(b-c)}{bc}+\frac{b(c-a)}{ca}+\frac{c(a-b)}{ab}\right\}=0$$

$$\therefore k=0.$$

$$\therefore x=a,\ y=b \text{ and } z=c.$$

EXERCISE 92.

Solve the following equations :—

1. $4x+3y-24=0$ and $5x-2y-7=0$.
2. $x+11y-67=0$ and $10x+2y-130=0$.
3. $11x-7y-41=0$ and $7x-5y-19=0$.
4. $\frac{x}{2}+\frac{y}{5}-11=0$ and $\frac{x}{7}+\frac{y}{4}-7=0$.
5. $\frac{12}{x}-\frac{4}{y}-2=0$ and $\frac{15}{x}-\frac{8}{y}-1=0$.
6. $ax+y-p=0$ and $x-ay-q=0$.
7. $5x+6y+8z=0$, $3x+4y+6z=0$ and $x+5y+16z=3$.
8. $4x-4y+2z=0$, $2x-10y+7z=0$ and $3x+4y+5z=35$.
9. $x+y+z=0$, $ax+by+cz=0$
   and $a^2x+b^2y+c^2z=(b-c)(c-a)(a-b)$.
10. $x+y+z=0$, $ax+by+cz=0$
    and $bcx+cay+abz=(a-b)(b-c)(a-c)$.
11. $ax+by+cz=0$, $bcx+cay+abz=0$ and $x+y+z=1$.
12. $(b-c)x+(c-a)y+(a-b)z=0$, $ax+(a-c)y-cz=0$
    $x+y+z=a+b+c$.
13. $x+y+z=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, $a^2x+b^2y+c^2z=a+b+c$,
    and $a^3x+b^3y+c^3z=a^2+b^2+c^2$.
14. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$, $\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$
    and $ax+by+cz=a^2+b^2+c^2$.
15. $x+y+z=0$, $bcx+cay+abz=1$ and $(b+c)x+(c+a)y+(a+b)z=0$.
16. $x+y+z=a+b+c$, $bx+cy+az=a^2+b^2+c^2$
    and $cx+ay+bz=a^2+b^2+c^2$.
17. $x+y+z=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, $ax+by+cz=3$,
    and $a^2x+b^2y+c^2z=a+b+c$.
18. $x+y+z=a+b+c$, $bx+cy+az=ab+ac+bc$,
    and $cx+ay+bz=ac+ab+bc$.
19. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$, $ax+by+cz=a^2+b^2+c^2$.
    and $\frac{x-a}{b+c}+\frac{y-b}{c+a}+\frac{z-c}{a+b}=0$.

20. $x+y+z=0$, $\dfrac{x}{a+b}+\dfrac{y}{a+c}+\dfrac{z}{b+c}=0$,

and $\dfrac{x}{a-b}+\dfrac{y}{c-a}+\dfrac{z}{b-c}=2a+b+c$.

190. Method of Undetermined Multipliers.

Take the equations $ax+by+cz=d$ ..... (1)

$a_1x+b_1y+c_1z=d_1$ ..... (2)

$a_2x+b_2y+c_2z=d_2$ ..... (3)

Multiplying (1) by l, (2) by m, (3) by n and adding, we get

$(al+a_1m+a_2n)x+(bl+b_1m+b_2n)y+(cl+c_1m+c_2n)z$

$=dl+d_1m+d_2n$ ..... (4)

Since l, m and n, are quite arbitrary, we may assume them to be such that the co-efficients of x and y shall vanish;

hence $bl+b_1m+b_2n=0$ ..... (5)

$cl+c_1m+c_2n=0$ ..... (6)

and $(al+a_1m+a_2n)x=dl+d_1m+d_2n$ ..... (7)

From (5) and, (6) by cross multiplication,

$$\frac{l}{b_1c_2-b_2c_1}=\frac{m}{b_2c-bc_2}=\frac{n}{bc_1-b_1c}=k \text{ (suppose)}$$

$\therefore l=(b_1c_2-b_2c_1)k$, $m=(b_2c-bc_2)k$, $n=(bc_1-b_1c)k$.

Substituting these values of l, m, n, in (7), we have—

$\{a(b_1c_2-b_2c_1)+a_1(b_2c-bc_2)+a_2(bc_1-b_1c)\}x$

$=d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c)$.

$$\therefore x=\frac{d(b_1c_2-b_2c_1)+d_1(b_2c-bc_2)+d_2(bc_1-b_1c)}{a(b_1c_2-b_2c_1)+a_1(b_2c-bc_2)+a_2(bc_1-b_1c)}$$

In the same way we can find the value of y by supposing the co-efficients of x and z in (4) to vanish, and the value of z by supposing the co-efficients of x and y in (4) to vanish.

Ex. Solve the equations $x+2y+z=8$ ..... (1)

$2x-y+3z=9$ ..... (2)

$3x+y-\ldots z=2$ ..... (3)

Multiplying (1), (2) and (3) by l, m and n respectively, and adding we have—

$(l+2m+3n)x+(2l-m+n)y+(l+3m-n\ldots=8l+9m+2n$ ..... (4)

Equating the co-efficients of y and z to zero we have—

$l+2m+3n)x=8l+9m+2n$ 5

$2l-m+n=0$ 6

$l+3m-n=0$ ..... (7)

From (6) and (7) by cross multiplication,

$$\frac{l}{-2}=\frac{m}{3}=\frac{n}{7}=k \text{ (suppose).}$$

Substituting $-2k$, $3k$ and $7k$ for l, m and n in (5)

we have $(-2+6+21)x=-16+27+14.$

$$\therefore\ 25x=25.$$

$$\therefore\ x=1.$$

Similarly we can find $y=2$ and $z=3$.

Note.— In Practice, this method is not simpler than the one given in Art. 188 when the co-efficients are numerical.

CHAPTER XXVII.

HARDER PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

191. We shall work out some problems of harder type below.

Ex. 1. There is a number of three digits; the sum of the digits is 9, the digit in the units' place is twice the digit in the hundreds' place, and if 198 be added to the number, the digits are reversed; find the number.

Let x, y and z be the digits in the hundreds', tens' and units' place respectively;

then $100x+10y+z$ is the number,

and $100z+10y+x$ is the number formed when the digits are *reversed*.

By the question, $x+y+z=9$ ..... (1)

$z=2x$ ..... (2)

and $100x+10y+z+198=100z+10y+x$ ..... (3)

From (3), $99x-99z=-198$. $\therefore x-z=-2$,

but $z=2x$ $\therefore x-2x=-2$.

$\therefore x=2$ and $z=4$.

From (1), $y=9-6=3$.

Hence the required number is 234.

Ex. 2. A and B working together can do a certain work in 4 days. A and C can do it in $3\frac{3}{4}$ days, while B and C can do it in $5\frac{1}{7}$ days. How many days will A, B and C each take to do it alone?

Let x = the number of *days* A takes to do the work alone ;

y = the number of *days* B takes to do the work alone ;

and z = the number of *days* C takes to do the work alone ;

Hence in one day A does $\frac{1}{x}$ of the work.

B does $\frac{1}{y}$ of the work.

and C does $\frac{1}{z}$ of the work.

$\therefore$ in one day A and B together do $\left(\frac{1}{x}+\frac{1}{y}\right)$ of the work.

But as they take 4 days to do the whole work in one day they must do $\frac{1}{4}$ of the work.

Therefore $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$ .....(1)

Similarly $\frac{1}{x}+\frac{1}{z}=\frac{1}{3\frac{1}{3}}=\frac{5}{18}$ .....(2)

and $\frac{1}{y}+\frac{1}{z}=\frac{1}{5\frac{1}{4}}=\frac{7}{36}$ .....(3)

From (2) and (3), $\frac{1}{x}-\frac{1}{y}=\frac{1}{12}$;

and from (1), $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$;

$\therefore \frac{1}{x}=\frac{1}{6}$ $\therefore x=6$

and $\frac{1}{y}=\frac{1}{12}$ $\therefore y=12$

and from (2), $\frac{1}{z}=\frac{5}{18}-\frac{1}{6}=\frac{1}{9}$

$\therefore z=9$

$\therefore A$, B, C can do the work alone in 6, 12 and 9 days respectively.

Ex 3. A man has to travel a certain distance; when he has travelled 20 miles, he increases his speed 1 mile per hour; if he had travelled with his increased speed during the whole of his journey, he would have arrived 40 minutes earlier; but if he had kept on at his first rate he would have arrived 20 minutes later; how far had he to travel?

Let x = the distance of the journey in *miles*.

and y = rate per hour in *miles*.

At first he takes $\left(\frac{20}{y}+\frac{x-20}{y+1}\right)$ hours.

When he travels the whole distance at $y+1$ miles per hour he takes $\frac{x}{y+1}$ hours.

and at y miles per hour, he takes $\frac{x}{y}$ hours.

By the question, $\frac{20}{y}+\frac{x-20}{y+1}=\frac{x}{y+1}+\frac{2}{3}$ .....(1)

$$\text{and } \frac{20}{y}+\frac{x-20}{y+1}=\frac{x}{y}-\frac{1}{3} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \frac{x}{y+1}+\frac{2}{3}=\frac{x}{y}-\frac{1}{3}.$$

$$\therefore \frac{x}{y}-\frac{x}{y+1}=1 \text{ [From (1) and (2)]}$$

$$\therefore \frac{x}{y(y+1)}=1.$$

$$\therefore x=y(y+1).$$

Substituting $y(y+1)$ for x in (1) we get—

$$\frac{20}{y}+\frac{y(y+1)-20}{y+1}=\frac{y(y+1)}{y+1}+\frac{2}{3}.$$

$$\therefore \frac{20}{y}+y-\frac{20}{y+1}=y+\frac{2}{3}$$

$$\therefore \frac{20}{y}-\frac{20}{y+1}=\frac{2}{3}.$$

$$\therefore \frac{20}{y(y+1)}=\frac{2}{3}.$$

$$\therefore y(y+1)=\frac{3}{2}\times 20=20.$$

$$\therefore x=30, \text{ since } x=y(y+1).$$

$\therefore$ the distance $=30$ miles.

Ex. 4. A pound of tea and three pounds of sugar cost six shillings, but if sugar were to rise 50 per cent., and tea 10 per cent, they would cost seven shillings. Find the price of tea and sugar.

Let x = the price of a pound of tea in *shillings* ;

and y = the price of a pound of sugar in *shillings*.

When the price of tea *rises* 10 per cent.,

the price of a pound $= x+\frac{1}{10}x$ shillings ;

and when the price of sugar *rises* 50 per cent.,

the price of a pound $= y+\frac{1}{2}y$ or $\frac{3}{2}y$ shillings.

By the question, $x+3y=6$ . . . . . (1)

and $\frac{11}{10}x+3\times\frac{3}{4}y=7$ . . . . . (2)

From (1), $11x+33y=66$,

From (2), $11x+45y=70$.

$$\therefore 12y=4.$$

$$\therefore y=\tfrac{1}{3}.$$

and $x=6-1=5$.

17. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours: it also goes up-stream 40 miles and down-stream 55 miles in 13 hours. Find the rate of the stream and of the boat.

18. A boat's crew rowed $3\frac{1}{2}$ miles down a river and up again in 100 minutes. If the stream had been half as strong again, they would have taken $31\frac{1}{4}$ minutes longer. Find the rate of the stream.

19. A man rowing against a stream meets a log of wood which is being carried down by the current. He continues rowing in the same direction for a quarter of an hour longer, and then turns and rows down the stream overtaking the log $1\frac{1}{2}$ miles lower down than the point where he first met it. Find the rate at which the current flows.

20. A cistern is filled in 25 minutes by 3 pipes one of which conveys 8 gallons more, and another 7 gallons less than the third, every three minutes. The cistern holds 1,050 gallons. How much flows through each pipe in a minute?

21. A and B run a mile. First A gives B a start of 1 minute 30 seconds, and is beaten by 88 yards. At the second heat A gives B a start of 80 yards and beats him by 30 seconds. Find the rates of A and B per hour.

22. Two travellers A and B set out at the same time from two places P and Q respectively and travel so as to meet. When they meet, it is found that A has travelled 30 miles more than B and that A will reach Q in 4 days, and B will reach P in 9 days after the meeting. Find the distance between P and Q.

23. A boy when he started for school, a distance of 4 miles found that if he proceeded at his walking pace he would reach school just in time. But at the end of the first mile, he turned back and ran home to bring a book, and thence back to the end of the first mile, increasing his rate by 1 mile an hour. If by walking the remaining 3 miles at twice his walking pace he reached the school exactly in time, find his walking pace per hour.

24. Two post runners start at the same time from two towns and each proceeds at a uniform rate towards the other. They meet at a point 3 miles nearer one town than the other and they expect to finish the journey in 2 hours 30 minutes, and 3 hours 36 minutes respectively. Find the distance between the towns and the rates of the runners.

25 A and B set out together to walk round a field one mile in circumference. When A, who is the quicker walker has passed B twice, B turns and going in the opposite direction meets A at the starting point. The whole time since they started is 3 hours, and A has walked $2\frac{1}{4}$ miles since they started more than B. Find their rates of walking.

26. A mail cart was travelling from A to B at the rate of 10 miles an hour, and when only 7 miles from B was met by a man who started from B at the same time that the cart left A. The cart then went on to B where remaining 1 hour and 12 minutes, it returned overtaking the man 6 miles from A. Find the distance from A to B and also the rate of the man.

27. A rectangular court having been measured it was observed that, if it were 3 feet broader and 4 feet longer, it would contain 116 square feet more ; but if it were 4 feet broader and 5 feet longer, it would contain 113 square feet more. Find its dimensions.

28. The dimensions of a rectangular court are such that if the length where increased by 3 yards, and the breadth diminished by the same, its area would be diminished by 18 square yards, and if its length were increased by 3 yards, and its breadth increased by the same, its area would be increased by 60 square yards. Find its dimensions.

29. A room of which the floor is rectangular is such that the addition of a foot to the height will increase the area of the walls as much as the addition of a foot to both the length and breadth, the increase in each case being 60 square feet ; and if the floor be made square, the perimeter remaining the same as before, its area will be increased by 9 square feet. Find the length, breadth and height of the room.

30. Find a number of three digits, the last two alike, such that the number formed by the digits inverted may exceed twice the original number by 42 and also the number formed by putting the single figure in the midst by 27.

31. A sum of money is divided equally amoung a certain number of persons ; if there had been four more, each would have received a shilling less than he did ; if there had been five fewer, each would

have received two shillings more than he did ; find the number of persons and what each received.

32. Two plugs are opened in the bottom of a cistern containing 192 gallons of water ; after three hours, one of the plugs becomes stopped, and the cistern is emptied by the other in eleven hours ; had six hours elapsed before the stoppage, it would have required only six hours more to empty the cistern. How many gallons will each plug-hole discharge in an hour, supposing the discharge uniform ?

33. If there were no accidents it would take half as long to travel the distance from A to B by railroad as by coach ; but three hours being allowed for accidental stoppages by the former, the coach will travel the distance all but fifteen miles in the same time ; if the distance were two-thirds as great as it is, and the same time allowed for railway stoppages, the coach would take exactly the same time. Find the distance.

34. A and B run a mile. First A gives B a start of 44 yards and beats him by 51 seconds ; at the second beat A gives B a start of 1 minute 15 seconds, and is beaten by 88 yards. Find the times in which A and B can run a mile separately.

35. A and B start together from the foot of a mountain to go to the summit. A would reach the summit half an hour before B, but missing his way goes a mile and back again needlessly, during which he walks at twice his former pace, and reaches the top six minutes before B. C starts twenty minutes after A and B, and walking at the rate of two and one-seventh miles per hour, arrives at the summit ten minutes after B. Find the rates of walking of A and B and the distance from the foot to the top of the mountain.

36. A railway train after travelling for one hour meets with an accident which delays it one hour, after which, it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time ; had the accident occurred 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Required the length of the line, and the original rate of the train.

37. A railway train running from London to Cambridge meets on the way with an accident, which causes it to diminish its speed to $\frac{1}{m}$th of what it was before, and it is in consequence a hours late. If

the accident had happened b miles nearer Cambridge the train would have been c hours late. Find the rate of the train before the accident occurred.

38. A shop-keeper on account of bad book-keeping, knows neither the weight nor the prime cost of a certain article which he purchased He only recollects that if he had sold the whole at 30*s.* per lb. he would have gained £5 by it, and if he had sold it at 22*s.* per lb. he would have lost £15 by it. What was the weight and prime cost of the article?

39. The rent of a farm is paid in certain fixed numbers of quarters of wheat and barley; when wheat is at 55*s.* and barley at 33*s.* per quarter, the portions of rent by wheat and barley are equal to one another, but when wheat is at 65*s.* and barley at 41*s.* per quarter, the rent is increased by £7. What is the corn rent?

40. A train 60 yards long passed another train 72 yards long which was travelling in the same direction on a parellel line of rails in 12 seconds. Had the slower train been travelling half as fast again, it would have been passed in 24 seconds. Find the rates at which the trains were travelling.

41. A and B run a race round a two-mile course. In the first beat B reaches the winning post 2 minutes before A. In the second beat A increases his speed by 2 miles an hour, and B diminishes his by the same quantity, and A then arrives at the winning post 2 minutes before B. Find at what rate each ran in the first beat.

42. A and B run a mile. At the first beat A gives B a start of 20 yards and beats him by 30 seconds At the second beat A gives B a start of 32 seconds, and beats him by $9\frac{5}{11}$ yards. Find the rate per hour at which A runs.

43. A railway train after travelling an hour is detained 15 minutes after which, it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles further on, the train would have arrived 3 minutes sooner than it did. Find the original rate of the train and the distance travelled.

44. The time which an express train takes to travel a journey of 120 miles is to that taken by an ordinary train as 9 to 14. The ordinary train loses as much time in stoppages as it would take to travel 20 miles without stopping. The express train only loses half as much

time in stoppages as the ordinary train, and it also travels 15 miles an hour quicker. Find the rate of each train.

45. Two trains, 92 feet long and 84 feet long respectively, are moving with uniform velocities on parallel rails; when they move in opposite directions, they are observed to pass each other in one second and a half; but when they move in the same direction the faster train is observed to pass the other in six seconds; find the rate at which each train moves.

46. A general finds that his cavalry with half his artillery and infantry together, or his infantry with one-third of his cavalry and infantry together, or his infantry with one-fourth of his cavalry and artillery together, make up the same number of men, *viz*, 5,950; how many men were there in each arm of the service?

47. There is a certain number of three digits which his equal to 18 times the sum of its digits, and if 198 be subtracted from the number, the digits will be reversed; also the sum of the extreme digits is equal to twice the middle digit; find the number.

48. Three cases of goods cost Rs. 4,000; they are sold again at a profit of 2, 3, 4, per cent. respectively, and the whole profit is 3 per cent. on the total cost; if the first and second cases had been sold for Rs. 5 more each and the third for what it had cost, the profit would have been 2 per cent: what was the cost of each case?

49. In a school consisting of 3 classes, $\frac{2}{3}$ there were in the second class 5 per cent, and in the third class 10 per cent, more than the first class. In an examination, each boy in the first class occupied thrice, and in the second class twice as much of the examiner's time as each boy in the third. The examination then lasted 31 hours. In the next year the first class had doubled in number, but each boy only required $\frac{1}{3}$ of his former time, there were 10 boys more in the second class, and each boy in the third class occupied $\frac{1}{11}$ hour more than he did before. The examination now lasted 43 hours. How many boys were there in the school at first?

50. A person starts from A to walk to B, and after he has gone 16 miles, another starts from B for A and walks at double the rate of the former. Thirty-two minutes after they meet, the slower of the two halts for two hours, and then does the remainder of the distance in 4 hours and 48 minutes. The other proceeds without stopping to

A and immediately returns to B which he reaches 4 hours 24 minutes after the other. Find the distance between A and B, and the distance from B to the halting place.

51. A and B start from opposite corners of a square and run round in the same direction. If B stops for a certain time at every corner he will be caught by A when he is just commencing his $(n+1)$th circuit ; but if B runs continuously and A stops the same time at each corner, A will be caught when he is just commencing his $(m+1)$th circuit. Shew that A's velocity : B's velocity as $4+n^{-1} : 4+m^{-1}$.

52. A man purchases some garden land, and reserving $\frac{5}{11}$ths of it for his own use, sells the rest in such a manner that the actual cost to him for his own portion is reduced by a half. Had he sold it at Rs. 150 more an acre, he would have been a gainer by the whole extent of land which he reserved for himself. At how much an acre did he purchase the land ; and at how much did he sell it ?

(S. S. L. C. 1913.)

53. In repairing a rectangular tank, the length is diminished by 40 yards, and the breadth increased by 25 yards. The tank is still rectangular and the area remains the same ; but the perimeter is reduced by $\frac{1}{30}$th. Form equations to find the present dimensions of the tank and solve them ; and check your result graphically.

(S. S. L. C. 1913.)

54. A and B engaged in play : when A had lost Rs. 20 he had only one-third of the money which B had ; but by continuing to play he not only won back his Rs. 20, but also Rs 50 more with it, and then found he possessed half as much again as B. With what sum did they respectively begin ?

Reduce the conditions of the problem to simultaneous equations in two unknowns and solve the problem. (S. S. L. C. 1917.)

55. A walks half a mile per hour faster than B and three quarters of a mile per hour faster than C. To walk a certain distance C takes three quarters of an hour more than B, and two hours more than A. Find the rates of walking A, B and C. (Matric Madras, 1899.)

56. When the price of sugar rises 50 per cent. and the price of tea 10 per cent., the increase in the cost of 3 lbs. of tea and 4 lbs. of sugar, which together originally cost Rs. 3-8-0 is 12 as. What is the original price of tea ? (Madras Matric 1897.)

57. In a mile race between a bicycle and a tricycle, their rates were proportional to 5 and 4. The tricycle had a minute's start, but was beaten by 176 yards. Find the rates of each.

(Bom. Univer. Ent., 1892-93.)

58. The gross income of a certain man was £ 40 more in the second of two particular years than in the first, but in consequence of the income-tax rising from 4*d.* in the pound to 6*d.* in the pound in the second year, his net income after paying the tax was unaltered. Find his income in each year. (Bom. Univer. Ent., 1891-92.)

59. The forewheel of a carriage makes 12 revolutions more than the hind-wheel in 240 yards ; but only 4 revolutions when the circumference is increased by one-fourth and that of the hind-wheel by one-fifth. Find the circumference of each wheel.

60. Three motor cycles, *A*, *B* and *C* leave *P* for *Q*. *B* starts 40′ later than *A* and travels 5 miles an hour faster. *C* starts 50′ later than *B* and travels 10 miles an hour faster. If they all reach *Q* at the same time, find their respective speeds as well as the distance between P and *Q*.

61. A marksman firing at a target 600 yds. away hears the bullet strike the target 7·7 seconds after he fires. A by-stander 500 yds. away from the target and 800 yds. away from the marksman hears the bullet strike the target 5 seconds after hearing the report. Find the velocity of sound.

62. A man leaves home at 6 a. m. and walks to an adjoining village which he reaches when the clock in his friend's house there indicates 7-40 a. m. He spends half an hour there and returns home by a route $\frac{1}{3}$ as long again as the route by which he had gone, at a rate 2½ times as fast as he walked and reached home at 9 a. m. By how much was the clock in his friend's house wrong ?

63. A carrier charges 6 as. for transmission of parcels not exceeding a certain weight and on heavier parcels demands an additional charge for every seer above that weight. The charge for a parcel of 25 seers is 11 as. and that for 35 seers is twice that for a parcel of 19 seers. Find the scale of charges.

CHAPTER XXVIII.

QUADRATIC EQUATIONS.

192. Quadratic Equations are those into which the second power of the unknown quantity enters with or without the first power.

If the second power of the unknown quantity alone enters such equations are called *Pure Quadratic Equations* :

thus $x^2=25$ and $x^2-b=7$, are pure quadratic equations.

If the first power as well as the second power of the unknown quantity be involved, such equations are called *Affected Quadratic Equations* ; thus $x^2+2x=3$ and $ax^2+bx=c$, are affected quadratic equations.

193. Pure Quadratic Equations are solved in the same manner in every respect as simple equations, except that, at the conclusion the square root of each side of the equation has to be taken. The signs + and − are prefixed to the root, because the square root of a quantity may be either positive or negative.

Ex. 1. Solve $12x^2-16=284$.

$12x^2=284+16=300$,

$\therefore \quad x^2=\frac{300}{12}=25$.

$\therefore \quad +x=\sqrt{25}=+5$,

i.e., $x=+5$, $+x=-5$, $-x=+5$ and $-x=-5$, or -5.

i.e., *i. e.*, $x=+5$ or -5.

Hence it follows that when we extract the square root of the two sides of an equation, it is sufficient to put the double sign before the square root of the right hand side.

Ex. 2. Solve $2bx^2+a-4=cx^2-5+d-bx^2$.

$\therefore \; 3bx^2-cx^2=d-5-a+4$;

$\therefore x^2(3b-c)=d-a-1$;

$\therefore x^2=\frac{d-a-1}{3b-c}$;

$$\therefore x=\pm\sqrt{\left(\frac{d-a-1}{3b-c}\right)}$$

Ex. 3. Solve $x+\sqrt{x^2-a^2}=\dfrac{2a^2}{\sqrt{x^2-a^2}}$.

$x\sqrt{x^2-a^2}+x^2-a^2=2a^2$, or $x\sqrt{x^2-a^2}=3a^2-x^2$.

Squaring, $x^2(x^2-a^2)=(3a^2-x^2)^2$,

or $x^4-a^2x^2=9a^4+x^4-6a^2x^2$,

or $5a^2x^2=9a^4$,

or $5x^2=9a^2$,

or $x^2=\dfrac{9a^2}{5}$.

$\therefore\ x=\pm\dfrac{3a}{\sqrt{5}}$.

Ex. 4. Solve $\dfrac{b-\sqrt{b^2-x^2}}{b+\sqrt{b^2-x^2}}=a$.

$\therefore\dfrac{b-\sqrt{b^2-x^2}+b+\sqrt{b^2-x^2}}{b-\sqrt{b^2-x^2}-b-\sqrt{b^2-x^2}}=\dfrac{a+1}{a-1}$ (II of Art. 185).

or $-\dfrac{b}{\sqrt{b^2-x^2}}=\dfrac{a+1}{a-1}$.

$\therefore\dfrac{\sqrt{b^2-x^2}}{b}=\dfrac{a-1}{a+1}$

$\therefore\dfrac{b^2-x^2}{b^2}=\left(\dfrac{a-1}{a+1}\right)^2$

$\therefore\ 1-\dfrac{x^2}{b^2}=\left(\dfrac{a-1}{a+1}\right)^2$.

$\therefore\dfrac{x^2}{b^2}=1-\left(\dfrac{a-1}{a+1}\right)^2=\dfrac{4a}{(a+1)^2}$.

$\therefore x^2=\dfrac{4ab^2}{(a+1)^2}$.

$\therefore\ x=\pm\dfrac{2b\sqrt{a}}{(a+1)}$.

EXERCISE 95.

Solve the following equations :—

1. $(x+3)^2=6x+10$.
2. $x(x+1)+(b-3)(x+2)=4$.
3. $\dfrac{2}{1+x}+\dfrac{2}{1-x}=5$.
4. $(x+a)^2+(x-a)^2=3a^2$.

5. $\frac{x-1}{x}-\frac{3x}{x-1}=2.$

6. $5x-\frac{12}{x}=0.$

7. $\frac{x+2}{2b+1}=\frac{x+3}{3x+1}.$

8. $\frac{x-2}{x+2}+\frac{x+2}{x-2}=2.\frac{x+3}{x-3}.$

9. $\sqrt{x^2+16}+\sqrt{x^2+7}=9.$

10. $\sqrt{4x^2+9}-\sqrt{4x^2-7}=2.$

11. $\sqrt{x^2+13}-\sqrt{x^2-11}=2.$

12. $\sqrt{1+x}+\sqrt{1-x}=\sqrt{2}.$

13. $x+\sqrt{a^2+x^2}=\frac{na^2}{\sqrt{a^2+x^2}}.$

14. $\frac{x+8}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}.$

15. $\frac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\frac{x}{2a}.$

16. $\frac{3x+\sqrt{9x^2-4}}{3x-\sqrt{9x^2-4}}=4.$

17. $\frac{a}{x}+\frac{\sqrt{a^2-x^2}}{x}=\frac{x}{b}.$

18. $\frac{2x+\sqrt{4x^2-1}}{2x+\sqrt{4x^2-1}}=4.$

19. $\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}=\frac{b}{c}$

20. $\frac{\sqrt{a^2-x^2}-\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{b^2+x^2}}=\frac{c}{d}$

21. $\frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}}=\frac{a}{x^2}.$

22. $\frac{2}{x+\sqrt{2-x^2}}+\frac{2}{x-\sqrt{2-x^2}}=x.$

23. $25x\{\sqrt{(1+x^2)}-x\}=3.$

24. $\sqrt{(a+x)}-\sqrt{(a-x)}=\sqrt{\left(\frac{a^2+2ab}{b+\frac{1}{2}a}\right)}.$

25. $\sqrt{\frac{x+2}{x-2}}+\sqrt{\frac{x-2}{x+2}}=4.$

194. Solution of Adfected Quadratic Equations

1. By completing the square.

Ex. 1. Solve $x^2-5x+6=0$

Transposing we have $x^2-5x=-6.$

To complete the square in terms of x add the square of half the co-efficient of x to both sides of the equation.

$$x^2-5x+\left(\frac{5}{2}\right)^2=\left(\frac{5}{2}\right)^2-6$$

$$\therefore\left(x-\frac{5}{2}\right)^2=\frac{1}{4}$$

Extracting the square root $x-\frac{5}{2}=\pm\frac{1}{2}$.

$$x-\frac{5}{2}=+\frac{1}{2};\qquad \therefore x=3$$

$$\text{and } x-\frac{5}{2}=-\frac{1}{2}\qquad \therefore x=2$$

$\therefore$ the roots of the equation are 3, 2.

From the preceding we derive the rule for the solution of an adfected quadratic equation :—

"*By transposition and reduction arrange the equation so that the terms involving the unknown quantity are alone on one side, and the co-efficient of* x^2 *is* $+1$; *add to both sides of the equation the square of half the co-efficient of* x, *and extract the square root of both sides.*"

Ex. 2. Solve $-3x^2+36x-105=0$.

Changing the signs, $3x^2-36x+105=0$.

Dividing by 3, and transposing the independent term to the other side $=x^2-12x=-35$

Adding to both sides the square of half the co-efficient of x, *i.e.*, $\left(\frac{12}{2}\right)^2$, *i. e.* 36.

$x^2-12x+36=36-35=1$ or $(x-6)^2=1$.

Extracting the square root of both sides, we get $x-6=\pm1$.

$\therefore x=6+1$ or $6-1$, *i.e.*, 7 or 5.

Ex. 3. Solve $x^2+px+q=0$.

Transposing, $x+px=-q$.

Adding to both sides the square of half the co-efficient of x *i.e.*, $\left(\frac{p}{2}\right)^2$, i.e., $\frac{p^2}{4}$,

we have $x^2+px+\frac{p^2}{4}=\frac{p^2}{4}-q$ or $\left(x+\frac{p}{2}\right)^2=\frac{p^2-4q}{4}$

$$\therefore x+\frac{p}{2}=\pm\frac{\sqrt{p^2-4q}}{2}.$$

$$\therefore x=\frac{-p\pm\sqrt{p^2-4q}}{2}$$

Ex. 4. Solve $x^2-2cx=a^2$.

$$\therefore x^2-2cx+\left(\frac{2c}{2}\right)^2=a^2+\left(\frac{2c}{2}\right)^2,$$

$$\text{or } x^2-2cx+c^2=a^2+c^2$$

$$\text{or } (x-c)^2=a^2+c^2.$$

$$\therefore x-c=\pm\sqrt{a^2+c^2}.$$

$$\therefore x=c\pm\sqrt{a^2+c^2}.$$

Applying the formula (A), $x=\frac{2c\pm\sqrt{4c^2+4a^2}}{2}$

$$=c\pm\sqrt{c^2+a^2}.$$

The most general form of an adfected quadratic equation is $ax^2+bx+c=0$.

This can be also solved by completing the square as in the previous examples.

Take the equation

$$ax^2+bx+c=0.$$

Divide both sides of the equation by a, the co-efficient of x^2, thus making the co-efficient of x^2 unity; then we have

$$x^2+\frac{bx}{a}+\frac{c}{a}=0.$$

Transposing the independent term, we have

$$x^2+\frac{bx}{a}=-\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ to both sides to make the left hand side expression a perfect square, we have

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}.$$

i.e., $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Extracting the square root, we have

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4a}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2+4ac}}{2a}.$$

∴ The two roots of the equation

are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$

This result may be used as a formula to write down the roots of the quadratic equation.

II. By formula.

Ex. 5. Solve $x^2-5x+6=0$

In this equation $a=1$, $b=-5$, $c=6$.

Putting these values in the formula.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

we have

$$x = \frac{+5+\sqrt{(-5)^2-4\times 6\times 1}}{2\times 1}$$

$$= \frac{+5 \pm \sqrt{1}}{2}$$

$$x = \frac{5+1}{1}$$

$$= 3 \text{ or } 2$$

Compare these answers with those obtained for Ex 1, worked above.

Ex. 6. Solve $-3x^2+36x-105=0$.

Hence $a=-3$, $b=36$, $c=-105$

Putting these values in the formula we have

$$x = \frac{-36+\sqrt{(36)^2-4\times -3\times -105}}{2\times -3}$$

$$= \frac{-36 \pm \sqrt{1296 - 1260}}{-6}$$

$$= \frac{-36 \pm \sqrt{36}}{-6}$$

$$= \frac{-36 \pm 6}{-6}$$

$$= 5 \text{ or } 7.$$

(The sign of the equation may be changed before the application of formula and a, b, c may then be put $= 3, -36, 105$ respectively.)

Ex. 7. Solve $abx^2 - x(a^2 + b^2) + ab = 0$.

Applying the formula, we have

$$x = \frac{(a^2 + b^2) \pm \sqrt{(a^2 + b^2)^2 - 4a^2b^2}}{2ab}$$

$$= \frac{(a^2 + b^2) \pm \sqrt{(a^2 - b^2)^2}}{2ab}$$

$$= \frac{(a^2 + b^2) \pm (a^2 - b^2)}{2ab}$$

$$= \frac{2a^2}{2ab} \text{ or } \frac{2b^2}{2ab}$$

$$= \frac{a}{b} \text{ or } \frac{b}{a}.$$

Ex. 8. Solve $2x^2 - x - 5 = 0$.

Applying the formula, we have

$$x = \frac{1 \pm \sqrt{1 + 40}}{4}$$

$$= \frac{1 \pm \sqrt{41}}{4}$$

$$\frac{1 + 6{\cdot}4}{4} \text{ nearly.}$$

$$= \frac{7{\cdot}4}{4} \text{ or } -\frac{5{\cdot}4}{4} \text{ nearly.}$$

$\therefore x = 1{\cdot}85$ or $-1{\cdot}35$ nearly.

Ex. 9. Solve $3x^2 + 2x + 4 = 0$.

Applying the formula we have,

$$x = \frac{-2 \pm \sqrt{4 - 48}}{6} = \frac{-2 \pm \sqrt{-44}}{6}$$

Here since the number under the root sign is negative, the square root cannot be found.

Ex. 10. Solve $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$.

$\therefore 6\{x^2+(x+1)^2\} = 13x(x+1)$.

$\therefore 6x^2+6x^2+12x+6=13x^2+13x$.

$\therefore -x^2-x+6=0$, or $x^2+x-6=0$.

Here $=a$ 1, $b=1$ and $c=-6$.

$\therefore x=\frac{-1+\sqrt{1+4\times 6}}{2}=\frac{-1+5}{2}=2$ or -3

Ex. 11. Solve $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{x-c}$.

Multiplying both sides by $(x-a)(x-b)(x-c)$,

we have $(x-b)(x-c)+(x-a)(x-c)=(x-a)(x-b)$,

or $x^2-(b+c)x+bc+x^2-(a+c)x+ac=x^2-(a+b)x+ab$,

or $x^2-2cx+bc+ac-ab=0$.

$\therefore x=\frac{2c\pm\sqrt{4c^2-4(bc+ac-ab)}}{2}$ [by formula (A).]

$=c+\sqrt{c^2-bc-ac+ab}$

$=c+\sqrt{(c-a)(c-b)}$

Ex. 12. Solve $\frac{x+1}{x-1}+\frac{x-1}{x+1}=\frac{a+1}{a-1}+\frac{a-1}{a+1}$.

Putting y for $\frac{x+1}{x-1}$ and b for $\frac{a+1}{a-1}$,

we have $y+\frac{1}{y}=b+\frac{1}{b}$.

$\therefore y^2-\left(b+\frac{1}{b}\right)y+1=0$.

$\therefore y=\frac{b+\frac{1}{b}\pm\sqrt{\left(b+\frac{1}{b}\right)^2-4}}{2}=\frac{b+\frac{1}{b}+\left(b-\frac{1}{b}\right)}{2}$

$=b$ or $\frac{1}{b}$.

Hence $\frac{x+1}{x-1}=\frac{a+1}{a-1}$ or $\frac{x+1}{x-1}=\frac{a-1}{a+1}$.

$\therefore \frac{x+1+x-1}{x+1-x+1}=\frac{a+1+a-1}{a+1-a+1}$ or $\frac{x+1+x-1}{x+1-x+1}=\frac{a-1+a+1}{a-1-a-1}$

$\therefore x=a$ or $x=-a$.

III. By factorisation.

Ex. 13. Solve $(3x-1)(4x+1)=0$.

The equation $(3x-1)(4x+1)=0$ states that the product of the two factors $(3x-1)$ and $(4x+1)$ must $=0$; and it is required to find what values must be substituted for x to produce this result.

Now it is evident that if *one* only of the factors be made equal to 0, then the product of the two must $=0$, whatever the other factor be; hence, if we can find those values of x which make each factor separately equal to 0, both of those values will be roots of the equation.

Therefore, to solve the equation,

(i) put $3x-1=0$. $\therefore\ 3x=1$. $\therefore\ x=\frac{1}{3}$.
(ii) put $4x+1=0$. $\therefore\ 4x=-1$. $\therefore\ x=-\frac{1}{4}$.

That is, $\frac{1}{3}$ and $-\frac{1}{4}$ are the two roots of the equation.

Generally, if an equation can be put in the form $(x-a)\times(x-b)=0$, the roots of the equation can be at once determined by putting each factor separately $=0$.

Ex. 14. Solve $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}$.

Squaring both sides.

$x+3+x+8+2\sqrt{(x+3)(x+8)}=25x$.

Transposing, $2\sqrt{(x+3)(x+8)}=(23x-11)$.

Squaring, $4(x+3)(x+8)=(23x-11)^2$
or $4x^2+44x+96=529x^2-506x+121$.
or $525x^2-550x+25=0$ or $21x^2-22x+1=0$.
or $(21x-1)(x-1)=0$,

$\therefore$ (i) $21x-1=0$. $\therefore\ 21x=1$. $\therefore\ x=\frac{1}{21}$.
(ii) $x-1=0$. $\therefore\ x=1$.

Therefore the roots of the equation are 1 and $\frac{1}{21}$.

Ex. 15. Solve $\dfrac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\dfrac{5}{11}$.

$$\therefore \frac{x-\sqrt{x+1}+x+\sqrt{x+1}}{x+\sqrt{x+1}-x+\sqrt{x+1}}=\frac{5+11}{11-5}.$$

$$\frac{x}{\sqrt{x+1}}=\frac{8}{3}.$$

$\therefore\ 3x=8\sqrt{x+1}$. $\therefore\ 9x^2=64(x+1)$.

$\therefore 9x^2-64x-64=0$ or $(9x+8)(x-8)=0$.

$\therefore\ x=8$ or $-\frac{8}{9}$.

Applying the formula (A), we have—

$$x = \frac{64 \pm \sqrt{(64)^2 + 36 \times 64}}{18} = \frac{64 + 80}{18} = 8 \text{ or } -\frac{8}{9}.$$

EXERCISE 96

Solve the equations 1 – 10 both by completing square and by formula

1. $x^2 - 2x = 5$.
2. $x^2 - 44x = 120$.
3. $x^2 + 32x = 320$.
4. $x^2 + 7x = 8$.
5. $x^2 + 19x = 20$.
6. $x^2 + 111x = 3400$.
7. $x = \frac{5}{3} + \frac{1}{12}x^2$.
8. $2x = 4 + \frac{6}{x}$.
9. $2x^2 + 1 = 11(x + 2)$.
10. $\frac{3}{4}(x^2 - 3) = \frac{1}{8}(x - 3)$.

Solve the following equations by factorisation :—

11. $x^2 - 7x + 12 = 0$.
12. $x^2 - x - 156 = 0$.
13. $x^2 + x(a + b) + ab = 0$.
14. $x^2 - ax(b + c) + a^2bc = 0$.
15. $3x^2 - 14 + 16 = 0$.
16. $30x^2 - 7x - 56 = 0$.
17. $abx^2 - x(a + b) + 1 = 0$.
18. $x - \frac{x^2 - 8}{x^2 - 5} = 2$.
19. $\frac{x + 2}{x - 1} - \frac{4 - x}{2x} = 2\frac{1}{2}$.
20. $\frac{12}{5 - x} + \frac{4}{4 - x} = \frac{32}{x + 2}$.
21. $\frac{5x}{x + 4} - \frac{3x - 2}{2x - 3} = 2$.
22. $\frac{1}{3} + \frac{1}{x + 3} + \frac{1}{2x + 3} = 0$.
23. $\frac{3x - 7}{x} + \frac{4x - 10}{x + 5} = 3\frac{1}{2}$.
24. $\frac{x}{a} + \frac{a}{x} = \frac{a}{b} + \frac{b}{a}$.
25. $x + \frac{1}{x} = a + \frac{1}{a}$.
26. $x^3 + 2ax = b^2 - a^2$.
27. $x^2 + a(1 + 3b)x + 3a^2b = 0$.
28. $a^2(x - b)^2 = (x - a)^2$.
29. $\frac{1}{x + a} + \frac{1}{x + b} = \frac{1}{c + a} + \frac{1}{c + b}$.
30. $\frac{1}{a} + \frac{1}{a + x} + \frac{1}{a + 2x} = 0$.
31. $\frac{x - a}{x - b} + \frac{x - b}{x - a} = \frac{p}{q}$.
32. $\frac{(x - a)^2 + (x - b)^2}{(a - x)(b - x)} = \frac{5}{2}$.
33. $\frac{x + a - b}{x - a + b} = \frac{a}{b}\left(\frac{x + a + 5b}{x + 5a + b}\right)$.
34. $\frac{x + a + 2b}{x + a - 2b} = \frac{b - 2a + 2x}{b + 2a - 2x}$.
35. $\frac{(x + a)^2 + (x - b)^2}{(x + a)^2 - (x - b)^2} = \frac{a^2 + b^2}{2ab}$.
36. $\frac{a}{x - a} + \frac{b}{x - b} = \frac{2c}{x - c}$.
37. $\frac{1}{x - a} + \frac{1}{x - b} = \frac{1}{a} + \frac{1}{b}$.

38. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}=\dfrac{a^2+b^2}{ab}$.

39. $abx^2-(a^2+b^2)x+ab=0$.

40. $cx+\dfrac{ac}{a+b}=(a+b)x^2$.

41. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3$.

42. $x+4\sqrt{x}=21$.

43. $x-2\sqrt{(x^2+x+5)}-14=0$.

44. $\sqrt{2x+a}+\sqrt{x-a}=\sqrt{b}$.

45. $\sqrt{2x+9}+\sqrt{x-4}=\sqrt{6x+1}$

46. $\sqrt{x+4}-\sqrt{2x-9}=\sqrt{x-1}$.

47. $\sqrt{3x+4}+\sqrt{2x+2}=\sqrt{10x-11}$.

48. $\sqrt{5x+3}-\sqrt{x+5}=\sqrt{x-7}$.

49. $\dfrac{\sqrt{a}+x}{b}=\dfrac{a}{\sqrt{a+x}+b}$.

50. $\sqrt{5-x}+\sqrt{5+x}=\dfrac{12}{\sqrt{5+x}}$.

51. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{x+3}{x-3}$.

52. $\dfrac{x^2+2x+2}{x+1}+\dfrac{x^2+8x+20}{x+4}=\dfrac{x^2+4x+6}{x+2}+\dfrac{x^2+6x+12}{x+3}$.

53. $x=\cfrac{3}{4-\cfrac{3}{4-\cfrac{3}{4-x}}}$.

54. $\dfrac{x-2}{x-4}-\dfrac{x-1}{x-3}=\dfrac{x-2}{x-1}-\dfrac{x-3}{x-2}$.

55. $1-x+x^2=\dfrac{1-b}{1+b}1(+x^2+x^4)$

56. $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}=\dfrac{4x-7}{2x-4}$.

57. $\dfrac{m^2-n^2}{m^2-n^2}=x+\dfrac{mn}{(m^2-n^2)x}$.

58. $\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}=\dfrac{x}{a}$.

59. $\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}=ax$.

60. $\dfrac{1-\sqrt{(x^2-1)}}{1+\sqrt{(x^2-1)}}=\dfrac{x-\sqrt{(x^2+8)}}{x+\sqrt{(x^2+8)}}$.

195. Relations between the Roots and Co-efficients.

Let a and β be the roots of $ax^2+bx+c=0$; then

$a=\dfrac{1}{2a}(-b+\sqrt{b^2-4ab})$ and $\beta=\dfrac{1}{2a}(-b-\sqrt{b^2-4ac})$.

We have $a+\beta=\dfrac{1}{2a}(-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac})=-\dfrac{b}{a}$

and $a\beta=\dfrac{1}{2a}(-b+\sqrt{b^2-4ac})\times\dfrac{1}{2a}(-b-\sqrt{b^2-4ac})$

$=\dfrac{1}{4a^2}\{b^2-(b^2-4ac)\}=\dfrac{c}{a}$.

Hence the following relations :—

i. *When the co-efficient of* x^2 *is unity, the sum of the roots = the co-efficient of the second term with the sign changed.*

ii. *The product of the roots is equal to the last term*

196 *If* α *and* β *be the roots of* $x^2+px+q=0$, *then the expression* $x^2+px+q=(x-\alpha)(x-\beta)$.

Since $p=-(\alpha+\beta)$ **and** $q=\alpha\beta$.

$\therefore\ x^2+px+q=x^2-(\alpha+\beta)x+\alpha\beta=(x-\alpha)(x-\beta)$.

Hence we can resolve any quadratic expression into two factors. We can also form the equation of which the roots are given.

We shall work out a few examples on the preceding articles.

Ex 1. Find the sum and product of the roots of—

$$5x^2-7x+9=0.$$

If p and q be the roots, $p+q=\frac{7}{5}$ and $pq=\frac{9}{5}$...(Art. 195)

Ex 2. Form the equation whose roots are—3 and 8.

The equation is $\{x-(-3)\}\{x-8\}=0$ or $x^2-5x-24=0$.

Ex. 3. Resolve into factors $12x^2-5x-3$.

The expression $=12(x^2-\frac{5}{12}x-\frac{1}{4})$.

The roots of the equation $x^2-\frac{5}{12}x-\frac{1}{4}=0$ are

$$\tfrac{1}{2}(\tfrac{5}{12}\pm\sqrt{\tfrac{25}{144}+1})=\tfrac{1}{2}(\tfrac{5}{12}\pm\tfrac{13}{12})=\tfrac{3}{4}\text{ or }-\tfrac{1}{3}.$$

$\therefore$ the factors of $x^2-\frac{5}{12}x-\frac{1}{4}$ are $(x-\frac{3}{4})$ and $(x+\frac{1}{3})$.

$\therefore$ the factors of $12(x^2-\frac{5}{12}x-\frac{1}{4})$ are

$$12(x-\tfrac{3}{4})(x-\tfrac{1}{3})=\frac{12(4x-3)(3x+1)}{4\times 3}=(4x-3)(3x+1).$$

Ex. 4. If α and β be the roots of $ax^2+bx+c=0$, find the values of $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$, in terms of a, b, and c.

We know from Art 195 $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{b}{a}\right)^2-2\frac{c}{a}=\frac{b^2}{a^2}-\frac{2c}{a}.$$

$$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\left(-\frac{b}{a}\right)^3-3\frac{c}{a}\left(-\frac{b}{a}\right)$$

$$=-\frac{b^3}{a^3}+\frac{3bc}{a^2}.$$

Ex. 5. Given *one* root of the equation $(b-c)x^2+(c-a)x+a-b=0$ to be 1, find the *other* root.

The product of the roots is $\frac{a-b}{b-c}$ and one root is 1.

$\therefore$ the other root is $\frac{a-b}{b-c}$.

Ex. 6. If α and β be the roots of the equation $ax^2+bx+c=0$ find the equation whose roots are $2\beta-\alpha$, $2\alpha-\beta$.

The sum of the roots of the new equation $=\alpha+\beta$.

The product of the roots

$$=(2\beta-\alpha)(2\alpha-\beta)$$
$$=5\alpha\beta-2(\alpha^2+\beta^2)$$

But since α, β are the roots of $ax^2+bx+c=0$

$$\alpha+\beta=-\frac{b}{a}$$

$$\alpha\beta=\frac{c}{a}.$$

$$\therefore (2\beta-\alpha)(2\alpha-\beta)=5\alpha\beta-2(\alpha^2+\beta^2)$$
$$=5\frac{c}{a}-2\{(\alpha+\beta)^2-2\alpha\beta\}$$
$$=\frac{5c}{a}-2\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)$$
$$=\frac{9c}{a}-\frac{2b^2}{a^2}$$
$$=\frac{9ac-2b^2}{a^2}$$

$\therefore$ The new equation is

x^2-x(the sum of the roots)+the product of the roots $=0$

i.e., $x^2-x\left(-\frac{b}{a}\right)+\frac{9ac-2b^2}{a^2}$

i.e., $x^2+\frac{bx}{a}+\frac{9ac-2b^2}{a^2}=0$

i.e., $a^2x^2+abx+9ac-2b^2=0$.

Ex. 7. If one root of $ax^2+bx+c=0$ be three times the other, find the relation between a, b, c. Let d be one root then $3d$ is the other root

$$\therefore d+3d \text{ or } 4d=-\frac{b}{a} \quad \ldots\ldots(1)$$

$$d.\ 3d \text{ or } 3d^2=\frac{c}{a} \quad \ldots\ldots(2)$$

From (1) $d = -\frac{b}{4a}$

Substituting this in (2) we have

$$3 \times \left(-\frac{b}{4a}\right)^2 = \frac{c}{a}$$

i.e., $\frac{3b^2}{16a^2} = \frac{c}{a}$

$\therefore\ 3b^2 = 16ac.$

EXERCISE 97.

Write down the *sum* and *product* of the roots of—

1. $x^2 - 5x + 1 = 0$
2. $3x^2 + 7x + 3 = 0$.
3. $4x^2 + 3 = 0$.
4. $5x^2 + 7x = 0$.
5. $(a+1)x^2 + (a^2-1)x + a^3 + 1 = 0$.
6. $(a^2 - b^2)x^2 + (b^2 - c^2)x + c^2 - a^2 = 0$.

Given one of the roots of the following equations, state the other root and find m.

7. $x^2 - 7x + m = 0$, one root is 4.
8. $x^2 - 9x + m = 0$,, ,, 2.
9. $x^2 - 3x + m = 0$,, ,, 8.
10. $x^2 - 5x + m = 0$,, ,, -4.
11. $x^2 + 2x + m = 0$,, ,, -11.
12. $x^2 + mx + 2 = 0$,, ,, -1
13. $2x^2 + x + m = 0$,, ,, $-\frac{3}{2}$.
14. $6x^2 - 13x + m = 0$,, ,, $\frac{7}{3}$.
15. $5x^2 + mx - 2 = 0$,, ,, -1.
16. $12x^2 + mx + 3 = 0$,, ,, $\frac{3}{4}$.

Form the *equation* whose roots are—

17. 4 and -3.
18. $-a$ and b
19. $-a+1$ and $a+1$.
20. $a-b$ and $b-c$.
21. $2+\sqrt{3}$ and $2-\sqrt{3}$.

If α and β be the roots of $ax^2 + bx + c = 0$, find in terms of a, b, c, the values of—

22. $\frac{1}{\alpha} + \frac{1}{\beta}$.
23. $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.
24. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.
25. $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$.
26. $\alpha^2 - \alpha\beta + \beta^2$.
27. $\alpha^2 + \alpha\beta + \beta^2$.
28. $\alpha^4 + \alpha^2\beta^2 + \beta^4$.

If α and β be the roots of

29. $3x^2 - x - 2 = 0$, find the value of $(2\alpha + \beta)(2\beta + \alpha)$.

30. $2x^2-4x+1=0$ find the value of $(2\alpha-\beta)(2\beta-\alpha)$

31. $4x^2-2x+4=0$, find the value of $(3\alpha+\beta)(3\beta+\alpha)$.

32. $x^2+3x-4=0$, find the value of $(3\alpha-\beta)(3\beta-\alpha)$.

If α, β be the roots of the following equations, find the equations whose roots are

33. $\alpha+\beta$, $\alpha\beta$; $2x^2-2x+1=0$.

34. $\frac{1}{\alpha}$, $\frac{1}{\beta}$; $x^2-7x+12=0$.

35. $\frac{1}{\alpha}+\frac{1}{\beta}$; $\frac{1}{\alpha\beta}$; $3x^2-6x+2=0$.

36. $\frac{\alpha+\beta}{2}$, $\frac{\alpha-\beta}{2}$, $x^2-5x+6=0$.

37. Find the condition that one root of the equation $ax^2+bx+c=0$ may be double the other.

38. Find the relation between the co-efficients of the equation $x^2+bx+c=0$ when one root is three times the other.

39. Find the condition that one root of the equation $x^2+px-q=0$ may be four times the other.

40. Find the value of m if one root of the equation $mx^2-2x+3=0$ is 3 times the other ; find the other root.

Resolve into *factors* :—[See Exercise 17 for more examples].

41. $(a^2-b^2)x^2+(b^2-c^2)x+c^2-a^2$.

42. $x^2+(a-c)x+(a-b)(b-c)$.

43. $(b+c-2a)x^2+(c+a-2b)x+a+b-2c$.

44. $a(b-c)x^2+b(c-a)x+c(a-b)$.

45. $x^2+(a+b)^2x+ab(a+b)^2$.

46. $x^2+(a-b)^2x-ab(a-b)^2$.

Nature of the roots:—

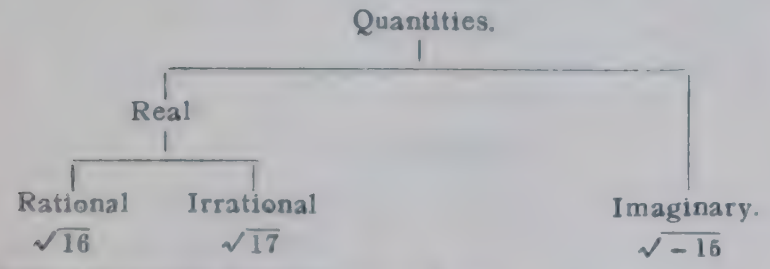
It has been observed that while applying the formula for the roots of a quadratic equation in Ex 6, 8 and 9, the number under the square root was an exact square in the case of Ex. 6, was not a perfect square but positive in Ex. 9 (hence the roots are approximate) and in Ex. 9, the number was negative where square root could not be attempted at all.

We thus came across three different sorts of numbers, with a sign of square root, *viz.* (1) numbers whose square roots can be exactly

found, (2) numbers whose square roots can be found only approximately, (3) numbers whose square roots it is impossible to find.

The first two kinds of quantities are called real quantities and the third is called an unreal impossible or imaginary quantity.

Of the two kinds of real quantities the first is called rational and the second irrational. So the following will make the classification clear.



Taking the roots of $ax^2+bx+c=0$, *viz.*

$\frac{-b+\sqrt{b^2-4ac}}{2a}$, we observe that they involve a quantity under the root sign and therefore the nature of the roots depends on the nature of this expression, *i. e.*, b^2-4ac.

1. If b^2-4ac is positive or $b^2>4ac$ the roots are real and different.

2. If b^2-4ac is negative or $b^2<4ac$ the roots are imaginary.

3. If b^2-4ac is zero or $b^2=4ac$ the roots are real and equal.

We have seen therefore that the nature of the roots depends on the nature of the expression b^2-4ac of the typical equation $ax^2+bx+c=0$. b^2-4ac is called the discriminant of the equation and is generally denoted by Δ.

Ex 1. To find the nature of the roots $2x^2+2x+1=0$ without actually solving it.

The discriminant or $\Delta=2^2-4\times2\times1$

$=4-8$

$=-4$.

$\therefore$ Δ is negative, the roots are imaginary.

Ex. 2 To find the condition that the roots of $x^2+px+q=0$ may be real and equal.

Here the discriminant $=p^2-4q$.

In order that the roots may be equal $p^2 - 4q$ should be equal to zero, *i.e.*, $p^2 = 4q$.

In this case the expression $x^2 + px + q$ will be a perfect square.

Ex. 3. Find m in order that the roots of $25x^2 - (m+1)x + 4 = 0$ may be equal.

The discriminant of the equation is $(m+1)^2 - 400$

In order that the roots may be equal

$$(m+1)^2 - 400 = 0$$

$$(m+21)(m-19) = 0$$

$$\therefore m = -21 \text{ or } 19.$$

EXERCISE 98.

State the discriminant in the following equations :—

1. $x^2 + 4x - 1 = 0$.
2. $2x^2 - 5x + 10 = 0$.
3. $7x^2 + 10x - 3 = 0$.
4. $6x^2 + bx + a = 0$
5. $ax^2 + bx + a = 0$.
6. $(x+1)^2 = 3x + 2$.
7. $x(x+3) = 5$.
8. $(x+1)(2x-1) = (x-1)(x+2)$.
9. $(a+b)x^2 - 2bx + a - b = 0$.
10. $(x-a)(x-b) = (x-c)^2$.
11. $abx^2 + x(a^2 + b^2) + ab = 0$.

Find the nature of the roots of the following equations without actually solving them.

12. $5x^2 - 8x + 3 = 0$.
13. $3x^2 + x + 2 = 0$.
14. $2x^2 - x - 1 = 0$.
15. $4x^2 - 12x + 9 = 0$.

For what values of m will the following equations have equal roots ?

16. $4x^2 - x + m = 0$.
17. $9x^2 + 2x + m = 0$
18. $m^2x^2 + 2(m+3)x + 4 = 0$.
19. Find the value of m which will make $4x^2 + x(m+1) + 1$ a perfect square.
20. Find a relation between a, b, c in order that $(b-c)x^2 + (c-a)x + (a-b)$ may be a perfect square.

CHAPTER XXIX.

EQUATIONS SOLVED LIKE QUADRATICS.

197. Equations of the form $x^{2n} \pm px^n - q = 0$.

Ex. 1 Solve $x^4 - 6x^2 = 16$

$\therefore x^4 - 6x^2 - 16 = 0.$

Putting y for x^2, we have $y^2 - 6y - 16 = 0.$

$\therefore y = \frac{1}{2}(6 \pm \sqrt{36 + 64}) = \frac{1}{2}(6 + 10) = 8 \text{ or } -2$

$\therefore x^2 = 8 \text{ or } -2.$

$\therefore x = \pm\sqrt{8} \text{ or } \pm\sqrt{-2}.$

Ex. 2 Solve $x^3 - 4x^{\frac{3}{2}} = 21.$

$\therefore x^3 - 4x^{\frac{3}{2}} - 21 = 0.$

Putting y for $x^{\frac{3}{2}}$, we have $y^2 - 4y - 21 = 0.$

$\therefore y = \frac{1}{2}(4 \pm \sqrt{16 + 84}) = \frac{1}{2}(4 \pm 10) = 7 \text{ or } -2.$

$\therefore x^{\frac{1}{2}} = 7 \text{ or } -3.$

$\therefore x = (7)^{\frac{2}{3}} \text{ or } (-3)^{\frac{2}{3}}.$

Ex. 3. Solve $\sqrt{a+x} - 2\sqrt[4]{a+x} = 4.$

$\therefore \sqrt{a+x} - 2\sqrt[4]{a+x} - 4 = 0.$

Putting y for $\sqrt[4]{a+x}$, we have $y^2 - 2y - 4 = 0.$

$\therefore y = \frac{1}{2}(2 \pm \sqrt{4 + 16}) = 1 \pm \sqrt{5}.$

$\therefore \sqrt{a+x} = 1 \pm \sqrt{5}.$

$\therefore a + x = (1 \pm \sqrt{5})^2, \quad x = (1 \pm \sqrt{5})^2 - a.$

Ex. 4. Solve $4^x + 2^{x+1} = 80$

$\therefore 2^{2x} + 2 \cdot 2^x - 80 = 0.$

Putting y for 2^x, we have $y^2 + 2y - 80 = 0$

$\therefore (y + 10)(y - 8) = 0$

$\therefore y = 8 \text{ or } -10.$

$\therefore 2^x = 8 \text{ or } -10.$

$\therefore 2^x = 2^3.$

$\therefore x = 3.$

11. Equations of the form $\mathbf{X^{1n} \pm pX^n = q}$ (in which **X** represents any simple or quadratic expression involving the unknown quantity).

Ex. 5. Solve $3x^2+15x-2=\sqrt{x^2+5x+1}$.

$\therefore\ x^2+5x-\frac{2}{3}-\frac{1}{3}\sqrt{x^2+5x+1}=0.$

$\therefore\ x^2+5x+1-\frac{1}{3}\sqrt{x^2+5x+1}-\frac{5}{3}=0.$

Putting y for $\sqrt{x^2+5x+1}$, we have $y^2-\frac{1}{3}y-\frac{5}{3}=0$.

$$\therefore y=\tfrac{1}{2}\left(\frac{1}{3}\pm\sqrt{\frac{1}{9}+\frac{20}{3}}\right)$$

$$=\frac{1}{2}\left(\frac{1}{3}\pm\frac{\sqrt{61}}{3}\right)=\frac{1\pm\sqrt{61}}{6}\quad=p\ (\text{suppose}).$$

$\therefore\sqrt{x^2+5x+1}=p.$

$\therefore x^2+5x+1=p^2.$

$\therefore x^2+5x+1-p^2=0.$

$$\therefore\ x=\tfrac{1}{2}\{-5\pm\sqrt{25-4(1-p^2)}\}$$

$$=\tfrac{1}{2}\{-5\pm\sqrt{(21+4p^2)}\}.$$

Ex. 6. Solve $(x^2+5x)^2-8x^2-40x-84=0$.

$$\therefore (x^2+5x)^2-8(x^2+5x)-84=0.$$

Putting y for x^2+5x, we have $y^2-8y-84=0$.

$\therefore\ (y-14)(y+6)=0.$

$\therefore\ y=14$ or -6.

$\therefore x^2+5x=14$ or -6.

$\therefore\ x^2+5x-14=0$ or $x^2+5x+6=0$.

$\therefore\ (x+7)(x-2)=0$ or $(x+2)(x+3)=0$.

$\therefore\ x=-7$ or 2 or -2 or -3.

III. Equations of the form $(x+a)(x+b)(x+c)(x+d)=p$ when the sum of any two of the quantities a, b, c, d, is equal to the sum of the remaining two.

Ex. 7. Solve $(x+1)(x+2)(x+3)(x+4)=24$.

Since $1+4=2+3$, we multiply together the first and last factors, and the second and third;

hence $(x^2+5x+4)(x^2+5x+6)=24$.

Putting y for x^2+5x+4,

we have $y(y+2)=24$.

$\therefore\ y^2+2y-24=0.$

$\therefore\ (y+6)(y-4)=0.$

$\therefore\ y=4$ or -6.

$\therefore\ x^2+5x+4=4$ or -6.

$\therefore\ x^2+5x=0$ or $x^2+5x+10=0$.

$\therefore\ x=0$ or -5 or $\frac{1}{2}(-5\pm\sqrt{-15})$

Ex. 8. Solve $(x+3)(x+5)(x-4)(x-6)=280$.

Since $3-4=5-6$,

we multiply $(x+3)$ and $(x-4)$ together

and $(x+5)$ and $(x-6)$ together.

Hence $(x^2-x-12)(x^2-x-30)=280$.

Putting y for x^2-x-12, we have $y(y-18)-280=0$.

$\therefore\ y^2-18y+280=3$.

$\therefore\ (y-28)(y+10)=0$.

$\therefore\ y=28$ or -10.

$\therefore\ x^2-x-12=28$ or -10.

$\therefore\ x^2-x-40=0$ or $x^2-x-2=0$.

$\therefore\ x=\frac{1}{2}(1+\sqrt{161})$ or $\frac{1}{2}(1\pm\sqrt{9})=\frac{1}{2}(1\pm\sqrt{161})$, or 2 or -1.

IV. Reciprocal Equations. Every equation of the form $ax^4+bx^3+cx^2+bx+a=0$ in which the co-efficients of the terms equidistant from the beginning and end are equal is called *a reciprocal equation*, and may be solved as a quadratic by dividing both sides of the equation by x^2.

Ex. 9. Solve $x^4-3x^3+4x^2-3x+1=0$.

Dividing by x^2, we have $x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0$.

$$\text{or}\quad x^2+\frac{1}{x^2}+2-3\left(x+\frac{1}{x}\right)+2=0$$

$$\text{or}\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+2=0.$$

Putting y for $x+\frac{1}{x}$,

we have $y^2-3y+2=0$.

$\therefore\ (y-2)(y-1)=0$

$\therefore y=2$ or 1.

$\therefore\ x+\frac{1}{x}=2$ or 1.

$\therefore\ x^2-2x+1=0$ or $x^2-x+1=0$.

$\therefore\ x=1,\ 1,\ \frac{1}{2}(1\pm\sqrt{-3})$.

Ex. 10 Solve $x^4+x^3-x^2-x+1=0$

Dividing by x^2, we have $x^2+x-1-\frac{1}{x}+\frac{1}{x^2}=0$.

$$\text{or } x^2+\frac{1}{x^2}-2+x-\frac{1}{x}+1=0.$$

$$\text{or} \left(x-\frac{1}{x}\right)^2+\left(x-\frac{1}{x}\right)-1=0.$$

Putting y for $x-\frac{1}{x}$, we have $y^2+y+1=0$.

$$\therefore\ y=\frac{1}{2}(-1\pm\sqrt{-3}).$$

$$\therefore\ x-\frac{1}{x}=\frac{1}{2}(-1+\sqrt{-3})=p \text{ (suppose.)}$$

$$\therefore\ x^2-px-1=0.$$

$$\therefore\ x=\frac{1}{2}(p\pm\sqrt{p^2+4}).$$

Ex. 11. Solve $(x+1)^6=16(x^5+1)$.

Dividing by $x+1$ we have $(x+1)^5=16(x^4-x^3+x^2-x+1)$.

Reducing, $15x^4-20x^3+10x^2-20x+15=0$.

Dividing by $5x^2$, $3x^2-4x+2-\frac{4}{x}+\frac{3}{x^2}=0$.

$$\text{or } 3\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+2=0.$$

$$\text{or } 3\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)+4=0.$$

Putting y for $x+\frac{1}{x}$, we have $3y^2-4y-4=0$.

$$\therefore\ y=\frac{1}{6}(4\pm 8)=2 \text{ or } -\frac{2}{3}.$$

$$\therefore\ x+\frac{1}{x}=2 \text{ or } -\frac{2}{3}.$$

Whence we find $x=1, 1, \frac{1}{3}(-1\pm 2\sqrt{-2})$, besides -1 which corresponds to the factor $(x+1)$ which we took out at first.

V. Miscellaneous Examples.

Ex. 12 Solve $\sqrt{x^2-5x+4}+\sqrt{x^2-4x+5}=x+1$ .....(*A*).

Identically, $(x^2-5x+4)-(x^2-4x+5) \quad =-(x+1)$. .....(*B*).

Dividing (B) by (A), $\sqrt{x^2-5x+4}-\sqrt{x^2-4x+5}=-1$.....$(C)$.

Adding (A) and (C), $2\sqrt{x^2-5x+4}=x$.

$\therefore\ 4(x^2-5x+4)=x^2$.

$\therefore\ 3x^2-20x+16=0$.

$\therefore x=\frac{1}{6}(20\pm4\sqrt{13})=\frac{1}{3}(10\pm2\sqrt{13})$.

Ex. 13. Solve—

$\sqrt{x^2-6x+15}+\sqrt{x^2-6x+13}=\sqrt{(10)}-\sqrt{(8)}$.....$(A)$

Identically, $(x^2-6x+15)-(x^2-6x+13)=2=10-8$.....$(B)$.

Dividing (B) by (A), $\sqrt{x^2-6x+15}-\sqrt{x^2-6x+13}$

$=\sqrt{(10}+\sqrt{(8}$,.....(C).

Adding (A) and (C), $2\sqrt{x^2-6x+15}=2\sqrt{(10)}$.

$\therefore\ x^2-6x+15=10$.

$\therefore\ x^2-6x+5=0$.

$\therefore\ (x-5)(x-1)=0$.

$\therefore\ x=5$ or 1.

Ex. 14. *To find the cube roots of unity.*

Let $x=\sqrt[3]{1}$; then $x^3=1$.

$\therefore\ x^3-1=0$ or $(x-1)(x^2+x+1)=0$.

$\therefore\ x-1=0$.

$\therefore\ x=1$. Also $x^2+x+1=0$.

$\therefore x=\frac{1}{2}(-1\pm\sqrt{-3})$.

The cube roots are 1, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

If $\frac{-1+\sqrt{-3}}{2}$ be denoted by w,

Then $w^2=\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{1-3-2\sqrt{-3}}{4}=\frac{-1-\sqrt{-3}}{2}$.

Hence *one imaginary root is the square of the other.*

Also $1+w+w^2=1+\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{1}=1-1=0$

The sum of the roots $=0$.

The product of the roots, i.e., $1\times w\times w^2$ or w^3

$=1\times\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)=\frac{1+3}{4}=1$.

Ex. 15. Solve $\sqrt{x-a}+\sqrt{x-b}+\sqrt{x-c}=0$. If $p+q+r=0$, then $(p^2+q^2+r^2)^2=2(p^4+q^4+r^4)$.

Hence $(x-a+x-b+x-c)^2=2\{(x-a)^2+(x-b)^2+(x-c)^2\}$.

$\therefore 3x^2-2(a+b+c)x=a^2+b^2+c^2-2bc-2ac-2ab.$

$\therefore x=\frac{1}{3}\{a+b+c+2\sqrt{a^2+b^2+c^2-ab-ac-bc}\}$.

Ex. 16. Solve—

$$\frac{1}{6x^2-7x+2}+\frac{1}{12x^2-17x+6}=8x^3-6x+1,$$

or $$\frac{1}{(2x-1)(3x-2)}+\frac{1}{(3x-2)(4x-3)}=(2x-1)(4x-1)$$

or $$\frac{4x-3+2x-1}{(2x-1)(3x-2)(4x-3)}=(2x-1)(4x-1).$$

or $$\frac{2}{(2x-1)(4x-3)}=(2x-1)(4x-1).$$

$\therefore (2x-1)^2(4x-1)(4x-3)=2.$

Putting y for $2x-1$, we have

$y^2(2y+1)(2y-1)=2.$

$y^2(4y^2-1)=2$

$\therefore 4y^4-y^2-2=0.$

$\therefore y^2=\frac{1}{8}(1\pm\sqrt{33})$

$\therefore y=\pm\sqrt{\frac{1}{8}(1\pm\sqrt{33})}.$

$\therefore 2x-1=\pm\sqrt{\frac{1}{8}(1\pm\sqrt{33})}.$

$\therefore x=\frac{1}{2}\{1\pm\sqrt{\frac{1}{8}(1\pm\sqrt{33})}\}$.

EXERCISE 99.

Solve the following equations :—

1. $x^6-2x^3=48.$
2. $x^4-2x^3+x=132$
3. $2x^4-7x^2=99,$
4. $x-1=2+\frac{2}{\sqrt{x}}.$
5. $x^3+1=0.$
6. $(1+x^{-1})=a(1+x)^{\frac{1}{2}}.$
7. $5^2x+6\times5^x=775.$
8. $7^{3x}+2\times7^x=63.$
9. $11^x+21\times11^{2x}=143.$
10. $x^2+5x+3\sqrt{x^2+5x+7}+3=0.$
11. $x(x-3)+\sqrt{x^2+9x+3}=3(9-4x).$
12. $x^2-8x-2\sqrt{y^2-8x+10}=-5.$

13. $2x^2 - \sqrt{x^2 - 2x - 3} = 4x + 9$.

14. $x^2 - 7x + \sqrt{x^2 - 7x + 15} = 24$.

15. $x^2 + 3 = 2\sqrt{(x^2 - 2x + 2)} + 2x$.

16. $5\sqrt{x^2 + 5x + 28} = x^2 + 5x + 4$.

17. $x(x+1) + 3\sqrt{2x^2 + 6x + 5} = 25 - 2x$.

18. $x^2 - x + 3\sqrt{(2x^2 - 3x + 2)} = 7 + \frac{x}{2}$.

19. $3x^2 - 2ax + 5 = 2\sqrt{(3x^2 - 2ax + 4)}$.

20. $\frac{q+p}{x^{2pq}} = \frac{a^2 - b^2}{a^2 + b^2} \cdot \left(\frac{1}{x^p} + \frac{1}{x^q}\right)$

21. $(1+x)^{\frac{2}{5}} + (1-x)^{\frac{2}{5}} = (1-x^2)^{\frac{1}{5}}$

22. $x^5 - 10x^3 + 120x - 144 = 0$.

23. $ax^4 + bx^3 + cx^2 - bx + a = 0$.

24. $x^4 - 9x^3 + 81x + 81 = 0$.

25. $x^4 + x^3 - 4x^2 + x + 1 = 0$.

26. $x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) = \frac{142}{9}$.

27. $6x^4 - 35x^3 + 6x^2 - 35x + 6 = 0$

28. $(x-1)(x-2)(x-3)(x-4) = 120$.

29. $(x-2a)(x-3a)(x-4a)(x-5a) = 360a^4$.

30. $16x(x-1)(x-2)(x-3) = 9$

31. $(x+a)(x+2a)(x+3a)(x+4a) = c^4$.

32. $(x-3)(x-4)(x-5)(x-6) = 24$.

33. $(x+3)(x+8)(x+13)(x+18) = 51$.

34. $2x^2 + 2\sqrt{x^2 + 4x - 5} = 4x^2 + 8x + 5$.

35. $16x(x+1)(x+2)(x+3) = 9$.

36. $(x-1)(x-2)(x-3) = 24$.

37. $\sqrt{x^2 + 9} + \sqrt{x^2 - 9} = \sqrt{34} + 4$.

38. $\sqrt{x^2 + ax - 1} + \sqrt{x^2 + bx - 1} = \sqrt{a} + \sqrt{b}$.

39. $x(x+2)(x+3)(x+5) = 40$.

40. $(x^2 + 3x)^2 - 2x^2 - 6x - 8 = 0$.

41. $2(4x^2 - 3x)^2 - 28x^2 + 21x + 5 = 0$.

42. $(x^2 - 5x + 7)^2 - (x-2)(x-3) = 1$.

43. $(x^2 - x)(x^2 + x + 1) = 42$.

44. $\sqrt{3x^2 + 7x - 5} - \sqrt{x^2 + 7x + 3} = x - 2$.

45. $3^{x+1} + 9^x = 198$.

46. $x^4 + 2x^3 + 2x^2 + x = \frac{1}{2}$.

47. $(x+2)(x+4)(x+6)=(2x+1)(3x+2)(4x+3)$.

48. $\dfrac{1}{2x^2-5x+3}+\dfrac{1}{3x^2-5x+2}=6x^2-23x+21$.

49. $\dfrac{1}{3x^2+11x+10}+\dfrac{1}{6x^2+19x+15}=\dfrac{2x^2+9x+10}{150}$.

50. $(x+3)^3-2(x^2+3)=2x(x+1)^2$.

51. $(x^2+x+10)\surd(x^2+3x+16)=2(20-x)$.

52. $\dfrac{b^2-x^2}{b^2-m^2}=\dfrac{x}{m}\cdot\dfrac{a-m}{a-x}$.

53. $\left(x-\dfrac{a^2}{x}\right)\left(x-\dfrac{b^2}{x}\right)\left(x-\dfrac{c^2}{x}\right)=(x-a)(x-b)(x-c)$.

54. $\dfrac{x+b}{a}+\dfrac{x+a}{b}=\dfrac{a}{x+b}+\dfrac{b}{x+a}$.

55. $(x-4)^2-7=2\left(\dfrac{1}{x}-x\right)$.

56. $\dfrac{x+a+\surd(x^2+a^2)}{\surd x+\surd(x^2+a^2)}=2\surd x$.

57. $x^4-6x^2+8x-3=0$

58. $\dfrac{1}{15x^2-11x+2}+\dfrac{1}{10x^2-9x+2}=18x^2-9x+1$.

59. $\dfrac{4}{x^2-4x+3}-\dfrac{2}{x^2-3x+2}=x^2-3x+2$.

60. $\dfrac{(x+1)(x+2)}{(x-1)(x-2)}+\dfrac{(x-1)(x-2)}{(x+1)(x+2)}=\dfrac{(x+3)(x+4)}{(x-3)(x+4)}+\dfrac{(x-3)(x+4)}{(x+3)(x+4)}$.

CHAPTER XXX.

SIMULTANEOUS QUADRATIC EQUATIONS.

198. Miscellaneous Examples.

Ex. 1. Solve $x+y=8$, $xy=15$

We know $(x-y)^2=(x+y)^2-4xy$,

Hence $(x-y)^2=64-60=4$.

$\therefore x-y=\pm 2$

and $x+y=8$.

$\therefore 2x=10$ or 6.

$\therefore x=5$ or 3.

and $y=3$ or 5

Or the value of x in terms of y may be obtained from one of the equations and substituted in the other. A quadratic equation in y will be got, which can be solved.

If $x-y$ and xy be given, we can find the value of $x+y$ from $(x+y)^2=(x-y)^2+4xy$.

Ex. 2. Solve $x^3+y^3=25$ and $x+y=5$.

$x^3+y^3=(x+y)(x^2-xy+y^2)$

Hence $35=5(x^2-xy+y^2)$ $\therefore x^2-xy+y^2=(7)$ (1)

Since $x+y=5$ $\therefore x^2+2xy+y^2=25$.....(2)

Subtracting (1) from (2), $3xy=18$.

$xy=6$

and $x+y=5$.

Hence x and y can be found as in Ex 1

Ex. 3 Solve $xy=a^2$; $yz=b^2$ and $zx=c^2$.

Multiplying the three equations,

$xy\times yz\times zx=a^2b^2c^2$,

or $x^2y^2z^2=a^2b^2c^2$.

$\therefore xyz=\pm abc$.

Hence $x=\dfrac{xyz}{yz}=\dfrac{\pm abc}{b^2}=\pm\dfrac{ac}{b}$,

$y=\dfrac{xyz}{xz}=\dfrac{\pm abc}{c^2}=\pm\dfrac{ab}{c}$

and $z=\dfrac{xyz}{xy}=\pm\dfrac{abc}{a^2}=\pm\dfrac{bc}{a}$

Ex. 4. Solve $x(x+y+z)=a^2-yz$; $y(x+y+z)=b^2-xz$ and $z(x+y+z)=c^2-xy$.

We have $x(x+y+z)+yz=a^2$.

$\therefore x(x+y)+z(x+y)=a^2$.

$\therefore (x+y)(x+z)=a^2$ ..... (1)

Similarly from the other two equations

we can get $(y+x)(y+z)=b^2$ ..... (2)

and $(z+x)(z+y)=c^2$ ..... (3)

Multiplying (1), (2) and (3).

$(x+y)(x+z)(y+z)(y+x)(z+x)(z+y)=a^2b^2c^2$.

$\therefore (x+y)^2(y+z)^2(z+x)^2=a^2b^2c^2$.

$\therefore (x+y)(y+z)(z+x)=\pm abc$ ..... (4)

Dividing (4) by (1), (2) and (3) successively,

we have $y+z=\pm\frac{abc}{a^2}=\pm\frac{bc}{a}$

$z+x=\pm\frac{abc}{b^2}=\pm\frac{ac}{b}$

and $x+y=\pm\frac{abc}{c^2}=\pm\frac{ab}{c}$

From these three equations, the values of x, y and z can be found

Ex. 5. Solve $x(x+y+z)=a^2$ ..... (1)

$y(x+y+z)=b^2$ ..... (2)

$z(x+y+z)=c^2$ ..... (3)

Adding (1), (2) and (3),

we have $(x+y+z)^2=a^2+b^2+c^2$.

Hence $x+y+z=+\sqrt{(a^2+b^2+c^2)}$ ..... (4)

Dividing (1), (2) and (3) by (4) successively we get—

$x=+\frac{a^2}{\sqrt{a^2+b^2+c^2}}$, $y=+\frac{b^2}{\sqrt{a^2+b^2+c^2}}$, $z=+\frac{c^2}{\sqrt{a^2+b^2+c^2}}$

Ex. 6. Solve $x(z+y)=14$ ..... (1)

$y(x+z)=18$ ..... (2)

$z(x+y)=20$ ..... (3)

Adding (1), (2) and (3),

we have $2(xy+yz+zx)=52$

Hence $xy+yz+zx=26$ ..... (4)

Subtracting (1), (2) and (3) successively from (4) we get $yz=12$, $xz=8$ and $yz=6$.

From these the values of x, y and z can be got as in Ex. 3.

Ex. 7. Solve $xy+3(x+y)=11$ ..... (1)

$yz+3(y+z)=21$ ..... (2)

$zx+3(z+x)=15$ ..... (3)

From (1), $xy+3(x+y)+9=11+9=20$.

$\therefore (x+3)(y+3)=20$ ..... (*A*)

From (2), $yz+3(y+z)+9=21+9=30$.

$\therefore (y+3)(z+3)=30$ ..... (*B*)

From (3), $zx+3(z+x)+9=15+9=24$.

$\therefore (z+3)(x+3)=24$ ..... (*C*)

$(A)\times(B)\times(C)$ is $(x+3)^2(y+3)^2(z+3)^2=20\times30\times24$

$\therefore (x+3)(y+3)(z+3)=10\times12\times120$ ..... (*D*)

$\frac{D}{A}$ is $z+3=\frac{120}{20}=6$. $\therefore z=3$.

$\frac{D}{B}$ is $x+3=\frac{120}{30}=4$. $\therefore x=1$

$\frac{D}{C}$ is $y+3=\frac{120}{20}=5$. $\therefore y=2$.

EXERCISE 100.

Solve the following equations:—

1. $x+y=1$ and $x^3+y^3=1-3b+3b^2$.
2. $x+y=\frac{3}{\sqrt{2}}$ and $x^3+y^3=\frac{9}{2\sqrt{2}}$.
3. $x+y=7$ and $x^2+y^2=25$.
4. $x-y=b$ and $\frac{x}{y}=\frac{5}{4}$.
5. $x+y=4$ and $\frac{1}{x}+\frac{1}{y}=1$.
6. $x-y=2$ and $x^3-y^3=8$.
7. $x+y=12$, $x^2+y^2=74$.
8. $x+y=11$, $x^3+y^3=1001$.
9. $abxy=ax+by$, $acxz=ax+cz$ and $bcyz=by+cz$.
10. $xy+x+y=6$, $yz+y+z=12$ and $zx+z+x=18$.
11. $(x^2+y^2)(y^2+z^2)=65$, $(y^2+z^2)(z^2+x^2)=130$ and $(z^2+x^2)(x^2+y^2)=20$.
12. $x(px+qy+rz)=p(p^2+q^2+r^2)$, $y(px+qy+rz)=q(p^2+q^2+r^2)$ and $z(px+qy+rz)=r(p^2+q^2+r^2)$.
13. $x(y+z)=a$, $y(z+x)=b$ and $z(x+y)=c$.

14 $xy+x+y=11$, $yz+y+z=19$ and $zx+z+x=14$.

15 $x+y+z=\dfrac{a^2-yz}{x}=\dfrac{b^2-zx}{y}=\dfrac{c^2-xy}{z}$.

16. $\dfrac{y^2+z^2}{y^2z^2}=a^2$, $\dfrac{z^2+x^2}{z^2x^2}=b^2$ and $\dfrac{x^2+y^2}{x^2y^2}=c^2$.

199. Simultaneous Equations: where one or more may be of a degree higher than the first

We shall work out some examples.

Ex. 1. Solve $x^2+y^2=25$, $xy=12$.

From these $(x+y)^2=49$ and $(x-y)^2=1$,

$\therefore\ x+y=+7$ and $x-y=+1$.

Hence $\left.\begin{matrix}x+y=7\\x-y=1\end{matrix}\right\}$, $\left.\begin{matrix}x+y=-7\\x-y=1\end{matrix}\right\}$, $\left.\begin{matrix}x+y=7\\x-y=-1\end{matrix}\right\}$, $\left.\begin{matrix}x+y=-7\\x-y=-1\end{matrix}\right\}$

$\therefore\ x=4,-3,3,-4$ and $y=3,-4,4,-3$.

Ex. 2. Solve $x^4+y^4=97$, $xy=6$.

$$x^4+y^4=(x^2+y^2)^2-2x^2y^2.$$

$\therefore\ (x^2+y^2)^2-2\times 36=97.$

$\therefore\ x^2+y^2=25$

$\therefore\ (x^2+y^2)=25.$

$\therefore\ x^2+y^2=\pm 5$ and $xy=6$.

From these the values of x and y can be found as in the last example.

Ex. 3. Solve $x^5+y^5=33$, $x+y=3$.

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4=\frac{33}{3}=11,$$

or $(x^4+y^4+2x^2y^2)-xy(x^2+y^2)-x^2y^2=11,$

or $(x^2+y^2)^2-xy(x^2+y^2)-x^2y^2=11,$

but $x^2+y^2=9-2xy.$

$\therefore\ x+y=3.$

$\therefore\ (9-2xy)^2-xy(9-2xy)-x^2y^2=11.$

or $5x^2y^2-45xy+70=0,$

or $x^2y^2-9xy+14=0,$

or $(xy-7)(xy-2)=0.$

$\therefore\ xy=7$ or 2 and $x+y=3$.

Hence x and y can be found.

Note.—The following *identities* will help us in solving similar equations. The student will have no difficulty in verifying them.

i. $x^2+y^2=p^2-2q$
ii. $x^3+y^3=p^3-3pq$
iii. $x^4+y^4=p^4-4p^2q+2q^2$
iv. $x^5+y^5=p^5-5p^3q+5pq^2$
} if $x+y=p$ and $xy=q$.

i. $x^2+y^2=p^2+2q$
ii. $x^3-y^3=p^3+3pq$
iii. $x^4+y^4=p^4+4p^2q+2pq^2$
iv. $x^5-y^5=p^5+5pq^2+5pq^2$
} if $x-y=p$ and $xy=q$.

Ex. 4. Solve $x-y=211$ and $x-y=1$.

We know $x^5-y^5=p^5+5p^3q+5pq^2$, if $x-y=p$ and $xy=q$.

Since $x-y=1$, $x^5-y^5=1+5q+5q^2$.

$\therefore\ 5q^2+5q+1=211$

or $5q^2+5q-210=0$

or $q^2+q-42=0$

or $(q+7)(q-6)=0$

$\therefore\ q=-7$ or 6,

i.e., $xy=-7$ or 6 and $x-y=1$.

Hence x and y can be found.

Ex. 5. Solve $2x-y=3$.....(1)

$3x^2-6xy+y^2=1$.....(2)

From (1), $y=2x-3$. Substituting in (2)

$3x^2-6x(2x-3)+(2x-3)^2=1$.

$\therefore\ 5x^2-6x-8=0$.

$\therefore\ x=\frac{1}{10}(6\pm14)=2$ or $-\frac{4}{5}$.

Hence $y=1$ or $-\frac{23}{5}$.

200. Homogenous Equations: when the terms containing x and y are of the same degree in each.

Ex. 1. Solve $x^2+xy=21$, $y^2-xy=4$.

Let $y=vx$; then these equations become—

$x^2(1+v)=21$.....(1)

$\therefore\ x^2(v^2-v)=4$.....(2)

By division, $\frac{1+v}{v^2-v}=\frac{21}{4}$.

$\therefore\ 4+4v=21v^2-21v$.

$\therefore\ 21v^2-25v-4=0$.

$\therefore\ v=\frac{4}{42}(25\pm31)=\frac{4}{3}$ or $-\frac{1}{7}$.

Hence from (1), $x^2\left(1+\frac{4}{3}\right)=21$, or $x^2\left(1-\frac{1}{7}\right)=21$

$$\therefore\ x=\pm 3,\ \text{or}\ x=\pm\frac{7}{\sqrt{2}}.$$

Since $y=vx$, $\therefore\ y=\pm 4$ or $\pm\frac{1}{\sqrt{2}}$

Ex. 2. Solve $5xy-2x^2=42$; $3y^2+xy=18$.

Let $y=vx$; then these equations become—

$$x^2(5v-2)=42 \quad \ldots\ldots\ldots\ldots(1)$$

$$x^2(3v^2+v)=18 \quad \ldots\ldots\ldots\ldots(2)$$

By division, $\frac{3v^2+v}{5v-2}=\frac{3}{2}$

$$\therefore\ 6v^2+2v=15v-6.$$

$$6v^2-13v+6=0.$$

$$\therefore\ v=\frac{1}{12}(13\pm 5)=\frac{3}{2}\ \text{or}\ \frac{2}{3}.$$

Hence from (1), $x^2\left(\frac{15}{2}-2\right)=12$, or $x^2\left(\frac{10}{3}-2\right)=12$

$$\therefore\ x=\pm\sqrt{\frac{24}{11}}\ \text{or}\ x=\pm 3.$$

Since $y=vx$,

$$\therefore\ v=\pm\sqrt{\frac{54}{11}}\ \text{or}\ +2.$$

EXERCISE 101.

Solve the following equations :—

1. $y+\frac{\sqrt{y}}{x}=\frac{42}{x}$; $\frac{x^2}{3}+\frac{x}{2\sqrt{y}}=\frac{54}{y}$.
2. $(x^2+y^2)xy=30$; $x^4+y^4=82$.
3. $x^2+y^2=10$; $(x+1)(y+1)=8$.
4. $x^4+x^2y^2+y^4=21$; $x+y=7$.
5. $x^2+6xy+17y^2=33$; $3xy+16y^2=22$.
6. $xy-y^2+2x-y=2$; $x-y^2=1$.
7. $x^2+xy-8y=1$; $x-y=-8$.
8. $\frac{1}{x+1}+\frac{1}{1-y}=\frac{7}{12}$; $x-y=xy-13$.
9. $x(9-xy)=y(xy-36)$; $xy(3a+12y-xy)=108(x+y-3a)$.
10. $4y^2+3xy-x^2=0$; $(y-a)(x-y)=(x+2a)(x+y)$.
11. $x^2-2xy=7$; $2y^2-xy=-3$.

12. $x^2+y^2+x+y=26$; $4(x+y)=3xy$.

13. $6x+5y=\frac{6}{x}+\frac{5}{y}+29\frac{1}{3}$; $3x+4y=\frac{3}{x}+\frac{4}{y}+18\frac{2}{3}$.

14. $y+\sqrt{x^2-1}=2$; $\sqrt{x+1}+\sqrt{x-1}=\frac{2}{\sqrt{y}}$.

15. $x+\frac{3}{y}=2$, $y+\frac{3}{x}=-2$.

16. $x^2+y^2=2a^2$
$x+y:x-y=m:n$.

17. $x^2+y^2=a^2$; $x+y=b$.

18. $x-y=1$; $(x^2+xy+y^2)(x^3-y^3)=361$.

19. $2x+y=4x^2$; $3y-2x=y^2$.

20. $xy+y^2=3$; $x^4-y^4=6(x^2+3y^2)-27$.

21. $x+y=a$; $x^5+y^5=b$.

22. $(1+m^2)(x+y)=2m(1+xy)$; $(1+n^2)(x-y)=2n(1-xy)$.

23. $49(x^2+a^2)(y^2+b^2)=50(xy+ab)^2$; $(x^2-a^2)(y^2-b^2)=24(bx-ay)^2$.

24. $x^5-y^5=31$; $x-y=1$.

25. $4(x^2+y^2)-5xy+3(x+y)-37=0$;
$3(x^2+y^2)-2xy+4(x+y)-47=0$.

26. $3x-y=5$; $2x^2-4xy+y^2=1$.

27. $3x+2y=5$; $4x^2+9y^2=13$.

28. $x^2+xy=126$; $y^2+xy=198$.

29. $x^3-y^3=63$; $x-y=3$.

30. $x^2+xy+y^2=84$; $x-\sqrt{xy}+y=6$.

31. $x^4+x^2y^2+y^4=273$; $x^2+xy+y^2=9$.

32. $\sqrt{x}+\sqrt{y}=6$; $x+y=126$.

33. $(x^2+y^2)(x^3+y^3)=45$; $x+y=3$.

34. $x^2+2y^2=47$; $3x^2-5xy+xy^2=13$.

35. $x^3+xy^2=a^3$; $y^3+yx^2=b^3$.

36. $\frac{x^2}{a}+\frac{y^2}{b}=\frac{a^2}{x}+\frac{b^2}{y}=a+b$.

37. $\frac{x^2}{y}+\frac{y^2}{x}=9$; $\frac{1}{x}+\frac{1}{y}=\frac{3}{4}$.

38. $x^2-xy+y^2=24$; $y^2-2xy+15=0$.

39. $x^4+y^4=14x^2y^2$; $x+y=a$.

40. $\frac{a}{a+x}+\frac{b}{b+y}=1$; $x+y=a+b$.

201 We add a few examples of quadratic equations involving *three* unknown quantities.

Ex. 1. Solve $x+y+z=x^2+y^2+z^2=x^3+y^3+z^3=1$.

Since $x+y+z=1$.

$\therefore\ x^2+y^2+z^2+2(xy+yz+zx)=1.$

and $x^2+y^2+z^2=1$

$\therefore\ 2(xy+yz+zx)=0.$

$\therefore\ xy+yz+zx=0$.....(1)

Again, since $x^3+y^3+z^3-3xyz$.

$=(x+y+z)(x^2+y^2+z^2-xy-yz-zx).$

$\therefore 1-3xyz=1\times(1-0)=1.$

$\therefore 3xyz=0.$

$\therefore xyz=0$.....(2)

Now $(1-x)(1-y)(1-z)=1-(x+y+z)+(xy+yz+zx)-xyz$

$=1-1+0-0=0.$

$\therefore$ one of the factors $1-x$, $1-y$, $1-z$ must $=0$.

Let $1-x=0$; then $x=1$;

$\therefore\ y+z=0.\quad \because x+y+z=1$

and from (2), $yz=0$

$\therefore y=0$ and $z=0$

Ex. 2. Solve $x^2+2yz=13$.....(i)

$y^2+2xz=10$.....(ii)

and $z^2+2xy=13$.....(iii)

Adding, $x^2+y^2+z^2+2xy+2yz+2zx=36.$

$\therefore\ x+y+z=6$.....(iv)

Subtracting (iii) from (i), $x^2-z^2+2y(z-x)=0.$

$\therefore\ (x-z)(x+z-2y)=0.$

$\therefore\ x+z=2y.$

$\therefore$ from (iv), $3y=6.$

$\therefore\ y=2.$

$\therefore\ x+z=4$ and from (ii), $xz=\frac{1}{2}(10-4)=3.$

$\therefore\ x=3$ or 1 and $z=1$ or 3.

Ex. 3. Solve $x+y+z=6$; $x^3+y^3+z^3=36.$

and $(x+z)(y+z)=20.$

Since $x+y+z=6$,

$x^3+y^3+z^3+3\ (x+y)(y+z)(z+x)=216.$

But $x^3+y^3+z^3=36$ and $(x+z)(y+z)=20.$

$\therefore 36+3\times 20(x+y)=216.$

$\therefore x+y=3.$

$\therefore z=3.$

$\because x+y+z=6$

From the last equation, $(x+3)(y+3)=20$.

$\therefore xy+3(x+y)=11$.

$\therefore xy=11-9=2$ and $x+y=3$

$\therefore x=2$ or 1 and $y=1$ or 2.

Ex. 4. Solve $x+y+z=6$ .....(i)

$x^2+y^2+z^2=18$ .....(ii)

and $yz-x^2=5$ .....(iii)

From (ii) and (iii), $x^2+y^2+z^2+2(yz-x^2)=14+10$.

$\therefore (y+z)^2-x^2=24$.

$\therefore (y+z+x)(y+z-x)=24$;

but $x+y+z=6$.

$\therefore y+z-x=4$ and $x+y+z=6$.

$\therefore x=1$ and $y+z=5$,

and from (iii), $yz=6$.

$\therefore y=3$ and $z=2$.

Ex. 5. Solve $x^2-yz=a$ .....(i)

$y^2-xz=b$ .....(ii)

and $z^2-xy=c$ .....(iii)

$(x^2-yz)^2-(y^2-xz)(z^2-xy)=a^2-bc$.

$\therefore x(x^3+y^3+z^3-3xyz)=x^2-bc$.

Similarly $y(x^3+y^3+z^3-3xyz)=b^2-ac$.

and $z(x^3+y^3+z^3-3xyz)=c^2-ab$.

$$\therefore \frac{x}{a^2-bc}=\frac{y}{b^2-ac}=\frac{z}{c^2-ab}.$$

$$\therefore \text{each}=\pm\frac{\sqrt{x^2-yz}}{\sqrt{(a^2-bc)^2-(b^2-ac)(c^2-ab)}}.$$

$$=\pm\frac{\sqrt{a}}{\sqrt{a(a^3+b^3+c^3-3abc)}}.$$

$$=\pm\frac{1}{\sqrt{(a^3+b^3+c^3-3abc)}}.$$

$$\therefore x=\pm\frac{a^2-bc}{\sqrt{(a^3+b^3+c^3-3abc)}}.$$

Ex. 6. Solve $x^2+xy+y^2=19$ .....(i)

$y^2+yz+z^2=37$ .....(ii)

and $z^2+zx+x^2=28$ .....(iii)

From (i) and (ii), $x^2 - z^2 + y(x-z) = -18$.

$$\therefore (x-z)(x+y+z) = -18 \quad \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

From (i) and (iii), $y^2 - z^2 + x(y-z) = -9$.

$$\therefore (y-z)(x+y+z) = -9 \quad \ldots\ldots\ldots\ldots\ldots\ldots (v)$$

By division, $\frac{x-z}{y-z} = 2$.

$$\therefore x - z = 2y - 2z.$$

$$\therefore y = \frac{x+z}{2}.$$

Substituting in (ii), $\frac{(x+z)^2}{4} + \frac{z}{2}(x+z) + z^2 = 37$

$\therefore 7z^2 + 4xz + x^2 = 148$,

and $x^2 + zx + x^2 = 28$,

Let $z = vx$; these equations become

$x^2(7v^2 + 4v + 1) = 148 \quad \ldots\ldots\ldots\ldots\ldots\ldots (A)$

and $x^2(v^2 + v + 1) = 28 \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$

By division, $\frac{7v^2 + 4v + 1}{v^2 + v + 1} = \frac{148}{28} = \frac{37}{7}$

$$\therefore 12v^2 - 9v - 30 = 0.$$

$$\therefore 4v^2 - 3v - 10 = 0.$$

$$\therefore (v-2)(4v+5) = 0.$$

$$\therefore v = 2 \text{ or } -\tfrac{5}{4}.$$

Hence from (A), $x^2(28 + 8 + 1) = 148$.

$$\therefore x = \pm 2.$$

$$\therefore z = \pm \text{ and } y = \pm\frac{4+2}{4} = \pm 3.$$

From the other value of v we can get two more values for each of x, y and z.

EXERCISE 102.

Solve the following equations :—

1. $x^2 + y^2 + z^2 = 6$; $xy + yz + zx = 5$ and $xyz = 2$.
2. $\frac{a^2}{x} = \frac{b^2}{y} = \frac{c^2}{z} = x + y + z$.
3. $x + y + z = 7$; $x(y+z) = 12$ and $x^2 + y^2 + z^2 = 21$.
4. $x^2 + y^2 + z^2 = 1$; $ax^2 + by^2 + cz^2 = 0$;

and $bcx^2 + acy^2 + abz^2 = 0$.

5. $x+y+z=0$, $(b+c)\,x+(c+a)\,y+(a+b)\,z=0$;
and $x^3+y^3+z^3=3(a-b)\,(b-c)\,(c-a)$.

6. $x+y+z=6$; $x^2+y^2+z^2=14$; and $x^3+y^3+z^3=36$.

7. $x+y+z=9$; $x^2+y^2+z^2=29$; and $x^3+y^3+z^3=99$.

8. $x^3+y^3+z^3=36$; $(x+y)\,(y+z)\,(z+x)=60$;
and $(x+y+z)\,(xy+yz+zx)=66$.

9. $x^2+xy+y^2=c$; $y^2+yz+z^2=a$ and $z^2+zx+x^2=b$

10. $(y-a)\,(z-a)=bc$; $(z-b)\,(x-b)=ca$; $(x-c)\,(y-c)=ab$.

11. $x+y+z=0$; $ax+by+cz=0$
and $x^3+y^3+z^3=3\,(a-b)\,(b-c)\,(c-a)$.

12. $x(y+z-x)=a$; $y(x+z-y)=b$ and $z(x+y-z)=c$.

13. $x^{-2}y^{-1}z=a$; $x^{-1}yz^2=b$ and $x^2y^2z^3=c$.

14. $x^2+(y-z)^2=a$; $y^2+(z-x)^2=b$ and $z^2+(x-y)^2=c$.

15. $x^2+yz=y^2+zx=z^2+xy=c$.

16. $x+y+z=a+b+c$; $x^2+y^2+z^2=a^2+b^2+c^2$;
and $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=3$.

17. $\dfrac{y+z}{a}=\dfrac{z+x}{b}=\dfrac{x+y}{c}=\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}$.

18. $x^2+2yz=148$; $y^2+2xz=145$ and $z^2+2xy=148$.

19. $x^2y^2+y^2z^2+z^2x^2=49$; $x^2+y^2+z^2=14$ and $x(y+z)=9$.

20. $y+x+z=1$; $x^2+y^2+z^2=13$ and $x^3+y^3+z^3=19$

CHAPTER XXXI.

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

202. We shall solve some problems which lead to quadratic equations.

Ex. 1. Divide 15 into two parts, such that their product shall be 56.

Let x = one part; then $15 - x$ = the other part.

By the condition of the problem, $x(15 - x) = 56$.

or $x^2 - 15x + 56 = 0$.

$\therefore$ (' $(x - 8) = 0$.

$\therefore$ $x = 7$ or 8 = *one* part; and 8 or 7 = the *other* part.

Ex 2. Divide a given line into two parts such that the square on one part may be equal to the rectangle contained by the whole and the other part.

Let a denote the line, and x the length of one part; then $a - x$ is the length of the other part

By the question, $x^2 = a(a - x)$ or $x^2 + ax - a^2 = 0$.

$$\therefore x = \tfrac{1}{2}(-a \pm \sqrt{5a^2}) = \frac{a}{2}(1 \pm \sqrt{5}).$$

$\therefore \frac{a}{2}(-1 + \sqrt{5})$ is the length of one part.

The *negative* answer is the solution of the following problem:—*Produce a given line, so that the square on the given line may be equal to the rectangle contained by the whole line produced and the part produced.*

Ex. 3. A man buys a certain number of oxen for £ 100; if he had bought 5 more for the same money, each would have cost him £ 1 less; how many did he buy?

Let x = number of oxen.

then $\frac{100}{x}$ = price of each ox in *pounds*.

If he had bought 5 more, $\frac{100}{5 + x}$ would have been the price of each

By the question, $\frac{100}{5+x} = \frac{100}{x} - 1$ or $100\left(\frac{1}{x} - \frac{1}{5+x}\right) = 1$

or $x^2 + 5x - 500 = 0$

$\therefore\ x = 20$ or -25.

$\therefore$ the number of oxen $= 20$.

The latter root must be rejected.

Ex. 4 The length of a rectangular field exceeds its breadth by 33 yards. Its area is one acre ; find its dimensions.

Let x = the breadth in *yards* ;

then $x+33$ = the length in *yards*.

By the question, $x(x+33) = 4840$ (one acre equals 4840 square yards.)

or $x^2 + 33x - 4840 = 0$.

$= -88$ or 55

The negative value must be rejected.

The breadth $= 55$ yards and the length $= 88$ yards.

Ex. 5. A and B started at the same time for a place 300 miles distant. A travels a mile an hour faster than B, and arrives at his journey's end 10 hours before him ; find the rate per hour at which each person travelled.

Let x = *number of miles per hour* that B travels ,

then $x+1$ = *number of miles per hour* that A travels.

Now $\frac{300}{x}$ and $\frac{300}{x+1}$, denote the time in hours taken by B and A respectively.

By the question, $\frac{300}{x+1} = \frac{300}{x} - 10$

or $30\left(\frac{1}{x} - \frac{1}{x+1}\right) = 1$

$\therefore\ x^2 + x - 30 = 0.$

$\therefore\ (x+6)(x-5) = 0.$

$\therefore\ x = 5$ or -6.

The negative value must be rejected.

$\therefore$ B's rate is 5 miles per hour ; A's rate is 6 miles per hour.

Ex 6 A person sells a horse for £24 and gains as much per cent. as the horse cost him ; what did the horse cost him ?

Let x = the cost price of the horse in *pounds* :

then his gain $= x \times \frac{x}{100} = \frac{x^2}{100}$.

By the question, $x + \frac{x^2}{100} = 24$.

$\therefore x^2 + 100x - 2400 = 0$.

or $(x+120)(x-20) = 0$.

$\therefore x = -120$ or 20.

The negative value should be rejected.

The cost price of the horse is £20.

EXERCISE 103.

1. The difference of two numbers is 4, the difference of their squares is 40 ; find the numbers.

2. Find two *consecutive* numbers whose product is 420.

3. The difference of the cubes of two *consecutive* numbers is 217 : find them

4. Find two numbers whose sum, product and difference of squares, shall be equal.

5. Find two numbers each of which is the square of the other.

6. If three feet be taken from one side of a rectangle and added to the other side, its area is doubled : the sum of the sides is 7 feet ; find the sides.

7. There is a rectangular field of 1 acre, whose length exceeds its breadth by 66 yards ; find its dimensions.

8. There is a rectangular field whose length exceeds its breadth by 16 yards and it contains 960 square yards ; find its dimensions.

9. The area of a rectangular field is $7\frac{1}{2}$ acres and the sum of the lengths of the two adjacent sides exceeds the length of either diagonal by 110 yards ; find the length of the sides.

10. By selling a horse for £25 I lose as much per cent as it cost me ; what was its prime cost ?

11. A person bought a certain number of oxen for £240 and after losing 3 sold the rest for £8 a head more than they cost him, thus gaining £59 by the bargain ; how many did he buy ?

12. A person bought 2 flocks of sheep for £15, in one of which there were 5 more than in the other ; each sheep in each flock cost as

many shillings as there were sheep in the other flock; how many were there in each?

13. A and B distribute £5 each in charity; A relieves 5 persons more than B, and B gives to each 1 shilling more than A; how many did each relieve?

14. Find three numbers, such that if the first be multiplied by the sum of the second and the third, the second by the sum of the first and the third, and the third by the sum of the first and the second, the products shall be 26, 50 and 56 respectively.

15. I have to walk a distance of 144 miles, and I find that if I increase my speed $1\frac{1}{2}$ miles per hour, I can do the journey in 16 hours less than if I walk at my usual rate; find my usual rate of walking.

16. There are two square fields, the greater of which is 4 times the less, and the side of the greater is 20 yards longer than the side of the other; find the area of each field.

17. The sum of the reciprocals of two numbers is $\frac{7}{12}$ and the product of the numbers is 12; find them.

18. A number of soldiers are formed into a hollow square of 3 deep; if 324 men are added to the centre they will form a solid square; find the total number of men.

19. If a number consisting of two digits be divided by the sum of the digits, the quotient is the first digit, and the remainder the last; the product of the digit is 21; find the number.

20. The sum of the fraction and its reciprocal is $2\frac{16}{21}$; but if the numerator and the denominator be each increased by unity, the sum would then be $2\frac{1}{2}$; find the fraction.

21. A boat's crew can row both up and down a stream 16 miles long, flowing at the rate of 3 miles an hour, in 10 hours; find the rate of the crew in still water.

22. A number consists of three digits whose sum is 14; the square of the middle digit is equal to the product of the extreme digits, and if 594 be added to the number, the digits are *reversed*: find the number.

23. The united ages of a father and son amount to 64. Twice the father's age exceeds the square of the son's age by 8: find their ages

24. Two kinds of oranges are sold in the market, two more of one kind being given for a shilling than of the other; a score of the inferior sort costs $4d.$ more than a dozen of the superior sort; find the price of the oranges.

25. Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

26. The floor of a room contains 40 square yards, its height is 5 yards, and the length is 3 yards more than the breadth; find the area of the 4 walls.

27. A man buys a horse which he sells again for £56 and gains as many pounds in 100 as the horse cost him; how much did he give for the horse?

28. A and B started at the same time for a place distant 150 miles. A travels 3 miles an hour faster than B and arrives at his journey's end 8 hours 20 minutes before B; find their rates.

29. A person bought oxen for £33-15-0 which he sold again at £2-8-0 a hand, gaining thereby as much as one ox cost him; how many oxen did he buy?

30. Divide the number 30 into 2 parts such that their product may be equal to 125.

31. A and B set out from two towns 247 miles distant from each other. A travelled at the rate of 9 miles a day, and the number of days at the end of which they met was greater by 3 than the number of miles which B went in a day; find the number of miles each travelled.

32. The fore-wheel of a carriage makes 64 revolutions more than the hind-wheel in passing over a mile: but if the circumference of the fore-wheel be increased by 11 inches, it will make only 40 revolutions more than the hind-wheel in the same space, find the circumference of each wheel.

33. One man can reap a field in 5 days less than another, and if they work together they can do it in 6 days; find in what time each could do it alone.

34. A man has to travel a certain distance; when he has gone 20 miles he increases his speed 1 mile per hour; if he had travelled at this increased rate during the whole of his journey he would have

arrived 40 minutes earlier; but if he had kept on at his first rate he would have arrived 20 minutes later; find the length of the journey.

35. A and B engaged to reap equal quantities of wheat, and A began half an hour before B; they stopped at 12 o'clock and rested an hour, observing that half the work was done, B's part was finished at 7 o'clock and A's at a quarter before 10. Supposing them to have laboured uniformly, find when they began.

36. A cask P is filled with 50 gallons of water, and a cask Q with 40 gallons of wine, x gallons are drawn from each cask, mixed and replaced. The same operation is repeated; find x when there are $8\frac{1}{2}$ gallons of wine in P after the second operation.

37. Divide a given line into two parts such that twice the square on one part may be equal to the rectangle contained by the whole line and the other part.

38. Find that number whose square added to its cube is 9 times the next higher number.

39. A farmer wishes to enclose a rectangular piece of land containing 1 acre 32 perches with 176 hurdles, each two yards long; how many hurdles must he place in each side of the rectangle?

40. What are eggs a dozen when two more in a shilling's worth lower the price one penny per dozen?

41. A person rents a certain number of acres of land for £84; he cultivates 4 acres himself, and letting the rest for 10s. an acre more than he pays for it, receives for this portion the whole rent £84; find the number of acres.

42. A set off from London to York, and B at the same time from York to London, and they travel uniformly: A reaches York 16 hours and B reaches London 36 hours, after they have met on the road; find in what time each has performed the journey.

43. A vessel can be filled with water by two pipes; by one of these pipes alone the vessel would be filled 2 hours sooner than by the other; also the vessel can be filled by both pipes together in $1\frac{7}{8}$ hours; find the time which each pipe would take to fill the vessel.

44. A person buys a quantity of wheat which he sells so as to gain 5 per cent. on his outlay, and thus clears £16. If he had sold it at a gain of 5 shillings per quarter, he would have cleared as many

pounds as each quarter cost him shillings ; find how many quarters he bought, and what each quarter cost him.

45. Two workmen, A and B, were employed by the day at different rates, A, at the end of a certain number of days received £4-16-0, but B, who was absent six of those days, received only £2-14-0. If B had worked the whole time, and A had been absent six days, they would have received equal sums ; find the number of days, and what each was paid per day.

46. A and B run a race round a two-mile course. In the first heat B reaches the winning post 2 minutes before A. In the second heat A increases his speed 2 miles per hour, and B diminishes his as much ; and A then arrives at the winning post 2 minutes before B ; find at what rate each man ran in the first heat.

47. A person bought two pieces of cloth of different sorts ; the finer cost 4 shillings a yard more than the coarser and he bought 10 yards more of the coarser than of the finer. For the finer piece he paid £18, and for the coarser piece £16 ; find the number of yards in each piece.

48. A merchant sent to his agent a certain number of bags of coffee for sale, expecting them to realize £250, and after the agent's commission of 4 per cent. was paid, to yield a profit of 20 per cent. on the original cost. On the way 5 bags were lost, but the rest were sold at 10s. per bag above the estimated price. This raised the net profits £76-17-6 per cent. although the agent now received a commission of 5 per cent.; how many bags were there at first ?

49. The relative value of two sorts of mixed metals consisting of gold and silver is as 11 to 17. If the proportion of gold to silver in each had been doubled, their relative value would have been as 7 to 11. The value of gold to that of silver being as 13 to 1, find the proportion of gold to silver in each of the mixed metals.

50. A person bought one horse for £x and another for £y, he sold the first at a profit of x per cent. and the second at a loss of y per cent., and thus received $\frac{1}{5}$ as much again as he would have, had he sold the *first* at a profit of y per cent. and the *second* at a loss of x per cent. If he had bought x horses at £x each, and sold them at x per cent. profit and had bought y horses at £y each, and had sold them at y per cent. loss, he would have gained altogether £1,520 ; what did he give for each horse ?

CHAPTER XXXII.

Graphs of Quadratic Functions.

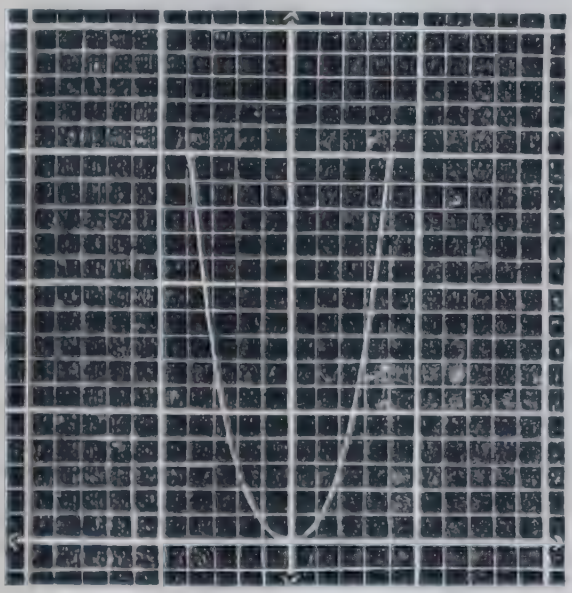
103. Graphs of linear algebraical functions have been already dealt with and in this chapter we shall deal with graphs of functions which involve the second degree of the variable. The simplest of them is $y=x^2$.

Ex. 1. To draw the graph of $y=x^2$.

The plotting of the graph representing $y=x^2$ may be done on the same lines as those adopted in the case of linear functions. A table of values of x and the corresponding values of y may be made thus :

If $x=$	0	1	2	3	4	−1	−2	−3	−4
$y=$	0	1	4	9	16	1	4	9	16

Points (0,0), (1,1), (2,4), (3,9), (4,16), (−1,1), (−2,4), (−3,9), (−4,16) are plotted and joined free hand. The figure thus obtained is shown in the diagram. In this case the same unit, *viz.*, 0·1" is used for both x and y.



It must be noticed that we get the same value of y for two values of x which are numerically the same but opposite in sign. Or putting conversely we may say that for any particular value of y we get two values of x which differ in sign.

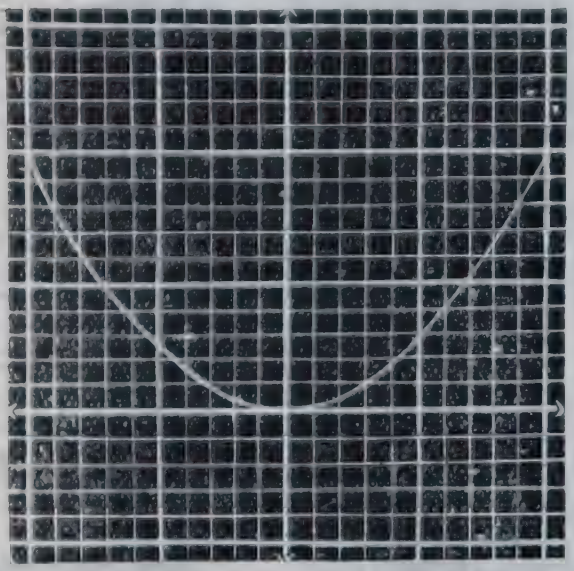
From the equation $x^2 = y$ we see that $x = \pm\sqrt{y}$, *i. e.*, $\sqrt{y}$ is either positive or negative. Thus the present curve may be used to find the square roots of arithmetical or numerical numbers. It is to be noticed that the curve is symmetrical about the y axis inasmuch as for every point on one half of the curve, there is a corresponding point on the other half, the two being equidistant from the axis of y. This curve is called a *parabola* and the line about which it is symmetrical is called the *axis* of the parabola. We also notice that this curve turns, *i. e.*, changes the direction in which it has been moving at a certain point which is the origin in this case and this point is called the turning point or the *vertex* of the parabola.

From the table of values of x and y it can be observed that as x increases numerically the value of y also increases but much more rapidly. The curve drawn in the diagram appears to be rather narrow and by taking different units for x and y, it may be made broader without changing its shape. Sometimes a portion of the curve can be made to appear almost as a straight line.

The curve may be plotted for values of x between + 1 and − 1 by taking fractional values of x:

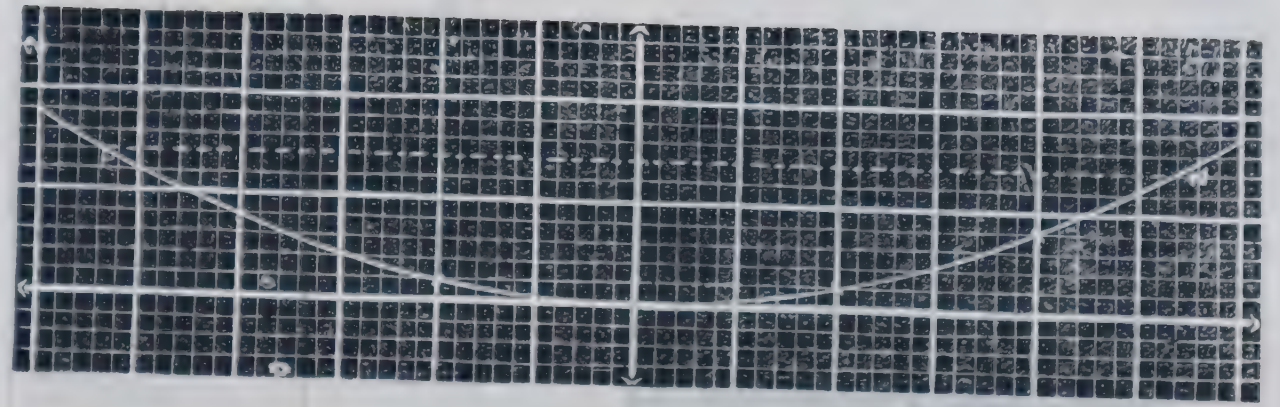
If $x=$	...	$\pm\cdot1$	$\pm\cdot2$	$\pm\cdot3$	$\pm\cdot4$	$\pm\cdot5$	$\pm\cdot6$	$\pm\cdot7$	$\pm\cdot8$	$\pm\cdot9$	±1
$y=$	...	·01	·04	·09	·16	·25	·36	·49	·64	81	1

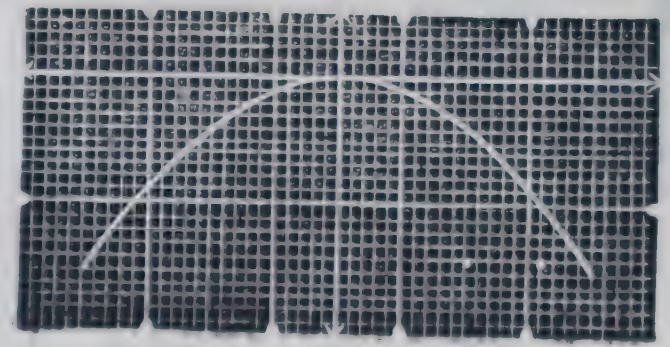
Let the unit for x and y be an inch.



We can plot the curve by taking a large unit for x and a small unit for y, for example, the unit 1'' for x and ·1'' for y. Taking the table previously made we have :

If $x=$ ...	0	1	2	3	4	−1	−2	−3	−4
$y=$ ...	0	1	4	9	16	1	4	9	16

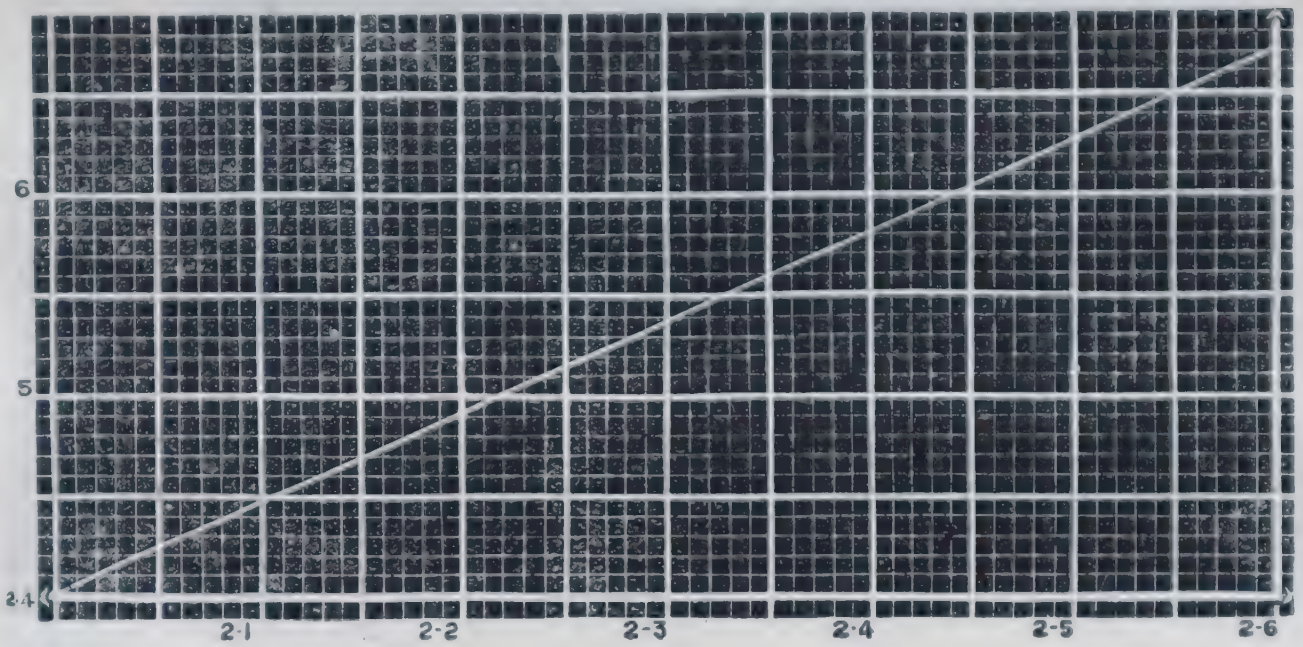




This curve can be used very conveniently to find the square roots of the numbers up to 9. For example: The square root of 7 can be read off from it thus:

Take a length on y axis which represents 7 units, *i e*, take a point 7'' from the origin and from it draw a parallel to the x axis to meet the curve. This parallel when produced cuts the curve in 2 points N and P. Read off the lengths of the abscissæ or the x co-ordinates of these points. These are + 2·6 and − 2·6 nearly. Therefore $\sqrt{7} = \pm 2{\cdot}6$ nearly.

Example:—To find the square root of 6, 7, correct to two decimals. (The procedure in this case is quite similar to that in the previous case.) Only suitable units should be taken on the x axis and on the y axis. And as we want the square root of 6 and 7 we are concerned with that portion of the curve mainly which contains points whose ordinates are 6 and 7. The origin may be marked as (2,4) as the required square root lies between 2 and 3, and 10″ may be taken to represent 1 unit on the x axis and 1″ on the y axis.



When read off as usual from the graph, it is found that

$\sqrt{5} = 2{\cdot}25$ numerically

$\sqrt{6} = 2{\cdot}45$ . . . .

To draw the graph of $y = -x^2$

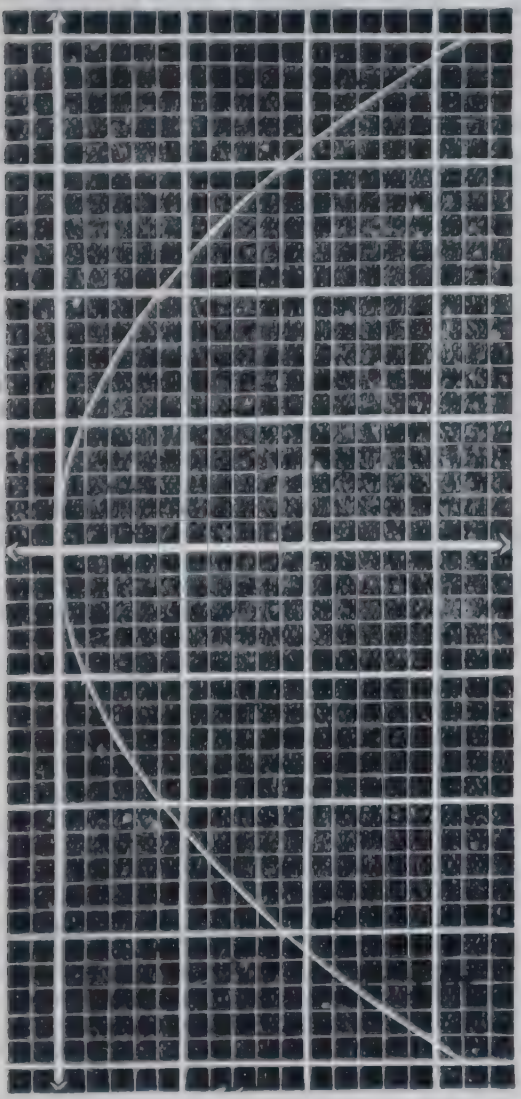
As before make a table of values of y corresponding to different values of x as shown below :—

When $x=$ ...	0	1	2	3	4	−1	−2	−3	−4
$y=$ ...	0	−1	−4	−9	−16	−1	−4	−9	−16

We notice here also that for every value of y there are two corresponding values of x which are numerically equal but opposite in sign.

Plotting the points given by the above table, we get the graph shown below.

It is to be noticed that, since every value of y is negative corresponding to any value of x whether positive or negative, all the ordinates are negative and consequently the graph lies entirely below the x axis.



The present graph is just similar to the graph of $y = x^2$ which lies entirely above the axis of x as all the ordinates are positive whether the abscissæ are positive or negative.

It can also be noticed that the graphs of $y = -2x^2$, $y = -3x^2$, etc., are similar to the graphs of $y = 2x^2$, $y = 3x^2$, etc., and that the former graphs lie below the x axis while the latter lie above it.

From the graphs that have been treated above the following points are noticed:—

(i) $y = ax^2$ represents a curve symmetrical about the axis, passing through the origin.

The curve which is called the parabola changes its direction in which it has been going at a certain point called the turning point or the vertex of the parabola.

(ii) The vertex of the parabola $y = ax^2$ is the origin.

(iii) The curve represented by $y = ax^2$ lies entirely above or below the x axis according as a is positive or negative. It is steep or flat according as a is an integer or a fraction.

EXERCISE 104.

Draw the graphs of:—

1. (i) $y = 2x^2$, (ii) $3y = x^2$, (iii) $y = 5x^2$, (iv) $y - x^2 = 0$.
2. Draw the graph of (i) $y = x^2$ from $x = -3$ to $x = +3$.
   (ii) $y = x^2$ from $x = -4$ to $x = +4$.
3. Use the graph of 2 (i) to find the values of (i) $(1{\cdot}5)^2$, (ii) $(2{\cdot}4)^2$, (iii) $(1{\cdot}8)^2$

and the graph of 2 (ii) to find the value of

(i) $\sqrt{3}$, (ii) $\sqrt{10}$, (iii) $\sqrt{12{\cdot}5}$, (iv) $\sqrt{13{\cdot}6}$.

4. Draw the graphs of $y = x^2$ to suitable scales to find the value of—

(i) $\sqrt{7}$ correct to two places of decimals

(ii) $\sqrt{12}$ „ two „ „

(iii) $\sqrt{18}$ „ two „ „

5. On the same diagram and to the same scale draw the graphs of $y = x^2$ and $y = 3x + 4$ and read off the co-ordinates of the points of intersection of the two graphs.

6. On the same diagram and to the same scale draw the graphs of (i) $y = 4x^2$, (ii) $y = 3x + 1$. Read off the x co-ordinates of the points of intersection of the two graphs.

7. Find the co-ordinates of the points of intersection of :—(i) $y = 2x^2$ and (ii) $y = 3x - 1$.

8. Find the abscissæ of the points of intersection of the graphs $2y = x^2$ and $x + 2y = 2$.

9. On the same diagram and to the same scale draw the graphs of $y = mx^2$.

When (i) $m = 1$, (ii) $m = 2$, (iii) $m = \frac{1}{2}$.

10. Find the co-ordinates of the points of intersection of $4y=x^2$ and $5x-4y=6$.

11. Draw the graphs of :—

(i) $y=-2x^2$, (ii) $2y=-3x^2$, (iii) $2y+x^2=0$, (iv) $y+3x^2=0$.

12. Draw the graph of:—

(i) $y=-x^2$ between $x=+3$ and $x=-3$.

(ii) $y=-x^2$ „ $x=+4$ and $x=-4$.

13. In the same diagram draw the graphs of :—
$y+x^2=0$ and $y=3-2x$ and read off the co-ordinates of the points of intersection. (The scale should be the same in the case of the two graphs).

14. In the same diagram and to the same scale draw the graphs of (i) $y=-4x^2$, (ii) $y+3x+1=0$ and read off the x co-ordinates of the points of intersection of the two graphs.

15. On the same diagram to the same scale draw the graphs of $y=ax^2$.

When $a=-2, -\frac{1}{2}, -\frac{1}{3}, 3, -4$.

16. Find the co-ordinates of the points of intersection of

(i) $2y+x^2=0$, (ii) $y+\frac{5}{2}x=3$.

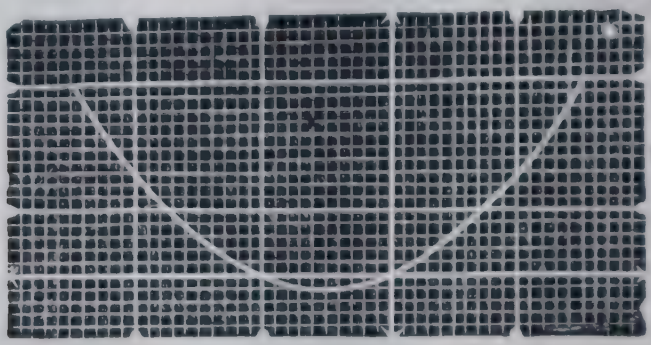
Graph of $y=ax^2+bx+c$.

Ex. 2. To draw the graph of $y=x^2+2x$.

As before a table of values of x and corresponding values of y has to be prepared $y=x^2+2x$.

When $x=$ ...	0	1	2	3	−1	−2	−3	−4	−5
$x^2=$ ...	0	1	4	9	1	4	9	16	25
$2x=$ ...	0	2	4	6	−2	−4	−6	−8	−10
$y=x^2+2x=$	0	3	8	15	−1	+0	3	8	15

Plot the points (0, 0), (1, 3), (2, 8), (3, 15), (−1, −1) (−2, 0) (−3, 3), (−4, 8) (−5, 15) and join them freehand.

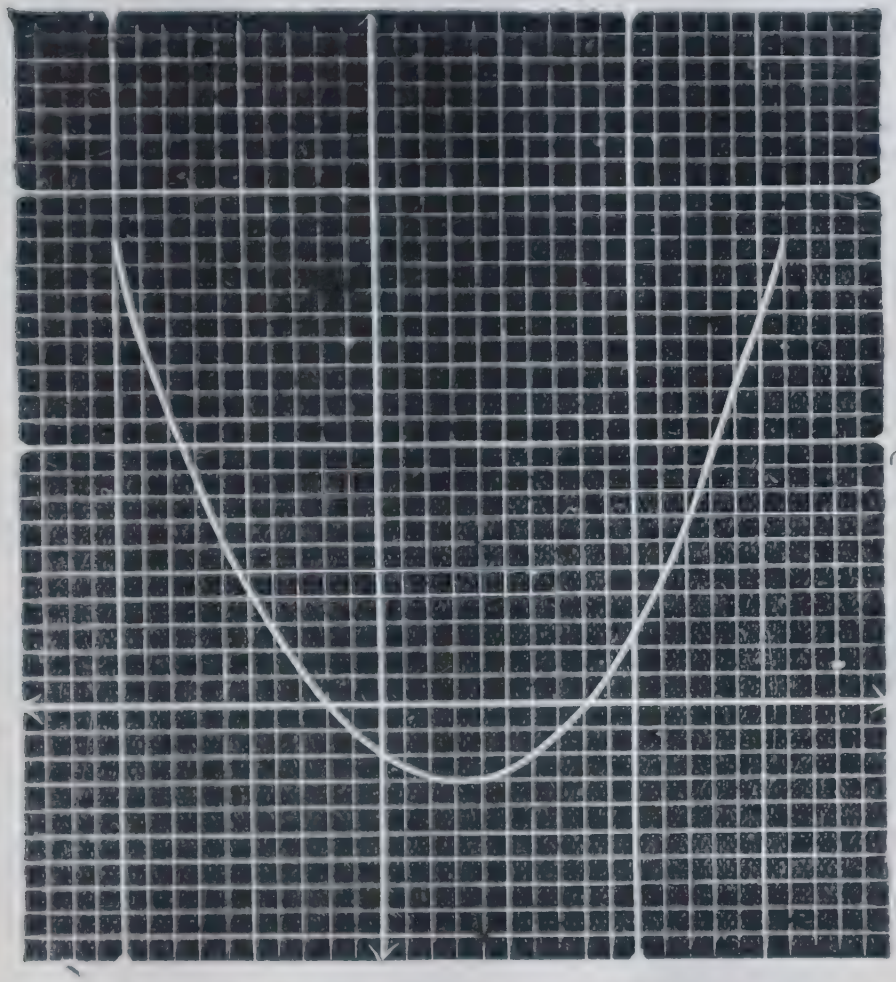


The graph obtained resembles one of the equation $y=ax^2$. In the latter case the graph lies entirely either above or below the x axis according as a is positive or negative while in the former a portion of the graph is above and the other below the x axis, the graph opening upwards.

In both the cases since the function of x does not contain a term independent of x, the graph passes through the origin. But in the case of the graph of $y=ax^2$ the *vertex* or the turning point is the origin while in the case of the graph of $y=x^2+2x$, the turning point is a point other than the origin. In both cases the curve is a *parabola* and symmetrical. The y axis is the line about which the curve is symmetrical in the case of $y=ax^2$ while the line of symmetry in the other case is a different line altogether.

It is to be observed that, when the graph is partly above and partly below the axis of x, some of the ordinates of the graph are positive and others are negative, and the ordinate at the turning point is *algebraically* the smallest. As the ordinate or y represents the function x^2+2x, it can be said that the value of the function is least or minimum when x takes that value which is the length of the abscissa of the turning point of the curve.

In the present graph since the algebraically least value of y is -1, the minimum value of the function is said to be -1 and it is so when $x=-2$. This means that though the function x^2+2x may take a very large number of values, it cannot take a value which is less than -1 whatever value may be given to x.



This is also observed from the graph as it has not descended below the turning point $(-2, -1)$ and for any value of x other than -2 the graph only ascends.

Ex. 3. To draw the graph of $y = 2x^2 - 3x - 2$. Make a table of corresponding values of x and y and plot the points whose co-ordinates are given by it and join them freehand :—

Let $2x^2 - 3x - 2 = y$

If $x=$	0	1	2	3	4	−1	−2	−3
$2x^2=$	0	2	8	18	32	2	8	18
$-3x=$	0	−3	−6	−9	−12	+3	+6	+9
$-2=$	−2	−2	−2	−2	−2	−2	−2	−2
$2x^2-3x-2=y=$	−2	−3	−0	7	18	3	12	25

Take 0·4" as unit for x and 0·1" as unit for y.

The following points are noticeable with regard to the graph shown above.

(i) The graph is a parabola and does not pass through the origin (the function contains a term independent of x).

(ii) As in the case of the graph of x^2+2x some portion of the graph is above and the other below the x axis, some of the ordinates being positive and others negative. This means that the function is positive for certain values of x and negative for other values, as ordinates represent different values of the function.

That portion of the graph, which lies between the points of intersection of the graph with the x axis, is negative and the rest of it is positive. Or the function is negative for all those values of x which lie between 2 and $-\frac{1}{2}$ and positive for other values, *i. e.*, for values of x which lie outside these values, and vanishes when $x=2$, $x=-\frac{1}{2}$.

(iii) The graph does not descend below a certain point and begins to ascend again from that point. This means that there is an algebraically smallest value of the ordinate for a certain value of the abscissa or that the function cannot take any value less than a particular value whatever may be the value of x.

In the case of the present graph, the graph turns at the point, $\frac{3}{4}$, −3·1 and does not go below it. The minimum value of the graph is −3·1 (nearly when $x=\frac{3}{4}$ or ·75.

To determine the turning point graphically the portion of the graph in the neighbourhood of it must be carefully and accurately plotted by taking values of x very close to each other, intermediate fractional values also being taken into consideration.

The ordinate becomes zero at two points on the graph, that is, at those points where the graph cuts the x axis. This is the same thing as saying that the function (as y represents it) becomes zero twice for two values of x which are given by the abscissæ of the points of intersection of the graph with x axis. Since $2x^2-3x-2=0$ becomes zero for two values of x as noticed from the figure, viz., $x=2$ and $x=-\frac{1}{2}$, 2 and $-\frac{1}{2}$ are the roots of the equation $x^2-3x-2=0$.

Generally, therefore, to find the roots of the equation $ax^2+bx+c=0$, draw the graph of the function ax^2+bx+c and observe at what points y becomes zero, in other words where the graph cuts the x axis ($y=0$ represents the x axis.) This graph enables us to find the value of the function corresponding to any values of x and the values of x corresponding to any value of y. To solve any equation of the form $f(x)=0$, $f(x)$ standing for any function of x of any degree, draw the graph of $f(x)$ and read the values of x or the abscissæ at the points of intersection of the graph with the x axis. If $y=f(x)$, this is the same thing as solving the two equations:—$y=f(x)$.

$$y=0.$$

which is done by drawing the graphs of the two equations and finding the values of the abscissæ of the points of intersection.

Thus to solve the quadratic equation

$ax^2+bx+c=0$, draw the graphs

of $y=ax^2+bx+c$ and

$y=0$ (the x axis itself)

and find the values of the abscissæ of the points of intersection of the two, *i. e.*, the intersection of the graph with the x axis.

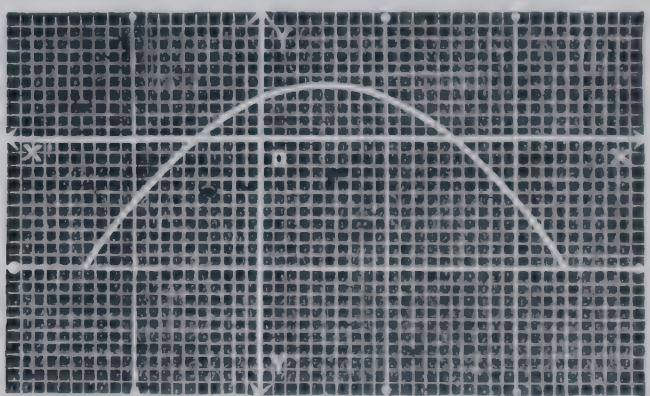
Ex. 4. To draw the graph of $3+2x-x^2$.

Let $y=3+2x-x^2$

Prepare a table of corresponding values of x and y.

$y=3+2x-x^2$

If $x=$	0	1	2	3	4	−1	−2	−3
$-x^2 =$	0	−1	−4	−9	−16	−1	−4	−9
$+2x =$	0	+2	4	+6	+8	−2	−4	−6
$+3=$	3	3	3	3	3	3	3	3
$-x^2+2x+3=y=$	3	4	3	0	−5	0	−5	−12



Unit for $x=0{\cdot}5''$ and for $y=0{\cdot}1''$

In this case the graph cuts the x axis at two points making the ordinate or the function zero, and the abscissæ of these points are 3 and −1

∴ The roots of the equation $3+2x-x^2=0$ are 3 and −1

The ordinates that lie between these points are all positive. Therefore the function is *positive* for values of x which lie between 3 and 1 and is *negative* for those outside 3 and −1.

It is to be noticed that the graph does not rise beyond a certain point and that the ordinate is *algebraically* the greatest there. Therefore the function attains its greatest or *maximum* value when x has the value that is given by the abscissa of this point. Thus the function $3+2x-x^2$ is *maximum* when $x=1$, the *maximum value* being 4.

It has been observed that in the case of the function $2x^2-3x-2$, we were able to find the *minimum* value of the function, while in the case of the function $3+2x-x^2$, the maximum value is obtained. Thus generally when the coefficient of x^2 is positive, the minimum and when it is negative, the maximum value of the function can be determined. The function ax^2+bx+c is negative or positive for those values of x which lie between the roots of the equation $ax^2+bx+c=0$, according as a, the co-efficient of x^2, is positive or negative and *vice versa*.

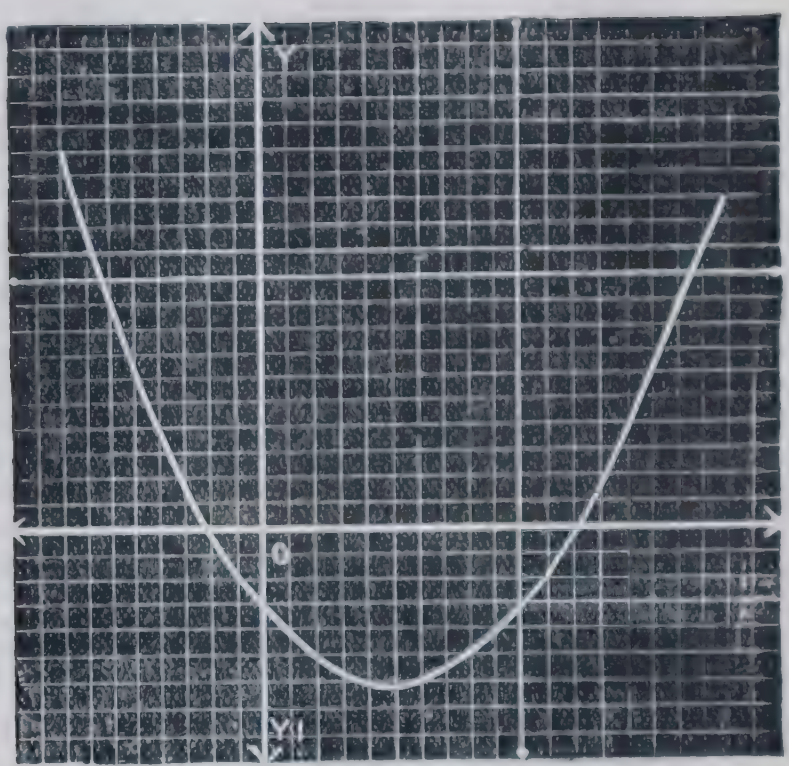
204. To find the roots of the equation $2x^2-5x-3=0$ *graphically* :

First method :—Draw the graph of the function $2x^2-5x-3$ as in the previous examples and read off the values of the abscissæ at the points of intersection of the graph with the x axis

$$y=2x^2-5x-3.$$

When $x=$	0	1	2	3	4	−1	−2	−3	. .
$2x^2=$	0	2	8	18	32	2	8	18	. .
$-5x-3=$	−3	−8	−13	−18	−23	2	7	12	. .
$y=2x^2-5x-3=$	−3	−6	−5	0	9	4	15	30	. .

Unit for $x=0{\cdot}4''$, and for $y=0{\cdot}1''$



Second Method (Double graph method).

$$2x^2 - 5x - 3 = 0$$

$$\therefore\ 2x^2 = 5x + 3$$

Let each side $= y$; then we have the following equations:—

$$y = 2x^2 \quad . \ . \ . \ . \ \text{(i)}$$

$$y = 5x + 3 \quad . \ . \ . \ \text{(ii)}$$

Solving the equation $2x^2 - 5x - 3 = 0$ is the same thing as solving the two equations

$$y = 2x^2 \text{ and } y = 5x + 3.$$

Draw the graphs of the above and find the values of abscissæ at the points of intersection of the two graphs. Equation (i) represents a parabola and (ii) a straight line.

$$y = 2x^2$$

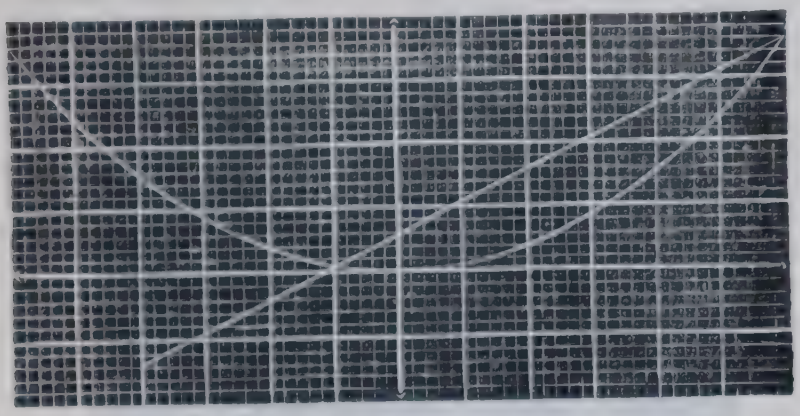
When $x=$	0	1	2	3	-1	-2	-3
$y=$	0	2	8	18	2	8	18

$y=5x+3$

When $x=$	0	1	2	3	-1	-2	-3
$y=5x+3=$	3	8	13	18	-2	-7	-12

Let the unit for $x=1{\cdot}0''$

$y=0{\cdot}1''$ in both the cases



Therefore the roots of the equation

$2x^2-5x-3=0$ are $3, -\frac{1}{2}$.

The equation $2x^2-5x-3=0$ may also be solved by drawing graphs of different sets of equations, *e. g.* $y=2x^2-5x$ and $y=3$.

or $y=2x^2-3x$ and $y=2x+3$ and so on.

To draw the graph of the function $6-7x-5x^2$ and use it to find the roots of the equation :

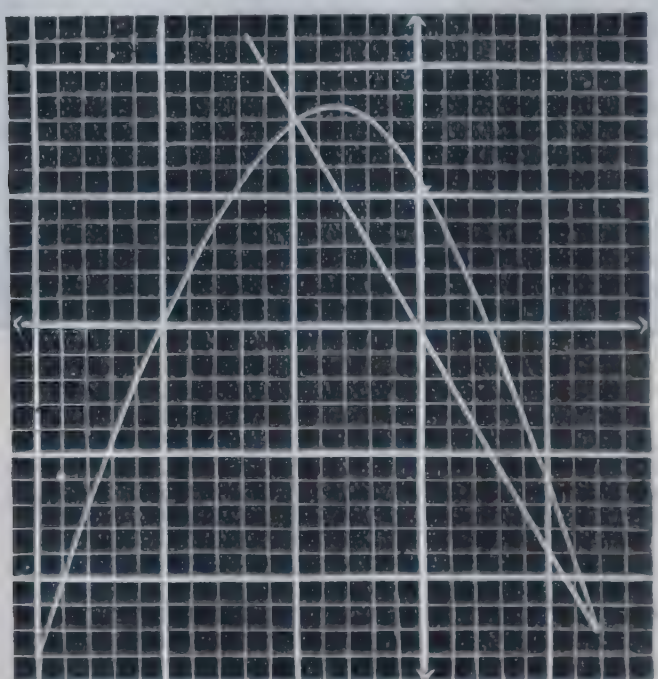
$5x^2-2x-7=0$

Let $y=6-7x-5x^2$.

$y=6-7x-5x^2$

When $x=$	...	0	1	2	3	−1	−2	−3
$6=$	...	6	6	6	6	6	6	6
$-7x=$	...	0	−7	−14	−21	7	14	+21
$-5x^2=$	...	0	−5	−20	−45	−5	−20	−45
$y=6-7x=5x^2=$	...	6	−6	−28	−60	8	0	−18

Unit for $x=0{\cdot}5''$ and for $y=0{\cdot}1''$.



The graph cuts the x axis at two points whose abscissæ are $+\frac{3}{5}$ and -2. Therefore the roots of the equation $6-7x-5x^2=0$ are $\frac{3}{5}$ and -2.

Since this equation is the same as the equation $5x^2+7x-6=0$ the roots of the latter equation are also $+\frac{3}{5}$ and -2 and can be obtained by the graph of the function $6-7x-5x^2$.

Though the graph shown in the figure does not represent the graph of the function $5x^2+7x-6$ yet the above graph can be used to find the roots of the equation $5x^2+7x-6=0$ as the graph of $5x^2+7x-6$ is the same as that of the function $6-7x-5x^2$ only inverted. The two graphs cut the x axis at the same points, therefore, the points of intersection of the graph of one function with the x-axis will give not only the roots of the equation formed by putting it equal to zero, but also of the function with its sign changed equal to zero.

To use this graph to find the roots of the equation $5x^2-2x-7=0$ This is the same as the equation $-5x^2+2x+7=0$ or $7+2x-5x^2=0$.

The left-hand side expression can be written so as to contain the function whose graph should be used to solve the equation :

thus $6-7x-5x^2+9x+1=0$

$\therefore\ 6-7x-5x^2=-9x-1$

Let each side $=y$, then the above equation is the same as the two simultaneous equations.

$$y=6-7x-5x^2 \quad . \ . \ . \quad \text{(i)}$$

$$y=-9x-1 \quad . \ . \ . \ . \quad \text{(ii)}$$

To solve these two equations, the two graphs should be drawn and abscissæ at the points of intersection of the two graphs will give the values of x which satisfy the equations (i) and (ii) or the given equation. But the graph of the first equation is ready for our use and so the graph of the second equation $y=-9x-1$ should be drawn in the same diagram with the same units.

$$y=-9x-1$$

If $x=$ ...	0	1	2	-1	-2	. . .
$y=$ ...	-1	-10	-19	8	17	. . .

The graph of this function which is a straight line is drawn in the same figure, and the abscissæ at the points of intersection of the parabola and the straight line are found to be -1 and $+\frac{7}{5}$.

Therefore the roots of the equation $5x^2-2x-7=0$ are -1 and $+\frac{7}{5}$.

It has been observed that in the case of the graph of ax^2+bx+c the minimum or the maximum value of the ordinate or the function can be obtained according as a (the co-efficient of x^2) is positive or negative. At this value of the ordinate the parabola turns and begins to go in a direction different from that in which it has been going.

In the case of the present graph the maximum value of the function (*i. e.*, the ordinate) is 8·5 nearly, and the corresponding value of x is—·7 nearly. Great care must be taken in plotting the portion of the graph that lies in the immediate neighbourhood of the turning point when the student has a rough idea of its position.

205. The maximum or minimum value of a function can be found **algebraically** also in a very simple manner.

Ex. 1. To find **algebraically** the **minimum** value of the function $2x^2-3x-2$.

$$2x^2-3x-2=2\left(x^2-\frac{3x}{2}-1\right).$$

$$=\left\{x^2-\frac{3x}{2}+\frac{9}{16}-\frac{9}{16}-1\right\}.$$

$$=\left\{(x-\tfrac{3}{4})^2-\frac{25}{16}\right\},$$

$$=2\,(x-\tfrac{3}{4})^2-\frac{25}{8}.$$

Thus by the above process the function is so written that it consists of a perfect square and a constant term by completing the square in x. As said above, the given function consists of an expression in x which is the variable part and a constant term.

Since x is a variable quantity $(x-\frac{3}{4})^2$ is the variable part of the function.

∴ The value of the function depends on the value of $(x-\frac{3}{4})^2$.

The function decreases as $(x-\frac{3}{4})^2$ decreases and it is least when $(x-\frac{3}{4})^2$ is least.

But $(x-\frac{3}{4})^2$ is a perfect square and is, therefore, always positive for all real values of x; and the least value which $(x-\frac{3}{4})^2$ can take is therefore zero,

$\therefore$ The function is least when $(x-\frac{3}{4})^2=0$

or when $x-\frac{3}{4}=0$

$\therefore$,, $x=\frac{3}{4}$

In that case the variable part vanishes and the function is equal to $-3\frac{1}{8}$.

$\therefore$ the minimum value of the function $=-\frac{25}{8}$ or $-3\frac{1}{8}$.

Then the value of $x=\frac{3}{4}$.

Therefore the turning point of the graph of the function is $(\frac{3}{4}, -3\frac{1}{8})$.

That is what was observed **graphically**.

Ex 2. To find the **maximum** value of the function $6-7x-5x^2$ **algebraically**.

As before put $y=6-7x-5x^2$

$$y=-(5x^2+7x-6)$$

$$=-5\left(x^2+\frac{7}{5}x-\frac{6}{5}\right)$$

$$=-5\left(x^2+\frac{7}{5}x+\frac{49}{100}-\frac{49}{100}-6\right)$$

$$=-5\left\{\left(x+\frac{7}{10}\right)^2-\frac{169}{100}\right\}$$

$$=-5\left(x+\frac{7}{10}\right)^2+\frac{169}{20}.$$

The function consists of two parts, one variable and the other constant. So the value of the function depends on the value of the variable part.

$-5\left(x+\frac{7}{10}\right)^2$ is the variable part.

$\left(x+\frac{7}{10}\right)^2$ is a perfect square, it is positive for all real values of x.

$\therefore -\left(x+\frac{7}{10}\right)^2$ is essentially negative for real values of x

The maximum value of a negative quantity is zero.

∴ The function is maximum for the maximum value of $-5\left(x+\frac{7}{10}\right)^2$ *i.e.*, when $-5\left(x+\frac{7}{10}\right)^2=0$

i.e., when $x+\frac{7}{10}=0$.

i.e., when $x=-\frac{7}{10}$ or $-0{\cdot}7$.

The value of the function when $x=-\frac{7}{10}$ is equal to $\frac{169}{20}$

∴ The maximum value of the function $=\frac{169}{20}$ or $8{\cdot}5$ nearly when $x=-0{\cdot}7$.

Therefore the turning point of the graph of the function $6-7x-5x^2$ is $(-0{\cdot}7,\ 8{\cdot}5)$ as was seen from the graph of the function.

For the sake of clearness the procedure to be adopted in finding out the minimum or maximum value of a quadratic function of the form ax^2+bx+c may be summed up as follows :—

(i) Divide the function by the co-efficient of x and multiply it by the same, *e g*, write it as

$$a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right).$$

(ii) Complete the square with the terms in x by adding and subtracting the square of half the co-efficient of x as

$$a\left\{x^2+\frac{bx}{a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right\}.$$

(iii) Transform the function so that it may contain two terms, one of which being a square and the other a constant term as

$$a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}$$

$$\text{or } a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}.$$

When $x=-\frac{b}{2a}$, the minimum or the maximum value of the function is obtained as $-\frac{b^2-4ac}{4a}$ according as a is positive or negative.

The principles involved in finding the minimum and maximum value of a function are chiefly (1) that the square of a real quantity is essentially positive and (2) that the least value of a perfect square is therefore zero.

206. To find algebraically the limits between which x must lie in order that the expression $2x^2-3x-2$ may be (1) positive, (2) negative.

Writing the expression as shown above by completing the square in terms of x and factoring it further we have

$$2x^2-3x-2=2\left\{\left(x-\frac{3}{4}\right)^2-\frac{25}{16}\right\}.$$

$$=2\left(x-\frac{3}{4}+\frac{5}{4}\right)\left(x-\frac{3}{4}-\frac{5}{4}\right).$$

$$=2\ (x+\tfrac{1}{2})\ (x-2).$$

Evidently the function vanishes when $x=-\frac{1}{2}$ or $x=2$.

$\therefore -\frac{1}{2}$ and 2 are the roots of the equation $2x^2-3x-2=0$.

Writing the factors as $x-$ one root and $x-$ the other root, we have the function $2x^2-3x-2=2\ (x--\frac{1}{2})\ (x-2)$.

The function is positive when

(i) both factors are positive, or

(ii) both factors are negative

That is when both of them are of the same sign.

(i) Both of them will be positive when x is $>-\frac{1}{2}$ and also >2.

But when x is >2, it is also $>-\frac{1}{2}$. So we can say that both the factors will be positive when $x>2$. Then the product and consequently the function is positive.

(ii) Similarly both factors will be negative when x is $<-\frac{1}{2}$ and <2,

But when x is $<-\frac{1}{2}$ it will be also <2. So we can say that the two factors will be negative when x is $<-\frac{1}{2}$.

Hence their product and therefore the function is positive when x is $<-\frac{1}{2}$.

Combining these two we have that the function is positive when x is $<-\frac{1}{2}$ and >2, *i. e.*, for all values of x which lie below $-\frac{1}{2}$ and above 2.

Again the function is negative when one factor is positive and the other is negative. This happens when x is $>-\frac{1}{2}$ and <2.

Therefore the function is negative for all values of x which lie between $-\frac{1}{2}$ and 2.

It has been seen that the function vanishes for the values $-\frac{1}{2}$ and 2 of x since the sign of the function changes from the positive to the negative and from the negative to the positive again.

Summing up the results we have that the function $2x^2-3x-2$ is *positive* for values of x which lie outside $-\frac{1}{2}$ and 2, *negative* for those which lie between $-\frac{1}{2}$ and 2, and *zero* when $x=-\frac{1}{2}$, and $x=2$.

207. To find algebraically the limits between which x must lie in order that the function $6-7x-5x^2$ may be (1) positive, (2) negative.

Writing the expression as one containing a perfect square and an independent term as shown above and factorising further we have

$$\begin{aligned}
6-7x-5x^2 &= -5\left(x^2+\frac{7}{5}x-\frac{6}{5}\right).\\
&= -5\left\{\left(x+\frac{7}{10}\right)^2-\frac{169}{100}\right\}.\\
&= -5\left\{\left(x+\frac{7}{10}+\frac{13}{10}\right)\left(x+\frac{7}{10}-\frac{13}{10}\right)\right\}.\\
&= -5\,(x+2)\left(x-\frac{3}{5}\right).\\
&= -5\,(x-(-2))\left(x-\frac{3}{5}\right).
\end{aligned}$$

Evidently -2 and $\frac{3}{5}$ are the roots of the equation

$$6-7x-5x^2=0.$$

Following the same argument as in the case of the previous function we see that the factors will be of like sign when x is $>\frac{3}{5}$ and <-2.

$\therefore$ The product of the two factors will be positive when x is $>\frac{3}{5}$ and <-2.

Since there is a negative sign before the product the function is negative when x is $>\frac{3}{5}$ and <-2.

Again, both factors will be of opposite sign and hence the product will be negative for values of x which lie between -2 and $\frac{3}{5}$.

As there is a negative sign before the product the function is therefore positive for these values of x.

It has already been observed that the function vanishes when $x=-2$ and $x=\frac{3}{5}$.

Summing up therefore we have that the function is positive for values of x which lie between -2 and $\frac{3}{5}$ (the roots of the equation $6-7x-5x^2=0$) and negative for those which lie outside -2 and $\frac{3}{5}$.

Generally, therefore, taking the function ax^2+bx+c which can be written as

$$a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

we can see that the function vanishes when

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

which are the roots of the equation $ax^2+bx+c=0$.

It is positive or negative according as a is positive or negative for values of x which lie outside the roots of the equation $ax^2+bx+c=0$.

And it is negative or positive according as a is positive or negative for values of x which lie between the roots of the equation $ax^2+bx+c=0$.

208. Circle. It has been observed in Chap. IX, that the distance of any point P whose co-ordinates are x, y from O, the origin, is given by $OP^2=x^2+y^2$ *i. e.*, $OP=\sqrt{x^2+y^2}$.

If this distance is constant, the point moves on a circle whose centre is O, (the origin,) and radius is $\sqrt{x^2+y^2}$, therefore if $x^2+y^2=a^2$, the graph represents a circle whose centre is the origin and radius is a.

To draw the **graph** of the equation $x^2+y^2=25$.

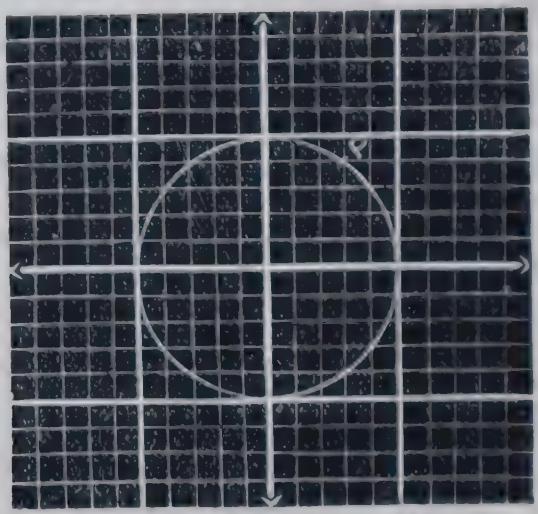
Describe a circle with the origin as centre and radius equal to 5 units.

Take any point P on the circumference. Its co-ordinates are 3 4

$\therefore O P^2=9+16=25$.

$\therefore$ The equation is satisfied by the co-ordinates of any point

The graph of the equation $x^2+y^2=25$.



The equation of a circle whose radius is c and the centre the origin is $x^2+y^2=c^2$.

Solve graphically the equation,

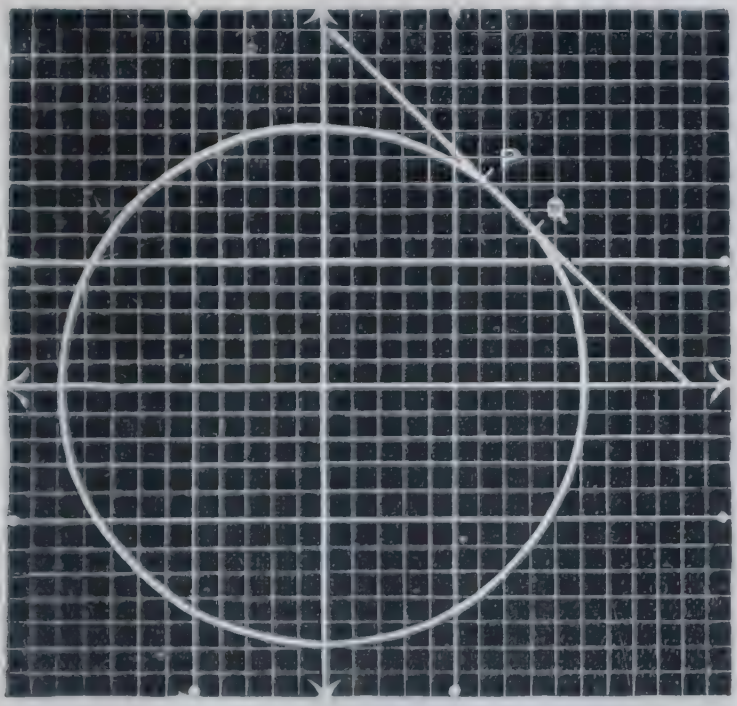
$$x^2+y^2=100 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$x+y=14 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

The graph of equation (i) is a circle whose centre is the origin and radius $\sqrt{100}$ or 10 units; and the graph of equation (ii) is a straight line. The values of x, y are the values of the co-ordinates of the points of intersection of the two graphs.

The graph of $x+y=14$ may be obtained by finding the intercepts on the x axis and y axis by putting $x=0$ and $y=0$ respectively in the equation; the intercepts are 14 and 14. Mark these points and join them by means of a straight line.

It can also be drawn by making a table of corresponding values of x and y and plotting the points thus obtained.



The points of intersection are P and Q. P is found to be (6, 8) and Q (8, 6)

$\therefore\ x=6$ or 8.

$y=8$ or 6.

ZERO AND INFINITE VALUES.

209. Two interpretations of zero. (1) It has been seen already that zero or 0 is such that $+x-x=0$. That is, if from a bag containing Rs. 15, all the Rs. 15 are removed, nothing will be left in the bag; we say that the bag contains zero rupees or absolutely nothing.

(2) Again if x is a variable quantity and takes the values 1, 10, 100, 1000, 10,000,.....then $\frac{1}{x}$ – 1, ·1, ·001, ·0001, ·00001,.....

That is as x increases, $\frac{1}{x}$ decreases and when x becomes very large without limit or infinitely large, *i.e.*, a quantity larger than any

that we can name, $\frac{1}{x}$ becomes very small or infinitely small. It becomes so small that its value becomes a negligible quantity; and this value of $\frac{1}{x}$ is also treated as zero, it is not actually zero as in the previous case but is so small that it can be altogether neglected.

The symbol to denote an infinitely large quantity or *infinity* is ∞.

Therefore $\frac{1}{x} = \frac{1}{\infty}$ when x is infinitely large and is equal to zero as seen above. Conversely $\frac{1}{x}$ when x is infinitely small can be represented as $\frac{1}{0}$ and is equal to ∞.

Hitherto we have regarded $+0$ and -0 as one and the same, but according to the same interpretation

$\frac{a}{+0} = +\infty$ and $\frac{a}{-0} = -\infty$ and conversely $\frac{a}{+\infty} = +0$ and $\frac{a}{-\infty} = -0$, $+a$ and $-a$ denoting infinitely large positive and negative quantities.

210. To draw the graph of $xy = 1$,

Put the equation in the form $y = \frac{1}{x}$

Make a table of corresponding values of x and y as shown below:—

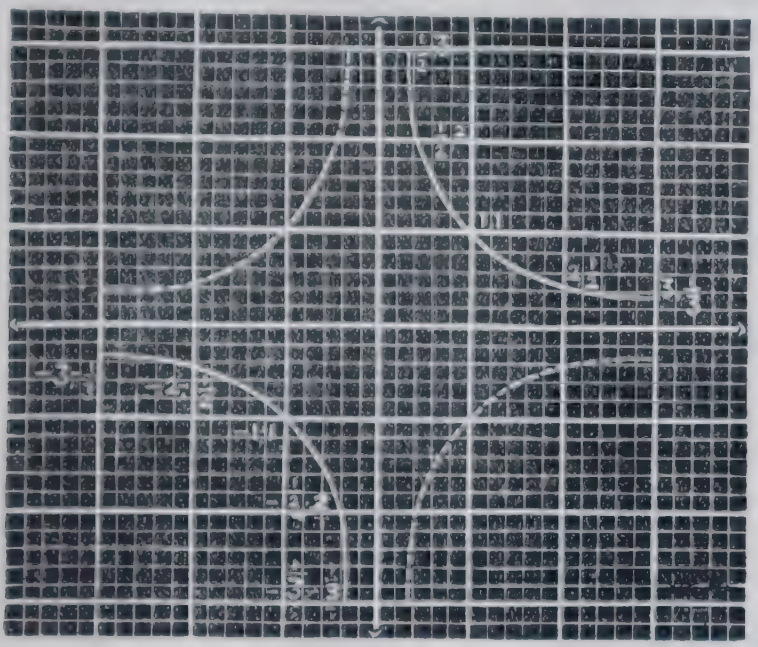
(i) Positive values of x.

When $x =$	0	1	2	3	4	...	a	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$y =$	$-a$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	...	0	2	3	4

(ii) Negative values of x.

When $x =$	-0	-1	-2	-3	-4	...	$-a$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$
$y =$	$-$	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	...	-0	-2	-3	-4

Plotting the points given by tables (i) and (ii) and joining them, we notice that we get two distinct branches of the graph, the first lying in the first quadrant and the second in the third quadrant, the two branches constituting the entire graph representing $xy = 1$. They are not two separate graphs but only two branches of one and the same graph. Such a graph is called *hyperbola*. It can also be noticed from the figure shown below that neither of the branches cuts the x axis or y axis and as we go on they approach them nearer and nearer. These axes are known as the *asymptotes* of the curve *hyperbola*. Unit 0·5'' in both axes.



The graph of $xy = -1$ also consists of two branches one lying in the second quadrant and the other in the fourth.

To solve **graphically** the equations

$$x + y = 3 \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$xy = 2 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The graph of equation (i) is a straight line and that of equation (ii) is a curve similar to the one treated above.

EXERCISE 105.

1. Draw the graphs of the following functions and in each case state the turning point :—

(Unit for $x = 0{\cdot}5''$ and $y = 0{\cdot}1''$.)

(i) $2x^2 + 3x$. (iv) $x^2 + 2x + 1$. (vii) $3 - 2x - x^2$.

(ii) $x^2 + 4x$. (v) $2x^2 - 3x + 1$. (viii) $x(4x + 3)$.

(iii) $x^2 - 2x$. (vi) $2 - x - 3x^2$. (ix) $x(1 - 3x)$.

2. Draw the graphs of the following functions and find graphically the minimum value in each case:—

(i) $x^2 - 4x + 9$. (ii) $x^2 - 4x - 1$. (iii) $4x^2 - 20x + 9$.

(iv) $4x^2 - 12x - 17$. (v) $2x^2 - 8x - 3$. (vi) $2x^2 - 4x - 3$.

(vii) $x^2 - 6x + 9$.

3. Draw the graphs of the following functions and find therefrom the maximum value in each case :—

(i) $1 + 2x - x^2$. (ii) $-(x^2 + 4x + 8)$.

(iii) $1 - 10x - 5x^2$. (iv) $x(2 - x)$.

4. Draw the graphs of :—

(i) $x^2 - 4x + 4$. (ii) $-(x^2 + 2x + 1)$.

Show that (i) is always positive and (ii) is always negative.

5. Draw the graph of $3x - 2 - x^2$ and determine the value of x when the function is maximum.

6. Solve graphically the equations :

(i) $x^2 - x - 2 = 0$. (ii) $x^2 - x - 6 = 0$.

(iii) $2x^2 - 5x - 3 = 0$. (iv) $x(7 - 2x) = 0$.

(v) $3 - 2x - x^2 = 0$. (vi) $2 - 5x - 3x^2 = 0$.

7. Find the graph of $y = (x + 1)(x + 3)$ and find the minimum value of $(x + 1)(x + 3)$; also find the values of x for which $y = 0$.

8. Show graphically that the function $x^2 - 4x + 6$ is positive for all values of x.

9. Draw the graphs of $y = x^2$ and $y = x + 2$; hence find the roots of the equation $x^2 = x + 2$.

10. Draw the graphs of $y = x^2$ and $y = 4x - 3$; hence find the roots of the equation $x^2 - 4x + 3 = 0$.

11. Draw the graph of $y = x^2$ and use it to solve the following equations.

(i) $x^2 - 3x + 2 = 0$. (ii) $x^2 + x - 2 = 0$.

(iii) $2x^2 - 5x + 2 = 0$. (iv) $2x^2 + x - 1 = 0$.

12. Draw the graph of $2x^2+x$ and use it to find the roots of the equation $2x^2+x-6=0$.

13. Draw the graph of $2x^2-x$ and use it to solve the equation $2x^2+x-6=0$.

14. Draw the graph of x^2+2x-3 and use it to solve the equation $x^2+2x+1=0$.

15. Draw the graph of $3+5x-2x^2$ and find for what value of x the function is (1) maximum, (2) zero. Use the graph for finding the roots of the equation $2x^2-5x-3=0$. Show graphically for what values of x the function $3+5x-2x^2$ is, (1) positive and (2) negative.

Find algebraically the maximum value of $3+5x-2x^2$.

16. Draw the graph of $2x^2+x-1$ and use this to solve the equation $2x^2-3x-2=0$.

17. Draw on the same diagram and to the same scale the graphs of $y=x^2$ and $2y-3x-2=0$ and hence read off the roots of the equation $2x^2-3x-2=0$.

18. Draw the graph of $y=(2-x)(3+x)$ and find the maximum value of $(2-x)(3+x)$; and hence use this to solve the equation $6-x-x^2=4$.

19. Show graphically that the function $2x^2-3x-9$ is positive for all real values of x which lie outside $x=-\frac{3}{2}$ and $x=3$. Find the minimum value of the function. Establish this algebraically.

20. Find graphically between what limits x should lie in order that $2-x-x^2$ may be positive. Find also the maximum value. Verify algebraically.

21. Draw the graph of $2x^2-x+1$ and deduce from it the nature of the roots of the equation $2x^2+1=x$. Verify algebraically.

22. Trace the graph of $4x^2-12x+9$ and deduce from it the nature of the roots of the equation $4x^2-12x+9=0$. Verify algebraically.

23. Show graphically that the equation $x^2+x+3=0$ cannot have real roots.

24. Show graphically that the function x^2-2x+3 is positive for all real values of x and find its minimum value. Verify algebraically.

25. Trace the graph of $-1\cdot25+x-x^2$ and show by the aid of this graph that the function is negative for all real values of x. Find its maximum value. Verify algebraically.

Solve the following pairs of equations graphically and verify in each case by substitution:—

26. $y = x^2 - 4$
$y = 2x + 1$.

27. $y = x^2 - x - 2$.
$y + 3x = 1$.

28. $x^2 + y^2 = 25$
$x - y = 1$.

29. $x^2 + y^2 = 100$
$x + y = 14$.

30. $x + y = 4$
$xy = 3$.

31. $x - y = 3$
$xy = -2$.

32. $x^2 + y^2 = 121$
$xy = 4$.

33. Construct a graph from which we can read off the time required by any given number of men to do a piece of work which 10 men can do in 10 days; from it read off the number of days in which 5 men can do it.

CHAPTER XXXIII.

ELIMINATION.

211. Take the two equations $ax+b=0$; and $a_1x+b_1=0$.

If these two equations are satisfied by the *same* value of x, then $-\frac{b}{a}=-\frac{b_1}{a_1}$; *i.e.*, $a_1b=b_1a$.

This relation is called the **Eliminant** of the two equations.

Similarly, if we have *three* equations involving *two* unknown quantities and if they are satisfied by the *same* values of the unknown quantities, we can find a relation between the other symbols involved, and the process of finding this relation is called the *Elimination* of the unknown quantities from the given equations.

Ex 1. Eliminate x from the equations

$ax^2+bx+c=0$, $a_1x^2+b_1x+c_1=0$.

If α be the value of x which satisfies both the equations, then we must have—

$$\left.\begin{aligned} a\alpha^2+b\alpha+c&=0\\ a_1\alpha^2+b_1\alpha+c_1&=0\end{aligned}\right\}$$

Hence, by the method of cross-multiplication,

$$\frac{\alpha^2}{bc_1-b_1c}=\frac{\alpha}{a_1c-ac_1}=\frac{1}{ab_1-a_1b}.$$

$$\therefore \frac{bc_1-b_1c}{ab_1-a_1b}=\left(\frac{a_1c-ac_1}{ab_1-a_1b}\right)^2.$$

$$\therefore (a_1c-ac_1)^2=(ab_1-a_1b)(bc_1-b_1c),$$

which is the required *Eliminant*.

Ex. 2. Eliminate x and y from

$ax+by+c=0$, $a_1x+b_1y+c_1=0$ and $x^2+y^2=1$.

From the first two, by cross-multiplication,

$$\frac{x}{bc_1-b_1c}=\frac{y}{ca_1-c_1a}=\frac{1}{ab_1-a_1b}$$

$$\therefore\ x=\frac{bc_1-b_1c}{ab_1-a_1b},\quad y=\frac{ca_1-c_1a}{ab_1-a_1b}.$$

Substituting these values in the third, we have—

$$\left(\frac{bc_1-b_1c}{ab_1-a_1b}\right)^2+\left(\frac{ca_1-c_1a}{ab_1-a_1b}\right)^2=1$$

or $(bc_1-b_1c)^2+(ca_1+c_1a)^2=(ab_1-a_1b)^2$,

which is the required *Eliminant*.

Ex. 3. Eliminate x, y, z from the equations

$x^2+y^2=cxy$, $y^2+z^2=ayz$, $z^2+x^2=bzx$.

We have $\frac{x}{y}+\frac{y}{x}=c$, $\frac{y}{z}+\frac{z}{y}=a$, $\frac{z}{x}+\frac{x}{z}=b$,

By multiplying together these equations, we get—

$$abc=\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+2.$$

$$\therefore abc=\left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{y}{z}+\frac{z}{y}\right)^2+\left(\frac{z}{x}+\frac{x}{z}\right)^2-4.$$

$\therefore abc=c^2+a^2+b^2-4$, which is the required *Eliminant*

Ex. 4. Eliminate x and y from

$x^3-y^3=ax-by$, $4xy=bx+ay$ and $x^2+y^2=1$.

From (1), $x^3-xy^2=ax^2-bxy$ }
From (2), $4xy^2 \quad =bxy+ay^2$ }.

$$x^3+3xy^2=a(x^2+y^2)=a. \quad \because \ x^2+y^2=1.$$

Similarly $b=y^3+3x^2y$.

$$\therefore a+b=(x+y)^3 \text{ and } a-b=(x-y)^3.$$

$\therefore (a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=(x+y)^2+(x-y)^2=2(x^2+y^2)$.

$\therefore (a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=2$, which is the required *Eliminant*.

Ex 5. Eliminate x, y, z from $\frac{y+z}{y-z}=a$, $\frac{z+x}{z-x}=b$, $\frac{x+y}{x-y}=c$

From (1), $\frac{y+z+y-z}{y+z-y+z}=\frac{a+1}{a-1}$.

$$\therefore \frac{y}{z}=\frac{a+1}{a-1}.$$

Similarly from (2), $\frac{z}{x}=\frac{b+1}{b-1}$

and from (3), $\frac{x}{y}=\frac{c+1}{c-1}$.

$$\therefore \frac{y}{z}\times\frac{z}{x}\times\frac{x}{y}=\frac{a+1}{a-1}\times\frac{b+1}{b-1}\times\frac{c+1}{c-1}.$$

$\therefore (a+1)(b+1)(c+1)=(a-1)(b-1)(c-1)$, which is the required *Eliminant*.

Ex. 6. Eliminate x, y, z, from

$x^2(y+z)=a^2$, $y^2(z+x)=b^2$, $z^2(x+y)=c^2$, $xyz=abc$.

Multiplying the first three equations, we have—

$x^2y^2z^2(x+y)(y+z)(z+x)=a^2b^2c^2$; but $x^2y^2z^2=a^2b^2c^2$.

$$\therefore (x+y)(y+z)(z+x)=1 \quad \text{.....(A)}$$

Again, adding the first three equations, we have—

$x^2(y+z)+y^2(z+x)+z^2(x+y)=a^2+b^2+c^2$ and $2xyz=2abc$.

$\therefore x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz=a^2+b^2+c^2+2abc$.

$\therefore (x+y)(y+z)(z+x)=a^2+b^2+c^2+2abc$ .....(B).

$\therefore$ from (A) and (B), $a^2+b^2+c^2+2abc=1$.

Ex. 7. Eliminate x, y, z from the equations

$$\frac{y}{z}-\frac{z}{y}=a,\ \frac{z}{x}-\frac{x}{z}=b,\ \frac{x}{y}-\frac{y}{x}=c.$$

Adding, $a+b+c=\dfrac{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}{xyz}$

$$=\frac{(x-y)(y-z)(z-x)}{xyz}.$$

Similarly $a-b-c=\dfrac{x(y^2-z^2)-y(z^2-x^2)-z(x^2-y^2)}{xyz}$

$$=\frac{(x+y)(z+x)(y-z)}{xyz}.$$

$$b-a-c=\frac{(x+y)(y+z)(z-x)}{xyz}.$$

$$c-a-b=\frac{(y+z)(z+x)(x-y)}{xyz}.$$

$\therefore (a+b+c)(b+c-a)(c+a-b)(a+b-c)$

$$=-\frac{(y^2-z^2)^2(z^2-x^2)^2(x^2-y^2)^2}{x^4y^4z^4}.$$

$$=-\left(\frac{y}{z}-\frac{z}{y}\right)^2\left(\frac{z}{x}-\frac{x}{z}\right)^2\left(\frac{x}{y}-\frac{y}{x}\right)^2$$

$$=-a^2b^2c^2.$$

$\therefore (a+b+c)(b+c-a)(c+a-b)(a+b-c)+a^2b^2c^2=0$ which is the required *Eliminant*.

EXERCISE 106.

Eliminate x from the equations :—

1. $\left.\begin{aligned} ax^2-bx+c&=0\\ bx+a&=0\end{aligned}\right\}$.

2. $\left.\begin{aligned} px^2-q&=0\\ rx^3-s&=0\end{aligned}\right\}$.

3. $\left.\begin{aligned} ax^2+bx+c&=0\\ px^3+qx+r&=0\end{aligned}\right\}$.

4. $\left.\begin{aligned} x^3+\frac{3}{x}&=4(p^3+q^3)\\ 3x+\frac{1}{x^3}&=4(p^3+q^3)\end{aligned}\right\}$.

5. $\left.\begin{aligned} ax^3-b&=0\\ px^3-q&=0\end{aligned}\right\}$.

6. $\left.\begin{aligned} lx^2+mx+n&=0\\ x+a&=0\end{aligned}\right\}$.

7. $x+\frac{1}{x}=p$, $x-\frac{1}{x}=q$.

8. $px+\frac{q}{x}=a$, $px+\frac{p}{x}=b$.

9. $ax^3+bx+c=0$, $a_1x^3+b_1x+c_1=0$.

10. $px^3+qx^2+r=0$, $lx^3+mx^2+n=0$.

11. $ax+A=bx+B=cx+C$.

12. $\frac{a}{x^3+a^2}=\frac{2b}{x^2+b^2}=\frac{4c}{x^2+c^2}$

Eliminate x and y from the equations :—

13. $x+y=a$, $x^3+y^3=b^3$, $x^5+y^5=c^5$.

14. $\frac{x^3-a^3}{y^3-b^3}=\frac{2a+3b}{3a+2b}$, $x^3-y^3=(a-b)^3$, $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$.

15. $ax+by=0$, $x+y+xy=0$, $x^2+y^2=1$.

16. $4(x^2+y^2)=ax+by$, $2(x^2-y^2)=ax-by$, $xy=c^2$.

17. $ax^2+b^2y=ay^2+b^2x=c^3$, $x+y=c$.

18. $x-y=a$, $x^2-y^2=b^2$, $x^3-y^3=c^3$.

19. $x+y=a$, $x^2+y^2=b^2$, $x^4+y^4=c^4$.

20. $p(x+y)=q$, $x-y=k(1+xy)$, $xyp=r$.

Eliminate x, y, z from the equations :—

21. $\frac{x}{y+z}=a$, $\frac{y}{z+x}=b$, $\frac{z}{x+y}=c$.

22. $\frac{ax}{by+cz}=\frac{by}{cz+ax}=\frac{cz}{ax+by}=\frac{1}{2}$.

23. $x+y+z=p$, $2(xy+yz+zx)=q^2$, $x^3+y^3+z^3=r^3$, $3xyz=s^3$.

24. $x^2(y-z)=a$, $y^2(z-x)=b$, $z^2(x+y)=c$, $xyz=d$.

25. $ay+bx=z$, $az+cx=y$, $bz+cy=x$.

26. $(x+y)^2=4xyc^2$, $(y+z)^2=4yza^2$, $(z+x)^2=4zxb^2$.

27. $(x-y)z^2=c^3$, $(y-z)x^2=a^3$, $(z-x)y^2=b^3$, $(x-y)(y-z)(z-x)=3abc$.

28. $\frac{x^2(y+z)}{a^3}=\frac{y^2(z+x)}{b^3}=\frac{z^2(x+y)}{c^3}=\frac{xyz}{abc}=1$

29. $\frac{x}{a}+\frac{a}{x}=\frac{y}{b}+\frac{b}{y}=\frac{z}{c}+\frac{c}{z}$, $xyz=abc$,

and $x^2+y^2+z^2+2(ab+ac+bc)=0$.

30. $(x+y-z)(x+z-y)=ayz$, $(x+z-y)(y+z-x)=czx$, $(y+z-x)(x+y-z)=bxz$.

CHAPTER XXXIV.

Miscellaneous Examples Worked Out.

Ex. 1 If $x(b-c)+y(c-a)+z(a-b)=0$,

then will $\dfrac{bz-cy}{b-c}=\dfrac{cx-az}{c-a}=\dfrac{ay-bx}{a-b}$.

We have $x(b-c)+y(c-a)+z(a-b)=0$)
Identically, $a(b-c)+b(c-a)+c(a-b)=0$)

$\therefore$ by cross-multiplication, $\dfrac{b-c}{cy-bz}=\dfrac{c-a}{az-cx}=\dfrac{a-b}{bc-ay}$.

$$\therefore \frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

Ex. 2. If $x=cy+bz$, $y=az+cx$ and $z=bx+ay$;

shew that $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

From the given equations, we have $x-cy-bz=0$.....(1)

$$cx-y+az=0.....(2$$

$$bx+ay-z=0.....(3)$$

$\therefore$ by cross-multiplication from (1) and (2),

$$\frac{x}{-ac-b}=\frac{y}{-bc-a}=\frac{z}{-1+c^2};\ \text{or}\ \frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}. \quad (A)$$

From (2) and (3), $\dfrac{x}{1-a^2}=\dfrac{y}{ab+c}=\dfrac{z}{ac+b}$.....(B)

From (1) and (3), $\dfrac{x}{ab+c}=\dfrac{y}{1-b^2}=\dfrac{z}{bc+a}$.....(C)

From (A) and (B), $\dfrac{x^2}{(1-a^2)(ac+b)}=\dfrac{z^2}{(1-c^2)(ac+b)}$.

$$\therefore \frac{x^2}{1-a^2}=\frac{z^2}{1-c^2}.$$

From (B) and (C), $\dfrac{x^2}{(1-a^2)(ab+c)}=\dfrac{y^2}{(1-b^2)(ab+c)}$.

$$\therefore \frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}.$$

Hence $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

Ex. 3. If $x+y+z=0$, $ax+by+cz=0$,
and $(b-c)x+(c-a)y+(a-b)z=0$
then shew that $a^2+b^2+c^2-ab-bc-ca=0$.

We have $x+y+z=0$ }
$ax+by+cz=0$ }

$\therefore$ by cross-multiplication, $\frac{x}{c-b}=\frac{y}{a-c}=\frac{z}{b-a}=$ (suppose).

Substituting the values of x,y,z in the third equation,

$-k\{(b-c)^2+(c-a)^2+(a-b)^2\}=0$.

$\therefore (b-c)^2+(c-a)^2+(a-b)^2=0$.

$\therefore a^2+b^2+c^2-ab-ac-bc=0$.

Ex. 4. If $\frac{a-m}{c}+\frac{a-n}{d}=\frac{1}{b}$, $\frac{b-m}{c}+\frac{b-n}{d}=\frac{1}{a}$,

shew that $\frac{x-m}{c}+\frac{x-n}{d}=\frac{x}{ab}$.

Subtracting the second from the first relation,

we have $\frac{a-b}{c}+\frac{a-b}{d}=\frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab}$.

$\therefore \frac{1}{c}+\frac{1}{d}=\frac{1}{ab}$ .....(1)

Adding the given relations,

$\frac{a+b-2m}{c}+\frac{a+b+2m}{d}=\frac{1}{b}+\frac{1}{a}=\frac{a+b}{ab}$;

but from (1), $\frac{a+b}{c}+\frac{a+b}{d}=\frac{a+b}{ab}$.

$\therefore -\frac{2m}{c}-\frac{2m}{d}=0$, *i.e.*, $-\frac{m}{c}-\frac{n}{d}=0$ .....(2)

Again from (1), $\frac{x}{c}+\frac{x}{d}=\frac{x}{ab}$ .....(3)

Adding (2) and (3), $\frac{x-m}{c}+\frac{x-n}{d}=\frac{x}{ab}$.

Ex. 5. If $x+y=1$, $ax+by=c$, $a^2x+b^2y=c^2$,
then $a^n x+b^n y=c^n$.

We have $x+y-1=0$ }
$ax+by-c=0$. }

$\therefore$ by cross-multiplication, $\frac{x}{b-c}=\frac{y}{c-a}=\frac{1}{b-a}$ .....(1)

Again $x+y-1=0$ }
$a^2x+b^2y-c^2=0$ }

$$\therefore \frac{x}{b^2-c^2}=\frac{y}{c^2-a^2}=\frac{a}{b^2-a^2} \quad \ldots\ldots(2)$$

From (1) and (2), $\frac{b-c}{b-a}=\frac{b^2-c^2}{b^2-a^2}$ and $\frac{c-a}{b-a}=\frac{c^2-a^2}{b^2-a^2}$

$\therefore\ b+a=b+c$ and $b+a=c+a$.

$\therefore\ a=b=c$.

$\therefore\ a^n x+b^n y=c^n x+c^n y=c^n(x+y)=c^n$.

$(\because\ x+y=1)$

Ex. 6. If $x=a(y+z)$, $y=b(z+x)$, $z=c(x+y)$,

then $\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}$.

$x-ay-az=0$ ..... (1)

$bx-y+bz=0$ ..... (2)

$cx+cy-z=0$ ..... (3)

From (1) and (2), $\frac{x}{-ab-a}=\frac{y}{-ab-b}=\frac{z}{-1+ab}$.

$\therefore \frac{x}{a(1+b)}=\frac{y}{b(1+a)}=\frac{z}{1-ab}$ ..... (A)

From (2) and (3), $\frac{x}{1-bc}=\frac{y}{b(1+c)}=\frac{z}{c(1+b)}$ ..... (B)

From (1) and (3), $\frac{x}{a(1+c)}=\frac{y}{1-ac}=\frac{z}{c(1+a)}$ ..... (C)

From (A) and (B), $\frac{x^2}{a(1+b)(1-bc)}=\frac{z^2}{c(1+b)(1-ab)}$

$\therefore \frac{x^2}{a(1-bc)}=\frac{z^2}{c(1-ab)}$,

From (B) and (C), $\frac{x^2}{a(1+c)(1-bc)}=\frac{y^2}{b(1+c)(1-ac)}$

$\therefore \frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ac)}$.

Hence $\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}$.

Ex. 7. If $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2$,

then $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Identically $(x^2+y^2+z^2)(a^2+b^2+c^2)$

$=(ax+by+cz)^2+(ay-bx)^2+(az-cx)^2+(bz-cy)^2$,

but $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2$.

$\therefore\ (ay-bx)^2+(az-cx)^2+(bz-cy)^2=0.$

Here the sum of three positive quantities is zero.

Therefore each of them must $=0$.

Hence $(ay-bx)^2=0.\quad\therefore\ ay-bx=0.\quad\therefore\ ay=bx.\quad\therefore\ \frac{y}{b}=\frac{x}{a}.$

$(az-cx)^2=0.\quad\therefore\ az-cx=0.\quad\therefore\ az=cx.\quad\therefore\ \frac{z}{c}=\frac{x}{a}.$

$(bz-cy)^2=0.\quad\therefore\ bz=cy=0.\quad\therefore\ bz=cy,\quad\therefore\ \frac{z}{c}=\frac{y}{b}$

$$\therefore\ \frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

If the sum of the squares of two or more real quantities be zero then each of the quantities must $=0$.

Ex. 8 If $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a(x^2-y^2)$;

and $xy-\sqrt{(1-x^2)(1-y^2)}=b(x^2+y^2-1)$; shew that $\frac{1}{a^2}+\frac{1}{b^2}=1$.

We have $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a(x^2-y^2)$....(1)

Identically, $x^2(1-y^2)-y^2(1-x^2)=x^2-y^2$.....(2)

Dividing (2) by (1), $x\sqrt{1-y^2}-y\sqrt{1-x^2}=\frac{1}{a}$.....(3)

Again, $xy-\sqrt{(1-x^2)(1-y^2)}=b(x^2+y^2-1)$....(A)

Identically, $x^2y^2-(1-x^2)(1-y^2)=x^2+y^2-1$.....(B)

Dividing (B) by (A), $xy+\sqrt{(1-x^2)(1-y^2)}=\frac{1}{b}$.....(C)

Squaring (3) and (C) and adding them, we have—

$$x^2(1-y^2)+y^2(1-x^2)-2xy\sqrt{(1-x^2)(1-y^2)}+x^2y^2$$

$$+(1-x^2)(1-y^2)+2xy\sqrt{(1-x^2)(1-y^2)}=\frac{1}{a^2}+\frac{1}{b^2}.$$

Reducing, we get $\frac{1}{a^2}+\frac{1}{b^2}=1$.

Ex. 9. If $a+b+c=0$

then $\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\times\frac{a^2+b^2+c^2}{2}$.

Since $a+b+c=0$,

$$a=-(b+c).$$

$$\therefore\ a^5=-(b+c)^5$$

$$\therefore\ a^5+b^5+c^5=-5b^4c-10b^3c^2-10b^2c^3-5bc^4.$$
$$=-5bc(b^3+2b^2c+2bc^2+c^3)$$
$$=-5bc\{(b^3+c^3)+2bc(b+c)\}$$
$$=-5bc(b+c)(b^2+c^2+bc)$$
$$=5abc\left\{\frac{2b^2+2c^2+2bc}{2}\right\}\qquad(\because\ b+c=-a).$$
$$\therefore\ \frac{a^5+b^5+c^5}{5}=abc\left\{\frac{(b+c)^2+b^2+c^2}{2}\right\}$$
$$=\frac{a^3+b^3+c^3}{3}\times\frac{a^2+b^2+c^2}{2}$$
$$[\because\ a^3+b^3+c^3=3abc \text{ and } (b+c)^2=a^2].$$

Ex. 10 If $x+y+z=1$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$,

and $ax+by+cz=a+b+c$,

find the value of $a^2x+b^2y+c^2z$.

From the first two, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=(x+y+z)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$$\therefore\ x\left(\frac{1}{b}+\frac{1}{c}\right)+y\left(\frac{1}{c}+\frac{1}{a}\right)+z\left(\frac{1}{a}+\frac{1}{b}\right)=0.$$
$$\therefore\ a(b+c)x+b(c+a)y+c(a+b)z=0\ldots\ldots\ldots\ldots(A)$$

From the third, $(a+b+c)(ax+by+cz)=(a+b+c)^2$.

$$\therefore\ a^2x+b^2y+c^2z+a(by+cz)+b(cz+ax)+c(ax+by)=(a+b+c)^2.$$
$$\therefore\ a^2x+b^2y+c^2z+a(b+c)x+b(c+a)y+c(a+b)z=(a+b+c)^2\ ;$$

but from (A), $a(b+c)x+b(c+a)y+c(a+b)z=0$.

$$\therefore\ a^2x+b^2y+c^2z=(a+b+c)^2.$$

Ex. 11. $\frac{x}{l(mb+nc-la)}=\frac{y}{m(nc+la-mb)}=\frac{z}{n(la+mb-nc)}$,

shew that $\frac{l}{x(by+cz-ax)}=\frac{m}{y(cz+ax-by)}=\frac{n}{z(ax+by-cz)}$.

We have $\frac{\frac{x}{l}}{mb+nc-la}=\frac{\frac{y}{m}}{nc+la-mb}=\frac{\frac{z}{n}}{la+mb-nc}$

$$\therefore\ \text{each equals } \frac{\frac{x}{l}+\frac{y}{m}}{2nc}=\frac{\frac{y}{m}+\frac{z}{n}}{2la}=\frac{\frac{z}{n}+\frac{x}{l}}{2mb}$$

$$\therefore\ \frac{mx+ly}{c}=\frac{ny+mz}{a}=\frac{lz+nx}{b}.$$

$$\therefore \frac{mxz+lyz}{cz} = \frac{nxy+mxz}{ax} = \frac{lzy+nxy}{by}.$$

$\therefore$ each $= \dfrac{mxz+lyz+nxy+mxz-lzy+nxy}{cz+ax-by}$ or $\dfrac{2mxz}{ax+cz-by}$.

Similarly each $= \dfrac{2nxy}{ax+by-cz}$ and $= \dfrac{2lyz}{by+cz-ax}$.

$$\therefore \frac{lyz}{by+cz-ax} = \frac{mxz}{ax+cz-by} = \frac{nxy}{ax+by-cz}.$$

Divide each by xyz

$$\therefore \frac{l}{x(by+cz-ax)} = \frac{m}{y(ax+cz-by)} = \frac{n}{z(ax+by-cz)}.$$

Ex. 12. Solve $4^1+x+4^1-x=10$.

We have $4.4^x+4.4^{-x}=10$.

$$\therefore 2.4^x+\frac{2}{4^x}=5$$

$$\therefore 2\cdot 4^{2x}-5\cdot 4^{x+2}=0.$$

$$\therefore 4^x=\tfrac{1}{4}(5+3)=2 \text{ or } \tfrac{1}{2}.$$

$$\therefore 2^{2x}=2^1 \text{ or } 2^{-1}.$$

$$\therefore 2x=1 \text{ or } -1.$$

$$\therefore x=\tfrac{1}{2} \text{ or } -\tfrac{1}{2}.$$

Ex. 13. Solve $x^{x+y}=y^{4a}$ and $y^{x+y}=x^a$.

From the first, $x = y^{\frac{4a}{x+y}}$ and from the 2nd, $x = y^{\frac{a}{x+y}}$

$$\therefore x = x$$

$$\therefore \frac{x+y}{4a}=\frac{a}{x+y}$$

$$\therefore (x+y)^2=4a^2.$$

$\therefore x+y=2a$ Substituting this value in the first, $x^{2a}=y^{4a}$.

$$\therefore x=y^2.$$

$$\therefore a+y=y^2+y=2a$$

$$\therefore y^2+y-2a=0.$$

$$\therefore y=\tfrac{1}{2}(-1+\sqrt{8a+1}).$$

$$x=2a-\tfrac{1}{2}(-1+\sqrt{8a+1})=\tfrac{1}{2}(4a+1+\sqrt{8a+1}).$$

Ex. 14. If x be a *proper* fraction, shew that—

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots\ldots} = (1+x)(1+x^2)(1+x^3)\ldots\ldots$$

Let $(1-x)(1-x^3)(1-x^5)\ldots\ldots = a$;

$(1+x)(1+x^3)(1+x^5)\ldots\ldots = b$,

$(1+x^2)(1+x^4)(1+x^6)\ldots\ldots = c$;

and $(1-x^2)(1-x^4)(1-x^6)\ldots\ldots = d$;

Then $(1-x^2)(1-x^6)(1-x^{10})\ldots\ldots = ab$

and $(1-x^4)(1-x^8)(1-x^{10})\ldots\ldots = cd.$

$\therefore (1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})\ldots\ldots = abcd$;

but $(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})\ldots = d.$

$\therefore abcd = d.$

$$\therefore \frac{1}{a} = bc.$$

$$\therefore \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots} = (1+x)(1+x^2)(1+x^3)\ldots\ldots$$

Ex. 15. (*a*) The product of the m*th* and n*th* terms in the quotient of 1 divided by $1-x$ is $\frac{1}{x^2}$, shew that $m^3+n^3=0$.

We have by division $\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\&c.$

It is plain that the m*th* term $= x^{m-1}$ and n*th* term $= x^{n-1}$

$\therefore$ the product of the m*th* and n*th* terms $= x^{m-1} \times x^{n-1}$

$$= x^{m+n-2}.$$

By the question, $x^{m+n-2} = \frac{1}{x^2} = x^{-2}.$

$\therefore m+n-2 = -2.$

$\therefore m+n=0.$

$\therefore (m+n)(m^2-mn+n^2)=0.$

$\therefore m^3+n^3=0.$

(*b*) The quotient obtained by dividing the r *th* term by the $(2r)$*th* term in the quotient of $\frac{1}{1-x^r}$ is $\frac{1}{x^9}$; find r

$$\frac{1}{1-xr} = 1+xr+(x^r)^2+(x^r)^3+\&c.$$

The r *th* term $= (xr)^{r-1}$ and $(2r)$*th* term $= (xr)^{2r-1}$.

$$\frac{\text{The } r \text{ th term}}{\text{The } (2r)\text{th term}} = \frac{xr^{(r-1)}}{xr^{(2r-1)}} = x^{-r^2}.$$

By the question, $x - x^{-1} = \frac{1}{x^3} = x$

$\therefore \quad -r^2 = -9.$

$\therefore \quad r^2 = 9.$

$\therefore \quad r = 3.$

Ex. 16. If the H. C. F. of :—

$ax^3 + bx^2 + (a+b)x + 1$ and $ax^3 + (b+1)x^2 + (a+b-1)x + 2$ be a *binomtal*, shew that $(b+1)(a+b-1) = (a+2b)^2$.

$$
\begin{array}{l|l}
ax\,ax^3 + bx^2 + (a+b)x + 1 & ax^3 + (b+1)x^2 + (a+b-1)x + 2 \;\Big|\; x \\
a+b \;\Big|\; ax^3 - ax^2 + ax & ax^3 + bx^2 + (a+b)x + 1 \;\Big|\; \frac{x}{a+2b} \\
\hline
x^2(a+b) - x(a+b) + a + b & x^2 - x + 1 \\
x(a+2b) - (a+b-1) & x^2 - \frac{x(a+b-1)}{a+2b} \\
 & \hline
 & x\left\{\frac{a+b-1}{a+2b} - 1\right\} + 1 \\
 & = \frac{-x(1+b)}{a+2b} + 1 \qquad - \frac{1+b}{(a+2b)^2} \\
 & \hline
 & -\frac{x(1+b)}{a+2b} + \frac{(a+b-1)(1+b)}{(a+2b)^2} \\
 & \hline
 & -\frac{(a+b-1)(1+b)}{(a+2b)^2} + 1.
\end{array}
$$

If $x(a+2b) - (a+b-1)$ be the H. C. F., the remainder must vanish. Therefore $\frac{(a+b-1)(1+b)}{(a+2b)^2} = 1.$

Hence $(a+b-1)(b+1) = (a+2b)^2.$

Ex. 17. Shew that $x^6 - 2x^4 + x^3$ is divisible by 36.

We have $x^6 - 2x^4 + x^2 = (x^3 - x)^2 = \{x(x^2-1)\}^2$

$= \{(x-1)x(x+1)\}^2.$

The product of any three consecutive numbers is divisible by $1 \times 2 \times 3$.

$\therefore$ $(x-1)x(x+1)$ is divisible by $(1 \times 2 \times 3)$ or 6.

$\therefore$ $\{(x-1)x(x+1)\}^2$ is divisible by 6^2 or 36.

Ex. 18. Shew that $x^n - nan-1x + (n-1)an$ is divisible by $(x-a)^2$.

We have the given expression

$= x^n - a^n - nan-1(x-a)$

$-(x-a)\{x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots\ldots+xa^{n-2}+a^{n-1}-na^{n-1}\}$.

$(x-a)\{x^{n-1}+x^{n-2}a+x^{n-3}a^2+\ldots+xa^{n-2}-(n-1)a^{n-1}\}$.

We have here $n-1$ terms and $(n+1)$ times a^{n-1}.

Take a^{n-1} along with each term containing x.

$(x-a)\{x^{n-1}-a^{n-1}+a(x^{n-2}-a^{n-2})+a^2(x^{n-3}-a^{n-3})+$

$\ldots\ldots+a^{n-2}(x-a)\}$.

$=(x-a)(x-a)\{x^{n-2}+x^{n-3}a+\ldots\ldots+a^{n-2}$

$+a(x^{n-3}+x^{n-4}a+\ldots\ldots+a^{n-3})+\ldots\ldots\ldots+a^{n-2}\}$.

Hence the given expression is divisible by $(x-a)^2$.

Revision Papers.

I

(1) It is known that $\frac{5x-4}{3}$ is greater than $\frac{3x+5}{4}$. If x is an integer, find the smallest possible value of x.

(2) Determine the values of a, b, c and d so that $a+b(x-1)+c(x-1)(x-2)+d(x-1)(x-2)(x-3)$ may be identically equal to $4x^3-9x^2+x+5$.

(3) Eliminate a, given $x=a+\frac{1}{a}$, $y^2=a^2+\frac{1}{a^2}$.

(4) Show that the roots of $(1+k^2)x^2-2(1+kl)x+1+l^2-0$ are imaginary

(5) Resolve into factors :—

$(b-c)(x^2+ax+a^2)+(c-a)(x^2+bx+b^2)+(a-b)(x^2+cx+c^2)$

(6) Solve :—

(1) $\frac{1}{x}+\frac{1}{y}+1=\frac{11}{xy}$; $\frac{1}{y}+\frac{1}{z}+1=\frac{19}{yz}$; $\frac{1}{z}+\frac{1}{x}+1=\frac{14}{xz}$

(2) $\frac{1}{x-3}-\frac{2}{x-4}+\frac{1}{x-6}=0$.

(7) If $x=ka-b-c$; $y=kb-c-a$, $z=kc-a-b$, prove that $x^3+y^3+z^3-3xyz=(k+1)^2(k-2)(a^3+b^3+c^3-3abc)$.

(8) A B C D is a rectangle having $AB=7''$ $BC=3''$; E, F are points on BC, CD, such that $CE=\frac{1}{3}$ $DF=x''$.

Express the area of $ABEF$ in terms of x.

Draw a graph of this function, and find from the graph the $ABEF$ is a minimum.

II

(1) If $x : y = 5 : 6$ find the value of $(a)\frac{x+y}{x-y}$ $(b)\frac{x^2+y^2}{x^2-y^2}$

(2) Show that the expression $4x^4+4x^3-8x^2-6x-1$ can be put in the form $(ax^2-bx)^2-(cx+d)^2$; determine the values of a, b, c, d, and solve the equation $4x^4+4x^3-8x^2-6x-1=0$.

(3) If $x+y+z=0$, show that x^2-4yz is a perfect square.
Hence show that $(2c-a-b)^2-4(2b-c-a)(2a-b-c)$ is a perfect square.

(4) Two variables x and y are connected by the equation $y=px^2+\frac{q}{x}$ where p and q are constants. When $x=3$, $y=3\frac{1}{2}$ and when $x=-3$, $y=4\frac{1}{2}$. Find the values of p and q and also the value of y when $x=-2$.

(5) If $\frac{x}{y}+\frac{y}{x}=l$, $\frac{1}{x}+\frac{1}{y}=m$, $x+y=n$, find a relation connecting l, m and n.

(6) Resolve into factors :—

$$x^3(y+z)+y^3(z+x)+z^3(x+y)+2(x^2y^2+y^2z^2+z^2x^2)+4xyz(x+y+z).$$

(7) When a man is standing on a plank bridge, its tendency to break is measured by the quantity $\frac{a}{b}(b-d)d$ where b feet is the length of the bridge, a lb. is the weight of the man and d feet is his distance from one end.

If a man weighing 135 lb stands on a bridge 15 feet long, draw a graph of the function connecting the tendency of the bridge to break and his varying distances from one end and find from the graph where the man is when the bridge is most likely to break.

(8) Two rectangles of equal area have their sides in the ratio of $a : b$ and $a^2 : b^2$ respectively and the perimeter of the latter exceeds that of the former by $2(a+b)$, show that the area of either is

$$\frac{a^2b^2}{(a-b)^4}.$$

III

(1) If $a=p^2-q^2$ and $b=(2p+q)q$, show that a^2+ab+b^2 is a perfect square.

(2) Express $5(x+2y)(3x-4y)$ as the difference of two squares.

(3) Express $(x-3)(x-4)$ in the form $p(x-2)^2+q(x-2)+r$.

(4) Resolve into factors :—

$$x^2y^2z^2+xy^2(x^2+y^2+z^2+1)+x^2y^2+y^2z^2+z^2x^2.$$

(5) Solve :—

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=12,\ \frac{y}{2}+\frac{z}{3}-\frac{x}{6}=8,\ \frac{x}{2}+\frac{z}{3}=10.$$

(6) Simplify : $p^2(p+5)^2+pq(p^2+q^2-6)+q^2(q+5)^2$, where p and q are the roots of the equation $x^2+5x-8=0$.

(7) The expression $9x^4-12x^3+ax^2-12x+9$ is a perfect square; find a.

(8) A boat which can row 6 miles per hour in still water, takes $3\frac{1}{4}$ hours to go to a place 10 miles down the river and come back. At what rate does the current flow ?

IV

(1) If $x^5-5ax-4b$ be divisible by $(x-p)^2$, show that $a^5=b^4$.

(2) If the expression lx^2+mx+n takes the values $\frac{1}{k}$, $\frac{1}{k+1}$, $\frac{1}{k+2}$ when x takes the values 0, 1, 2 find the values of l, m, n and prove that when $x=k+2$, the expression is equal to $\frac{1}{k+1}$.

(3) Find the H. C. F. of x^3-4x^2-4x-8 and x^8-16 and resolve it into two quadratic factors.

(4) Simplify :—

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

(5) Resolve into factors :—

(1) $3x^2-14xy-5y^2+5x+7y-2$

(2) $6x^2y^2-7xy^3-3y^4$

(6) Eliminate x, y, z from $x+y+z=a$; $x^2+y^2+z^2=b^2$; $x^3+y^3+z^3-3xyz=c^3$

(7) Express $8x^3-3x+2$ as a function of $2x+1$.

(8) Draw the graphs of $y=x^2$ and $y=3x+10$ in the same diagram; hence find the roots of the equation $x^2=3x+10$.

V

(1) A man receives $\frac{a}{b}$ of £1 and afterwards $\frac{b}{a}$ of £1 ; he then gives away £2. Show that he does not lose by the transaction.

(2) In the expression $3x^2 - x - 2$, put $y^2 - y$ for x; arrange your result in powers of y, and resolve it into its factors.

(3) If $v = u + ft$ and $S = ut + \frac{1}{2}ft.^2$ show that $v^2 = u^2 + 2fs$.

(4) Find the factors of :—

$(x+1)(x+3)(x+5)(x+7)+16$.

(5) Find the equation to the straight line joining the points $(-3, 4)$ and $(-4, 3)$.

(6) Solve :—

(1) $\sqrt{3x+4}+\sqrt{3x-5}=9$

(2) $xy = x+y$; $xz = 2(x+z)$; $yz = 3(y+z)$.

(7) If $(a-b)x^2+(c-a)x+b-c$ be a complete square, show that $2b = c+a$.

(8) A rides at the rate of 8 miles an hour; B walks at the rate of $3\frac{1}{2}$; B starts first, and after a certain interval A sets off to overtake him. When he had ridden 14 miles, his horse broke down, and he had to walk on, which he did at the rate of 4 miles an hour, overtaking B in 14 miles more. What start had B ?

VI

(1) If $p = (b-c)(a-x)$, $q = (c-a)(b-x)$, $r = (a-b)(c-x)$ show that $p^3 + q^3 + r^3 = 3pqr$.

(2) Find the co-efficients of x^2 and x in the product $(5x^2+3x+5)(2x^2+3x+4b)$. Find for what value of b the two co-efficients will be equal.

(3) Factorise :—

$9x^6y^2 - 576y^2 - 4x^3 + 256x^2$.

(4) Express $8x^3 - 4x^2 - 6x + 5$ as a function of $2x - 1$.

(5) Find the H. C. F. of :—

$2x^6 - 8x^4 + 12x^3 - 8x^2 + 2x$ and $3x^5 - 6x^3 + 3x$.

(6) Solve :—

(1) $\dfrac{x+1}{y-1} - \dfrac{x-1}{y} = \dfrac{6}{y}$; $x - y = 1$

(2) $xy = x - y$, $xz = 3(x+z)$; $yz = 2(y-z)$.

(7) Draw the graph of the function $2x^2 - 3x + 5$, and find from it the minimum value of the function.

(8) A cask which held 270 gallons was filled with a mixture of brandy, wine and water; there were 30 gallons of wine in it more than

of brandy and 50 of water more than there were of wine and brandy together. How many were there of each ?

VII

(1) Factorise : $y^2-6x^2-xy+7x+y-2$. If the expression $=0$, draw the graphs of the two straight lines and find their point of intersection.

(2) If $\frac{b^2-c^2}{x-y}=\frac{bc}{a}$ and $\frac{c^2-a^2}{y-z}=\frac{ca}{b}$, then $\frac{a^2-b^2}{z-x}=\frac{ab}{c}$

(3) What must be added to $(x-a)(x+a)(x+3a)(x+5a)$ to make it a perfect square ?

(4) If $\frac{x}{2a-b+2c}=\frac{y}{2b-c+2a}=\frac{z}{2c-a+2b}$ show that

$$\frac{2x+2y-z}{a}=\frac{2y+2z-x}{b}=\frac{2z+2x-y}{c}$$

(5) Solve :—

(1) $x+y+z=\frac{a^2}{x}=\frac{b^2}{y}=\frac{c^2}{z}$

(2) $3x^2-20x=5$

(6) Eliminate a, b, c from $(a+b)^2-4ax^2$; $(b+c)^2=4bcz^2$ and $(c+a)^2=acy^2$.

(7) Find the H. C. F. of x^3+5x^2+6 and $3x^3+120x+117$.

(8) A sets off from Madras to Bangalore and B at the same time from Bangalore to Madras and they travel uniformly. A reaches Bangalore 16 hours and B reaches Madras 36 hours after they have met on the road. Find in what time each has performed the journey.

VIII

(1) Divide $1+2x$ by $1-3x$ to 5 terms in the quotient.

(2) Express $3x^2-5x+1$ as a function of $x+2$.

(3) If $x=a(y+z)$, $y=b(z+x)$, $z=c(x+y)$, show that $ab+ac+bc+2abc=1$.

(4) Write down $x^4+\frac{97}{6}x^2+1-\frac{91}{12}x(x^2+1)$, as the product of four factors.

(5) Find the L. C. M. of x^4+4x^2-5 and $2x^3+x^2-8x+5$

(6) Find the condition that $x^4+ax^3+bx^2+cx+1$ may be a perfect square.

(7) Simplify :—

$$\frac{3}{2a-3}-\frac{2a+15}{4a^2+9}-\frac{2}{2a+3}+\frac{18(2a+15)}{81-16a^4}$$

(8) Draw the graph of $3x^2-2x-1$ and use it to find the roots of the equation $1+2x-3x^2=0$.

IX

(1) If $x=\frac{q^2+r^2-p^2}{2qr}$, $y=\frac{p^2-(qr)^2}{(q+r)^2-p^2}$ find the value of $\frac{x+y}{1-xy}$

(2) Prove that

$$(x+y+z)\{x^2+y^2+z^2-(y-z)^2-(z-x)^2-(x-y)^2\}=8xyz+(y+z-x)(x+z-y)(x+y-z)$$

(3) Solve :—

(1) $c(ax+by)=bxy$

$c(bx-ay)=axy$

(2) $a(x^2+1)=x(a^2+1)$

(4) Find the square root of :—

$$a^4+a^3b+\frac{a^2b^2}{4}-\frac{3a^2}{2}-\frac{3ab}{4}+\frac{9}{16}$$

(5) Reduce to its simplest form

$$\frac{(a^2+b^2)(c+d)^2+2(ac-bd)(bc-ad)}{(a^2-b^2)(c^2+d^2)}$$

(6) If $ax+by+gz=hx+by+fz=gx+fy+cz=0$, show that

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

(7) Construct a homogeneous and symmetric function of second degree in x, y which takes values 31, 208 for $(1,-2)$ and $(+2,5)$ of x and y respectively.

(8) Find algebraically the maximum value of $x-x^2$ and verify graphically.

X

(1) If $x=p+q+\frac{(p-q)^2}{4(p+q)}$, and $y=\frac{p+q}{4}+\frac{pq}{p+q}$, prove that $(x-p)^2-(y-q)^2=q^2$

(2) Find k when $x^2-y^2+8x+2y+k$ is resolveable into two linear factors.

(3) If $(x^2-y^2)z=(y^2-z^2)x$, then $\frac{x}{z}=\frac{x^2}{y^2}$.

Eliminate x from $x^3+\frac{1}{x^3}+3\frac{(x+1)}{x}=m$ and $x^3-\frac{1}{x^3}-3\frac{(x-1)}{x}=n$.

(4) Solve :—

(1) $x^2+xy=a^2$; $x^2-xy=b^2$

(2) $(1+x)(1+y)=10$; $x^2y+xy^2=18$.

(5) Find the L. C. M of $6x^4-7x^2+2$ and $2x^3+6x^2-x-3$

(6) Find the square root of

$$x^6+\frac{1}{x^6}-6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)-20.$$

(7) Find the values of p and q in order that the expression $2a^4-3a^3+pa^2+q$ may be divisible by a^2-3a+2.

(8) I bought a horse and carriage for Rs. 750 ; I sold the horse at a gain of 5 per cent. and the carriage at a gain of 20 per cent., making on the whole a gain of 16 per cent. Find the original cost of the horse.

XI

(1) If $(x+y+z)^2=3(xy+xz+yz)$, and x, y and z are real, show that $x=y=z$.

(2) Resolve into factors :—

(1) $3(6a^2+5a)^2-10(6a^2+5a)-8$

(2) $(a+b)^2(a^2-b^2)+(b+c)^2(b^2-c^2)+(c+a)^2(c^2-a^2)$

(3) Show that $(x^2+y^2)(p^2+q^2)=(xp+yq)^2+(xq-yp)^2$

Can the right-hand member be expressed in any other form ?

(4) Find the H. C. F. of a^3-5a^2+4 and $a^3-11a+10$.

(5) Solve the equations ;—

(1) $px+qy=2pq$, $qx-py=q^2-p^2$

(2) $\frac{3x-4}{x+1}=\frac{x-1}{3x+4}-\frac{1}{2}$.

(6) If $t=\frac{2}{2-w}$, $w=\frac{2}{2-z}$, $z=\frac{2}{2-y}$, $y=\frac{2}{2-x}$ find t in terms of x.

(7) The expression ax^2+bx+c takes the values 1, 2, 5 for 0, 1, 2 of x ; find a, b, c

(8) Draw the graph of $3x^2-2x$ and use this to find the roots of $3x^3-x-4=0$.

XII

(1) If $f(x)=3x^2-x+1$ show that $f(x+1)-2f(x)+f(x-1)=6$

(2) Resolve into factors :—

(1) $2x^2-21xy-11y^2-x+34y-3$

(2) $2(x^6+y^6)-xy(x^2+y^2)(2xy-3x^2+3y^2)$

(3) Express x^3-6x+5 as a function of $x-2$

(4) (*a*) Write down the most general form of a rational, integral, homogeneous, symmetrical function of the second degree in x, y and z. (*b*) Such an expression has the value 11 when $x=0$, $y=1$, $z=-1$ and the value 23, when $x=-1$, $y=2$, $z=1$. Determine the expression

(5) Simplify $(a+b+c)(x+y+z)+b+c-a)(z+y-x)+(a+b-c)(x+y-z)+(c+a-b)(z+x-y)$

(6) Solve :—

$$\frac{x-2a}{x-3a}+\frac{y-4b}{y-3b}=2$$

$$\frac{x+2a}{x+a}=\frac{y+5b}{y+3b}$$

(7) Find the square root of $(x^2+y^2)(p^2+q^2)+2(xp+yq)(xq-yp)$

(8) A man buys 570 oranges, some at 16 for a shilling and the rest at 18 for a shilling ; he sells them all at 15 for a shilling and gains 3*s*; how many of each sort does he buy ?

XIII

(1) If x^3+ax^2+bx+c be divisible by x^2+px+q, show that $p(p-a)=q-b$.

(2) If $a+b+c=0$, $a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz)$, show that $x+y+z=0$.

(3) If p and q be the roots of the equation $\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}=0$, find the value of $(p-a)(q-a)$.

(4) Solve ;—

$x+y+z=3$, $ax+by+cz=a+b+c$ and $bcx+acy+abz=ab+ac+bc$

(5) Simplify :—

$$\frac{bc(x-a)^2}{(a-b)(a-c)}+\frac{ac(x-b)^2}{(b-a)(b-c)}+\frac{ab(x-c)^2}{(c-a)(c-b)}$$

(6) Find algebraically the equation to the straight line which joins the origin with the point of intersection of the straight lines represented by $2x-y=1$, and $x+2y=8$.

(7) Factorise :—

$$x^4+y^4+z^4-(x+y+z)(x^3+y^3+z^3)+(xy+xz+yz)(x^2+y^2+z^2)$$

(8) Draw the graphs of $y=x^2+4$ and $y=2x+1$; hence find the roots of the equation $x^2+4=2x+1$.

XIV

(1) From the equation

$$\frac{2}{y-3}+\frac{4}{2-x}-\frac{8}{(x-2)}(y-3)=0$$ find the value of $\frac{x}{y}$

(2) Eliminate x and y from the equations

$$x+\frac{1}{x}=p, y+\frac{1}{y}=q \text{ and } xy+\frac{1}{xy}=r.$$

(3) Solve :—

$3x-2y=15$

$z-2x=1$

$x+y+z=5$

(4) Extract the square root of

$$a^2(a^2+b^2+c^2)+2a(b+c)(bc-a^2)+b^2c^2$$

(5) If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, show that $\frac{xy}{x+y}+\frac{yz}{y+z}+\frac{zx}{z+x}=\frac{x}{x-1}+\frac{y}{y-1}+\frac{z}{z-1}$

(6) Assuming that $(x+y+z)^3-x^3-y^3-z^3=a(x^3+y^3+z^3)+b(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+cxyz$, where a, b, c are constants, determine the values of a, b and c.

(7) Find the H. C. F. of :—

$5x^3+38x^2-195x-600$ and $4x^3-15x^2-38x+65$

(8) An officer can form his men into a hollow square 4 deep and also into another 3 deep. If the front in the latter formation contains 16 men fewer than in the former formation, find the number of men.

XV

(1) A is 27 years old, B is 7 years old. What is the least number of years after which the ratio of their ages will be less than 3 : 1?

(2) Resolve into partial fractions :—

$$\frac{18x+66}{(x+1)(x+5)(x+7)}$$

(3) If $\frac{1}{x^3}\left(\frac{y}{z}+\frac{z}{y}\right)=\frac{1}{y^3}\left(\frac{z}{x}+\frac{x}{z}\right)=\frac{1}{z^3}\left(\frac{x}{y}+\frac{y}{x}\right)$, show that each $=\frac{2}{xyz}$

(4) Draw the graph of the function $2+3x-2x^2$; find from it the roots of the equation $2x^2-3x-2=0$. Find also the maximum value of $2+3x-2x^2$.

(5) In the function $ax^2+2bxy+cy^2$, put $x=m+n$ and $y=m-n$ and arrange the result in descending powers of m. If the function $\equiv pm^2+2qmn+rn^2$, show that $pr-q^2=4(ac-b^2)$.

(6) Solve :—

(1) $(x-2a)^3+(x-2b)^3=2(x-a-b)^3$

(2) $(x-9)(x-7)(x-5)(x-1)=(x-2)(x-4)(x-6)(x-10)$

(7) If $2s=a+b+c$, and $2S^2=a^2+b^2+c^2$, show that $(S^2-a^2)(S^2-b^2)+(S^2-b^2)(S^2-c^2)+(S^2-c^2)(S^2-a^2)=4s(s-a)(s-b)(s-c)$.

(8) A person leaves £12670 to be divided among his five children and three brothers, so that after the legacy duty has been paid, each child's share shall be twice as great as each brother's. The legacy duty on a child's share being one per cent., and on a brother's share three per cent.; find the amounts they respectively receive.

CHAPTER XXXV.

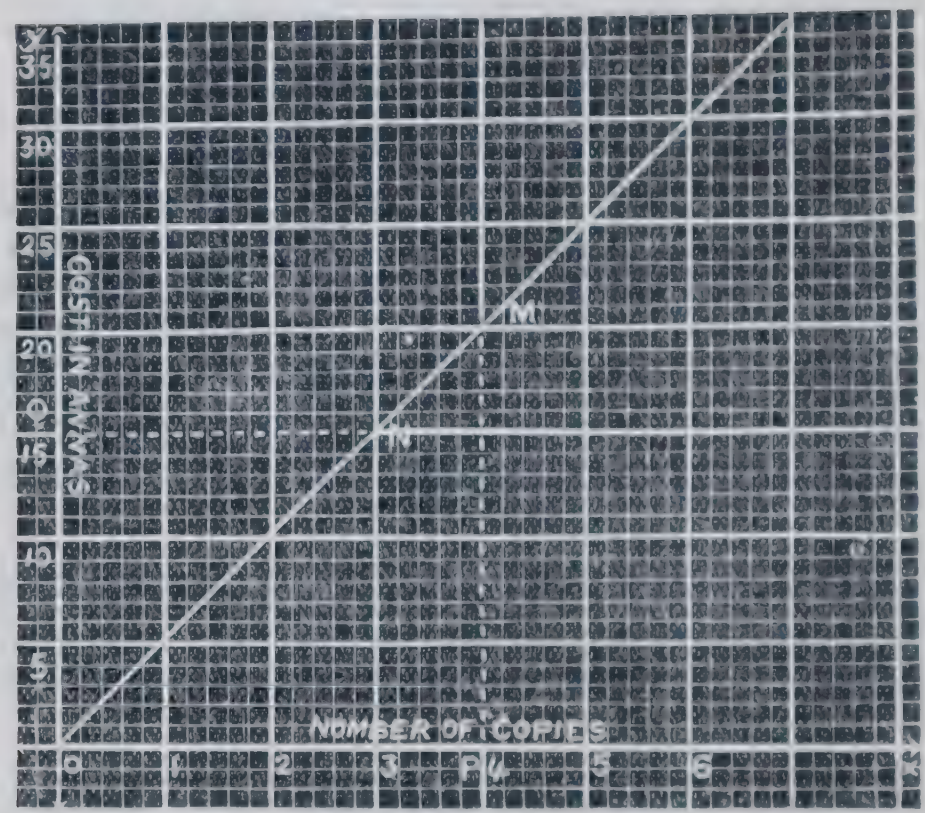
ARITHMETICAL PROBLEMS TREATED GRAPHICALLY.

If the cost of 1 copy of a book is 5 annas, the cost of 2 copies is 10 annas, of 4 copies, Re. 1-4 as. etc. The cost is directly proportional to the number of copies, that is, if the number of copies increases, the cost increases and *vice versa*. If the number of copies be x, the cost is $5x$ annas ; and if the cost is denoted by y we have then $y = 5x$. By drawing the graph of the equation $y = 5x$, we shall be able to find the relation between the number of copies and the cost. Since x denotes the number of copies and y the cost, we can use the x axis to measure the number of copies and the y axis to measure the cost in annas.

Since the equation does not involve an independent term, it passes through the origin (O, O) This means that the cost of zero number of copies is zero number of annas. We have to determine another point on the graph ; this can be got by giving the value of 1 to x ; then $y = 5$ Plot the point 1, 5.

In plotting this point suitable units for the axes must be chosen and the graph must be made as large as possible.

Since the negative side of the axes is not necessary, it need not be shown. the axes may be marked as shown in the figure below.



Here we notice that as x increases slowly y increases more rapidly. we can, therefore, use a larger unit for x than for y.

Let $\frac{1}{2}$ inch denote 1 copy.

" $\frac{1}{10}$ " " 1 anna.

The point (1, 5) is plotted and joined with the origin as it is also a point in the graph.

Thus the required graph is obtained. This will be useful to find the cost of books up to the number that may be denoted on the axis, and also to find the number of copies that can be bought with a given number of annas. This may, therefore, be considered as a ready reckoner to find the cost of a given number of copies and *vice versa*.

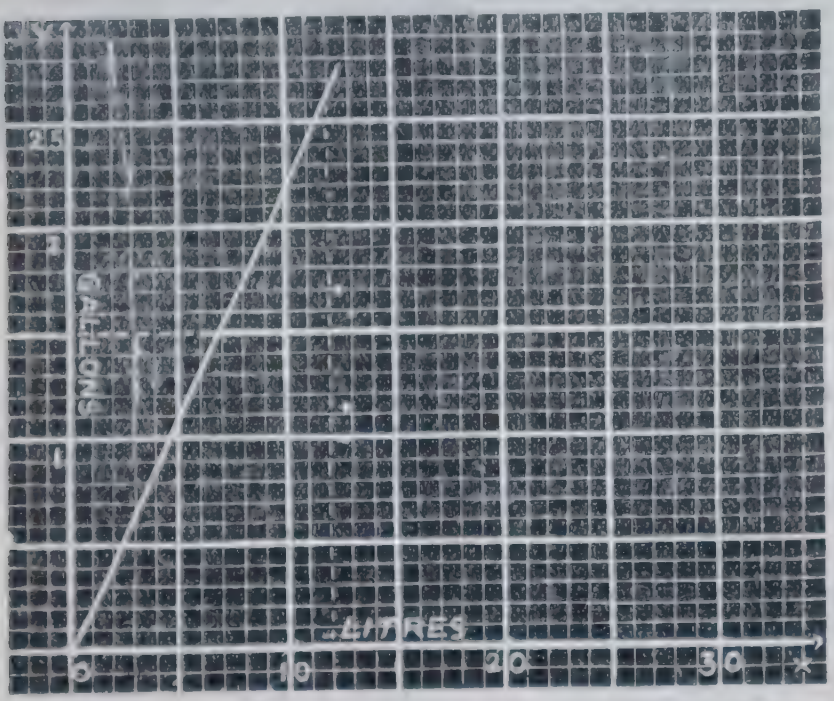
For example, suppose the cost of 4 copies is required. At the point P on the x axis which stands at the number 4 indicating the

number of copies erect a prependicular P M to it to meet the graph at M. The length of the ordinate measured in the unit chosen represents the number of annas, *i e.*, the cost.

Conversely, if the cost is given to be 15 annas the number of books can be read off by marking a point Q at 15 on the y axis and erecting a perpendicular to it to meet the graph at N. The abscissa represents the number of copies.

Note.—The student should not forget to state the units chosen clearly and graduate the axes before the points are plotted. It may be once more urged that the units must be chosen with reference to the size of the paper used and the corresponding value of the quantities involved.

Conversion Graph



Ex 1. Given that 10 litres = 2.2 gallons; draw a graph to convert gallons to litres or litres to gallons.

Express : (1) $1\frac{1}{2}$ gallons in litres.
(2) 12 litres in gallons.
Since 10 litres $= 2{\cdot}2$ gallons.
x $\therefore$ $= \frac{2{\cdot}2}{10}x$,,

If y denotes the number of gallons in x litres we have $y = \frac{2.2}{10}x$.

The graph representing the equation passes through the origin and when $x = 10$, $y = 2.2$. Therefore the graph can be obtained by joining the point whose co-ordinates are 10, 2.2 with the origin.

Let the x axis represent litres
and the y ,, ,, gallons

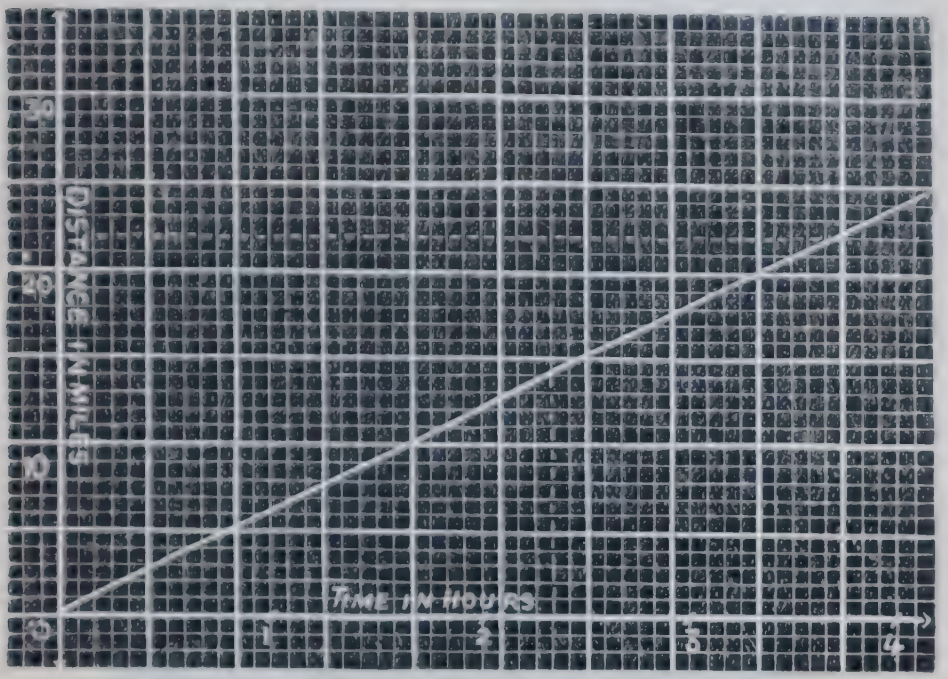
Take 1" on the x axis to 10 litres
1" ,, y ,, 1 gallon.
$1\frac{1}{2}$ gallons $= 7$ litres approximately.
12 litres $= 2{\cdot}6$ gallons ,,

The graph should be made as large as possible. It may be noted that we have represented the two quantities involved in the question by the two axes and the relation between the quantities given in the question is denoted by plotting a point ; this is joined to the origin.

As the graph is used to convert one quantity to another, it is called a *Conversion graph*.

Ex. 2. A man walks at the rate of 6 miles an hour ; draw a graph to show the relation between the time and distance. Find from it (1) how far he will go in 2 hrs. 20 min. (2) when he will reach a place which is 22 miles from the starting point.

Takes $O\,x$ and $O\,y$ two straight lines at right angles to each other. Measure time along Ox (1.2" for 1 hour) and distance in miles (1" for 10 miles) along $O\,y$. Since the distance is directly proportional to the time, we can get the graph by joining the point whose co-ordinates are 1, 6 with the origin (which will be a point on the graph as the equation is $y = 6x$). In drawing the straight line care must be taken to see that it is accurate by plotting one or two more points.



It can be seen from the graph that the distance done in 2 hrs. 20' = 14 miles.

Time taken to walk 22 miles = 3 hrs. 40 min.

(The correctness of answers may be verified by arithmetical calculation).

Ex. 3. In an examination marks are given to candidates with a maximum of 120; these will have to be raised so that the maximum may be 200. Show that this can be done by drawing a graph and read off from it the altered mark of a candidate who scored 60 before alteration and also the mark before alteration if the altered mark is 60.

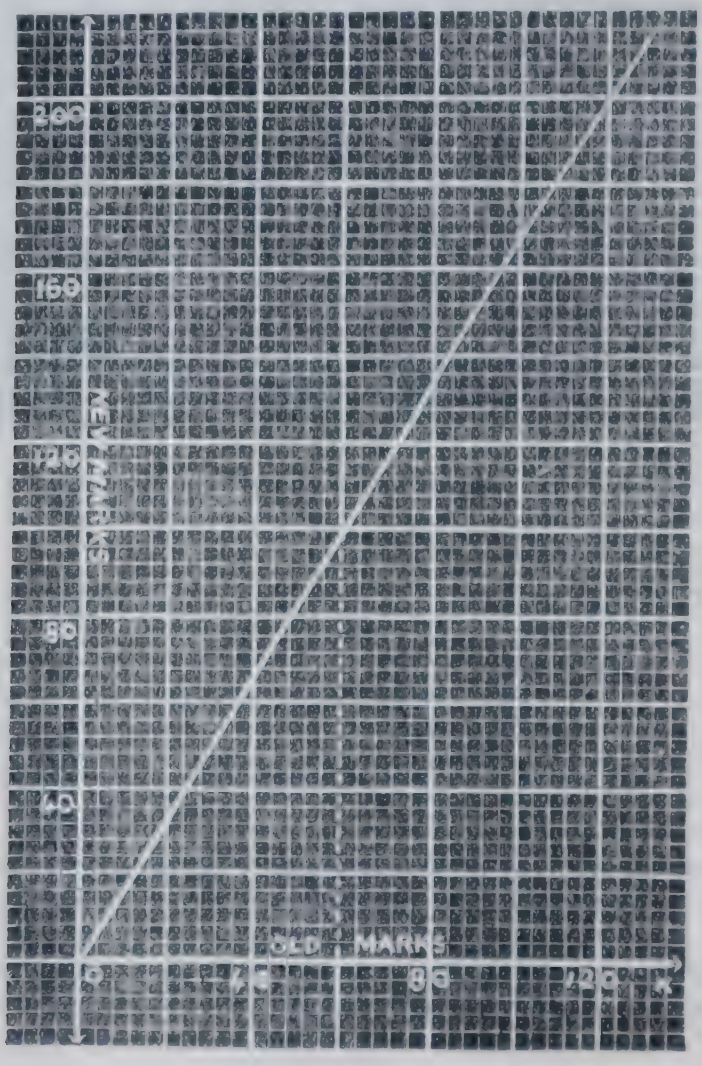
Take two straight lines $O\,x$, $O\,y$ at right angles to each other. Measure old marks on $O\,x$ (40 marks = 1") and new marks on $O\,y$ (40 marks = 1"). 100 marks before alteration correspond to 200 after it.

So plot the point 120,200; join this with the origin as O, O is also a point on the graph. The straight line thus obtained is the required graph.

If x denotes the number of marks in the old scale and y, the corresponding mark in the scale, the equation is $y=\frac{200}{120}x$, *i. e.*, $y=\frac{5}{3}x$.

(This passes through the origin) 60 marks (old scale) = 100 (new scale)

60 marks (new scale) = 36 (old scale)



EXERCISES.

1. If 10 articles cost Rs. 1-14 draw a graph to represent this, and find from it the cost of 32 books and also the number of articles that can be brought for Rs 1-2.

2. Draw a ready reckoner to show the price of sugar at Re. 1 per viss. Read off from it the price of $2\frac{1}{2}$ visses.

3. The selling price of an article includes a gain of 20 per cent on the cost price. Draw a graph to represent the relation between the selling price and the cost price of an article. Also find from it the cost price of an article whose (1) C. P. = Rs. 1-8. (2) S. P. = Rs 2-4.

4. Given that £ 1 = Rs 15, draw a graph to convert pounds to rupees and *vice versa*. Also find from it the English money equivalent to Rs. 24/-

5. Given that, a Madras maund = 25 lbs, draw a graph to convert Madras maunds to lbs. and *vice versa*. Find from it the number of (*a*) lbs. in $2\frac{1}{2}$ maunds, (*b*) maunds in $87\frac{1}{2}$ lbs.

6. Given that 5·5 cm = 2·2 in (approximately, draw a graph to express any number of centimetres in inches or inches in centimetres.

Express 2 in. in centimetres and 3 cms in inches.

(Take one inch as unit on each axis.)

7. Being given 1 kilogramme = 2·2 lbs. roughly draw a conversion graph for expressing kilogrammes in pounds and *vice versa*. Express 10 kilograms in lbs.

8. Given that the wages for a day's work for 8 hrs is 6 annas, draw a graph to find the wages for any fraction of a day. Find approximately (to the nearest pie) what should be paid to men who work 2, $3\frac{1}{2}$, 6 hrs. a day respectively.

9. A man walks at the rate of 5 miles an hour. Represent the relation between the time and distance by a graph and find from it the time taken to go 2 miles and the distance walked in 2 hrs. 22 min.

10. A man starts from P to walk to Q at a distance of 36 miles from P and reaches Q in $4\frac{1}{2}$ hrs Find by means of a graph, his rate of walking in miles per hour ; also find how far he will have gone from P in 1 hr. 20 min after starting.

11. In an examination the highest mark gained is 80 ; the marks gained by candidates will have to be raised so that the maximum may be 100. Draw a graph to show how this can be done and read off from it the altered marks equivalent to (1) 64. (2) 44.

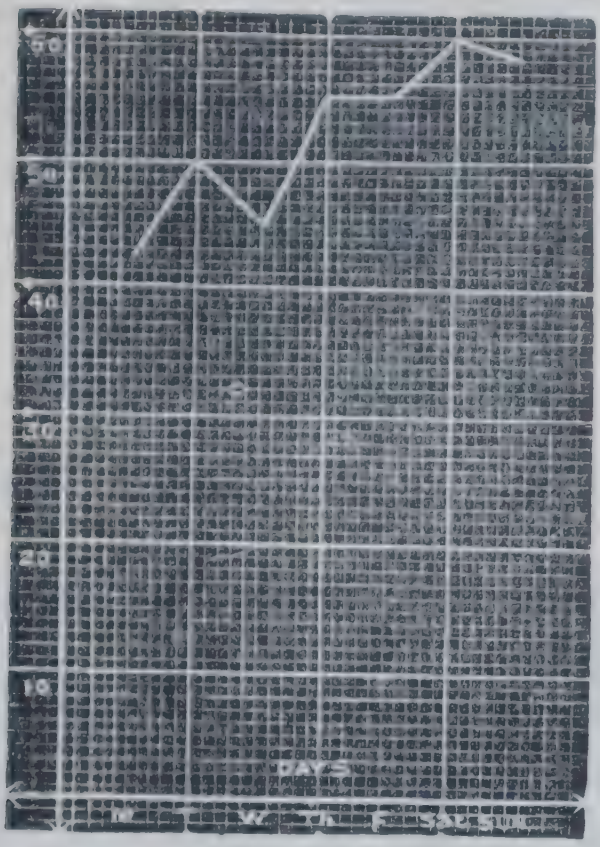
12. The papers are marked in an examination with a maximum of 210 , the marks gained by candidates will have to be reduced so that the maximum may be 120. Draw a graph to represent how this can be done and find from it the reduced marks of a candidate who gains (1) 140, (2) 100, originally.

Statistical Graphs.

Ex. 1. The daily sales of a merchant are given below : Exhibit them by means of a graph.

	Rs.
Monday	42
Tuesday	50
Wednesday	45
Thursday	55
Friday	55
Saturday	60
Sunday	58

Draw $O\,x$ and $O\,y$ two straight lines at right angles to each other. Represent days on the x axis and rupees on the y axis. Take 5 small divisions to represent each day on the x axis and one small division to represent one rupee on the y axis. Thus the sale on each day may be denoted by a point and from the given data all the seven points may be plotted as in the figure. Join the points thus plotted *successively* by means of straight lines. The broken line is the required graph.

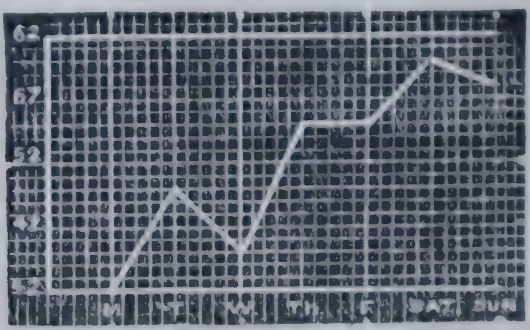


This graph enables us to see more clearly at a glance the rise and fall in the sales and also the days on which the merchant has the highest and lowest sales. We see that the level indicates that there is neither rise nor fall on Friday with reference to Thursday sales.

It is to be observed from the figure that no graph lies below the point 42 on the y axis and all the space of the squared paper is wasted. This waste we can avoid by not showing the divisions below 42 and beginning with 42 at the origin itself as we do not require the portion below 42 which is the least number of rupees in the data given.

We can also mark Monday at the origin so that the origin will be (Monday, 42) and 5 divisions after, Tuesday and so on. By this

arrangement so much paper is saved and the graph is lowered as in the figure.



In the previous chapters on the subject we were given the equation of our graph or the relation with which the co-ordinates of the points on the graph are connected. Sometimes though we were not given the equation, we were unable to determine the equation and thus to plot as many points on our graph as we pleased by choosing values of x and y that satisfy the equation.

In some cases we have to deal with only a limited number of corresponding values of the *two variables* that are obtained by *Observation* or experiment. We have in such cases to depend only on the data given for the correctness of our graph as there is no means of finding some more points on the graph. The data will be plotted by points which will be joined in succession by means of straight lines. We will then have a broken line as our graph to represent the variation in the values of the variables. For example, the temperature of a room observed at intervals during a certain period of time may be represented by a graph by plotting temperature against time.

In experimental work also we obtain a series of corresponding values of two variables, such as the length of a simple pendulum and the time of oscillation, we can plot one variable against the other and obtain as many points as are allowed by the data. These points are joined successively.

In both the above cases we cannot assume that the data given are absolutely free from error and irregularities as the data are after all the results of experiment or observation.

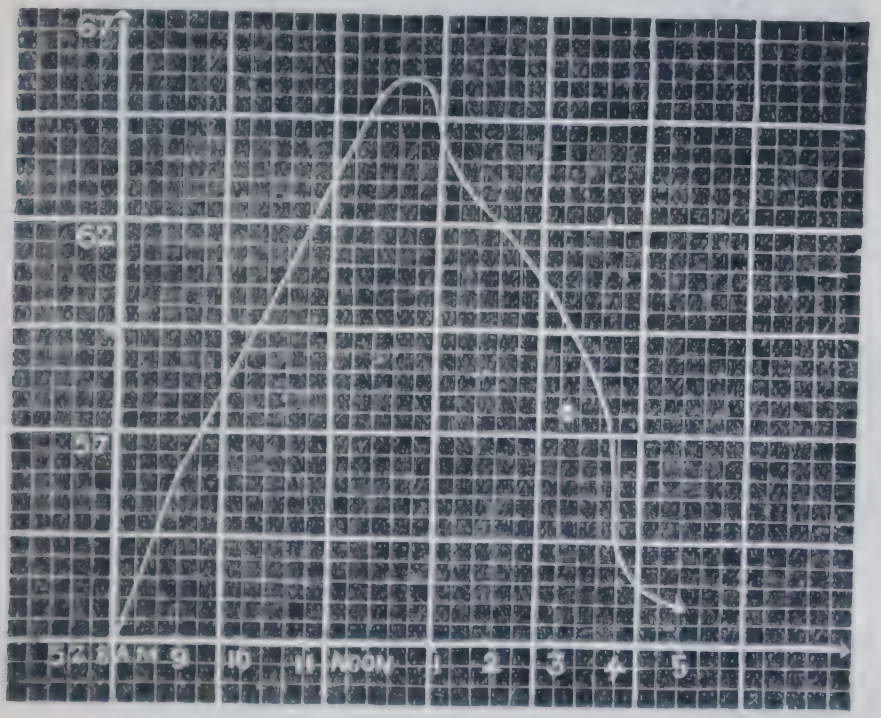
Therefore the position of a point cannot be supposed to be exactly determined. Generally, therefore, to exhibit the relation between such variables we draw a free hand continuous curve lying as *evenly* as possible between these points The curve is said to be a *smooth curve* It will pass through some of the points. while some of them lie outside it on one side and some on the other side.

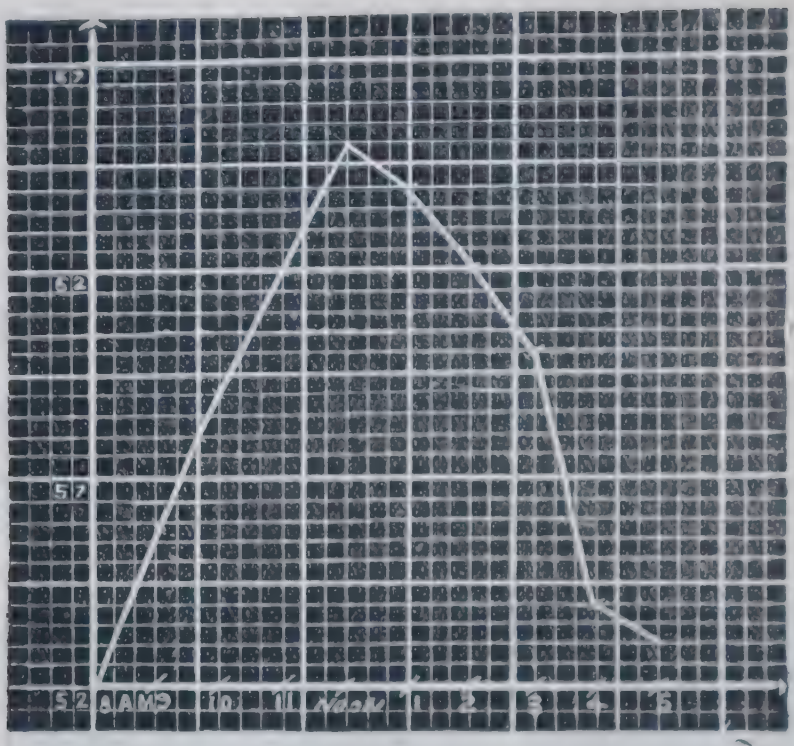
Ex 2 The temperature of a room taken at every hour from 8 A M to 5 P M is given in the following table. Construct a graph to show the variations of temperature

Time.	8 A. M.	9 A. M	10 A.M	11 A M	12 Noon	1 P M	2 P.M	3 P M.	4 P M	5 P M.
Temp.	52°	56°	59°	62°	65°	64°	62°	60°	54°	53°

Take $O x$ and $O y$ at right angles to each other to represent time and temperature respectively Take 3 small divisions to represent 1 hour on $O x$ and 2 small divisions to represent 1 degree.

The origin is marked 8 A. M , 52.





All the given data are marked by means of points which are joined by straight lines. We thus get a broken line for our graph as shown in figure (1) to represent the variations.

Unlike in the previous example, we can infer naturally that the changes in the temperature cannot be abrupt but should be only gradual. If for example the temperature at 12 noon is 65 and that at 2 P. M. is 64, we can naturally infer that the highest temperature will be somewhere between these hours. Therefore the graph or the curve may be made to rise to the highest point between these hours as shown in the figure.

We can use this graph to find the temperature at any intermediate time as in the case of a straight line. The student should note that such an *interpolation* is simply absurd in the case of the graph of *Ex.* 1' (regarding the sales) which is only a matter of accident. Boys

should remember that they should label the graph. the axes should be graduated and the scale clearly stated. Without these the graph becomes utterly useless It should be the aim of the student to make the graph as large as possible by using suitable units without unnecessary waste of paper. In choosing the scale, the student should do it with reference to the data given and the required thing in the given question.

EXERCISE

1. In the successive months a boy secured the following marks in Elementary Mathematics out of a maximum of 25.

8, 12, 13, 6, 15, 10. 4, 8, 12.

Exhibit this by means of a graph.

2. The following table gives the population of a country in millions. Make a chart to show this graphically :—

Year	1851	1861	1871	1881	1891	1901	1911
Population.	1·8	2·5	2·0	1·5	2·2	2·9	3·5

3. The following table gives the population of two countries *A* and *B* in millions Exhibit the variation in the population of the two countries in the same diagram :—

Year	1831	1841	1851	1861	1871	1881	1891
A	2.8	2·0	2·4	3·1	3 9	3·4	4·3
B	3·5	3 9	4 2	4·0	3·5	3·0	3·7

4. The height of an individual at different ages are given by the following table :—

Age	2	4	9	12	16	18	20	30
Height in inches	35	40	50	55	64	68	69	69½

Draw a graph showing the growth of the individual. Find his probable height at 14 years of age.

5. The following table gives the areas of circles with given diameters. Draw a graph showing the relation between the area and the diameter of a circle :—

Diameter	2	3	4	5	6
Area of a circle.	3 1	7 0	12 5	19 6	28 2

Find the area of a circle whose diameter is 4½ ft.

QUESTION PAPERS

OF

SECONDARY SCHOOL LEAVING CERTIFICATE

Public Examinations.

1911.

1 What is the remainder when a rational integral function $f(x)$ is divided by a binomial $x+a$?

Find without actual division the remainder when x^3-3x^2+2x-p is divided by $x+4$. For what value of p will the expression be exactly divisible by $x+4$?

2. Determine p and q so that the terms containing x^3 and x^4 may disappear in the expansion of

$$(1+px+qx^2)(1+x)^2.$$

Determine r so that $2x^2+3xy+2x-2y-r$ may be expressed as the product of two linear factors.

3. If $4x-5 \equiv A(x+3)+B(x-5)$, find A and B; and hence by resolving the denominators into factors, express $\dfrac{4x-5}{x^2-2x-15}$ as the sum of two fractions each with a binomial denominator.

4. I recently sold $\frac{1}{4}$ of my wet and $\frac{1}{3}$ of my dry lands and my assessment is reduced in consequence from Rs. 218 to Rs. 156. If the rates of assessment on these two kinds of land are respectively Rs. 8 and Rs. $1\frac{1}{2}$ an acre, find the extent of each which I now hold; and check the result graphically.

5. A cyclist started on a 16-mile journey; but after an hour his bicycle broke down and he had to walk the rest of his journey, which he did in $1\frac{1}{2}$ hours, after the accident. Had he accident happened after 12 miles' cycling, he would have reached his destination 18 minutes earlier. Find his rates of cycling and walking.

6. A stone projected vertically upwards ascends in every quarter second 2 feet less than in the previous quarter-second. If its ascent in the first quarter-second be 21 feet, find the height to which it rises in *n*

quarter seconds. Draw a graph showing the variation in the height as n increases. Also find the greatest height attained by the stone and the time that elapses before it attains that height.

1912

1. Simplify :—

$$\frac{\{(px+y)q+py-1\}^2-(x+y^2)(q-p)^2}{p^2x+2py-1}$$

and find the values of x and y so that the result may be exactly divisible by $(q-1)(q+2)$

2. Express $x^3-5x^2+10x-10$ as a function of p arranged in descending powers of p where $p=x-1$.

If $A(x-1)(x-2)+B(x-2)(x-3)+C(x-3)(x-4)$

$\equiv 8x-8$; determine A, B and C.

3. Given that $x=2t-1$ and $y+5=24t-16t^2$ express y as a function of x ; nd find graphically the value of x for which that function is a maximum and the value for which it vanishes.

4. The expenses of a hostel are partly constant and partly dependent upon the number of boarders. The total expenses in a month when there are 42 boarders amounted to Rs. 425 and in another month when there were 36 boarders to Rs. 370. Draw a graph to represent the expenses in a month for any number of boarders and find the expenses for the month in which there were 40 boarders.

5. A proposed railway line runs right through a rectangular field cutting the longer sides at an angle of 30° and taking up 2 acres or 0·32 of its entire area If the sides of the field are in the ratio of 5 : 2 find the length and breadth of the railway intercepted by the field.

6. The strength of an intermediate class is 90, and the student may, in the matter of fees due from them for a term be divided into 3 classes namely (1) those who pay Rs 58-12-0 in instalments, (2) those who pay Rs 50 in advance, (3) those who pay at a concessional rate of Rs. 29-6-0 If the number of these classes of students in order specified above are in *A. P.*, find them if the total amount of collection is Rs. 4,587-8 As.

1913

I. Carry the division of $2+x^5$ by $1-3x^2-x^3$ to 4 terms in ascending powers of x; and write down the remainder.

Hence find the value, correct to 3 places of decimals of $\frac{2+x^5}{1-3x^2-x^3}$ when $x=0{\cdot}1$.

II. In the expression $ax^2+2bxy+cy^2$ put $m-n$ for x and $m+n$ for y; and if the result $=pm^2+2qmn+rn^2$, show that $pr-q^2=4(ac-b^2)$.

III. (1) Prove that three quantities a, b and c are in A. P. or G. P. according as $\frac{a-b}{b-c}=\frac{a}{a}$ or $\frac{a}{b}$.

(2) if $\frac{1}{m+n}, \frac{1}{n+p}, \frac{1}{p+m}$ are in A. P., show that p^2, m^2, n^2 are also in A. P.

IV. A man purchases some garden land, and reserving $\frac{5}{11}$ths of it for his own use, sells the rest in such a manner that the actual cost to him for his own portion is reduced by a half. Had he sold it at Rs. 150 more an acre, he would have been a gainer by the whole extent of land which he reserved for himself. At how much an acre did he purchase the land; and at how much did he sell it?

V. In repairing a rectangular tank, the length is diminished by 40 yds and the breadth increased by 20 yds. The tank is still rectangular and the area remains the same; but the perimeter is reduced by $\frac{1}{20}$th. From equations to find the present dimensions of the tank and solve them and check your result graphically.

VI. The price of a box of mathematical instruments is 12 as. more at one firm than at another. If a headmaster is able to get 45 boxes more at the latter firm for Rs. 150 than at the former, what is the price of a box at each firm?

1914

I. (1) Factorise completely.

$3x^4+4x^3-19x^2-8x+12$.

(2) If the co-efficients of x^4 and x in the product $(2x^3+3x^2+ax-10)(3x^3-ax^2-10x+4)$ be equal to one another, find the value of a.

II. (1) If the sum of 15 terms of an A. P. of which the first term is 10 is 195, find the common difference.

(2) Simplify. $\frac{1}{2}-\left\{\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\ldots\frac{1}{3}n\right\}$.

III. Two pipes A and B running together can fill a cistern in $5\frac{5}{11}$ minutes; and, when running separately B takes two minutes longer than A to fill it. How long does each pipe running separately take to fill the cistern?

IV. (1) Prove that $(a-2)(a+1)(a+4)(a+7)+81$ is a perfect square.

(2) If $A(x+1)(x+2)+B(x-2)(x-3)\equiv C(x-1)+1$, find A, B, & C.

V. The perimeter of a rectangle is 44 inches. If its length be increased by 3 inches and its breadth be diminished by 2 inches, the area will remain unaltered. Form the equations to find its length and its breadth, and solve them. Check the result graphically by plotting the equations.

VI. A cyclist starts at 2 P. M. and travels at 11 miles an hour. A motorist travels at 16 miles an hour but starts 1 hour 40 minutes later than the cyclist. Draw a graph to represent the motion of each of these, and read off from it when and after how many miles from the starting point the motorist will overtake the cyclist. Verify the result algebraically.

1915.

I. (1) Prove that if $f(x)$ a rational integral function of x is divided by $x-a$, the remainder is $f(a)$.

(2) Resolve into factors :—

$a^3(b-c)+b^3(c-a)+c^3(a-b)$.

II. (1) Find the equation of straight line passing through the points (3, 4) and $(-1{\cdot}5, 2)$.

(2) If $a(y+z)+b(z+x)+c(x+y)=x+y+z=0$,

prove that $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$

III. (1) Write down a rational homogeneous expression of the most general form of the second degree in x, y and z.

(2) If the expression $ax^2+y^2+bz^2+2xy+2yz+czx$ be symmetrical with respect to x, y and z, find the value of a, b, c.

IV. (1) If $11x+7 \equiv A(x-3)+B(x+7)$, find the value of A and B.

Hence express $\dfrac{11x+7}{x^2+4x-21}$ as the sum of two fractions each with a binomial denominator.

(2) If a, b and c be in G. P., and x and y be the arithmetical means between a, b and b, c respectively show that

$$\frac{a}{x}+\frac{c}{y}=2.$$

V. Two towns A and B are 55 miles apart. A cyclist rides from A towards B at 6 miles an hour starting at 2 P.M. Another cyclist starts from B at 3 P.M. and rides towards A at the rate of 8 miles an hour. Draw a graph to represent the motion of each of these and read off from it when and where they meet. Also how far apart will they be at 4-30 P.M.?

VI. The forewheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yds. But if the circumference of each wheel be increased by 3 ft. it will make only 4 revolutions more than the hind wheel in the same distance. Find the circumference of each.

1916.

I. (1) If $y=f(x)=\dfrac{ax+b}{cx-a}$; prove that $f(y)=x$.

(2) Prove that $a^m \times a^n = a^{m+n}$ where m and n are positive integers.

Assuming the result interpret a^0.

II. (1) Simplify $\dfrac{x}{1+\dfrac{x}{2-x+\dfrac{9}{3-2x}}}$ and arrange the result according to ascending powers of x.

(2) Find a, b, c and d so that $x^3 - 4x^2 + 6x - 4$ may be expressed in the form $a(x-1)^3 + b(x-1)^2 + c(x-1) + d$.

(3) If $2s = a+b+c$ evaluate $s^3 - (s-a)^3 - (s-b)^3 - (s-c)^3$.

III. Draw the graph of $y = 3 - 4x - 4x^2$, thence find the roots of the equation $4x^2 + 4x - 3 = 0$. Show that the expression $3 - 4x - 4x^2$ is positive for all the values of x between 0·5 and −1·5 and negative for all real values outside these limits. Also find the maximum value of $3 - 4x - 4x^2$.

Also obtain algebraically the roots of equation $4x^2 + 4x - 3 = 0$ and maximum value of the expression $3 - 4x - 4x^2$.

IV. (1) If a^2, b^2, c^2, be in Arithmetical Progression prove that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.

(2) Find the terms in a G. P. whose sum is $6\frac{1}{4}$ and product 8.

V. Two trains start at the same time one from A to B, the other from B to A. When they meet, the train from A will have travelled 60 miles more than the train from B. The former train reaches destination $4\frac{1}{2}$ hours and the latter 8 hours after crossing. Find the distance between A and B and the rates with which the trains travel.

VI. A certain number of three digits exceeds the sum of the digits by 252. The number formed by reversing the digits exceeds the same sum by 450. If the number and that formed by reversing the digits be in the ratio of 4 : 7, find the number.

1917.

I. (1) What is the remainder when a rational integral function $f(x)$ is divided by $(x+a)$?

Resolve into factors :—

$a^3(b-c) + b^3(c-a) + c^3(a-b)$.

(2) Prove that :—

$(2a+x)(2a-x)(2a-3x)(2a-5x) + 16x^4$ is a perfect square.

II. (1) Find the co-efficient of x^2 in the product

$(1 - 3x + 4x^2)(3x - 2x^2 + 5x^3)$.

(2) Simplify $\frac{x+y}{1-xy}$ when $x = \frac{a+b}{1-ab}$ and $y = \frac{a-b}{1+ab}$

III. (1) An employer is given the option of being paid at a constant rate of Rs. 50 a month or of starting with Rs. 35 a month and receiving an increment of Rs. 3 to his pay monthly. Find for how many months he must serve in order that he may get the same amount which ever mode of payment he chooses.

(2) If the arithmetic mean between a and b is twice as great as the geometric mean, show that

$$\frac{a}{b} = 7 + 4\sqrt{3}.$$

IV. (1) Draw the graphs of $y = x^2$ and $y = 2x + 15$.

Hence find the roots of the equation $x^2 - 2x - 15 = 0$.

(2) Express $\frac{7x+11}{x^2+2x-15}$ as the sum of two fractions each with a linear denominator.

V. A railway train, after travelling for one hour, meets with an accident which delays it one hour; after which it proceeds at three-fifths of its former rate and arrives at the terminus three hours behind time. Had the accident happened 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Required the length of the line and the original rate of the train.

VI. The expenses of a hostel are partly constant and partly dependent upon the number of boarders. The total expenses in a month, when there were 44 boarders, amounted to Rs. 599, and in another month, when there were 48 boarders to Rs. 633. Draw a graph to represent the expenses in a month for any number of boarders; and find the expenses for the month in which there were 50 boarders. Find also the value of the constant.

1918.

I. (1) Find the remainder when $x^4 + 5x^3 - 9x^2 - ax - c$ is divided by $2x^2 - x - 6$; hence find the values of a and c for which the divisor is a factor of the dividend.

(2) If $a = \frac{2}{2-b}$; $b = \frac{2}{2-c}$; $c = \frac{2}{2-d}$; and $d = \frac{2}{2-a}$, find a relation between a and x.

II. (1) If $x^4+ax^3+bx^2+cx+d$ is a perfect square, then $8c=a(4b-a^2)$ and $(4b-a^2)^2=64d$.

(2) Prove that the sum of three fractions

$$\frac{b-c}{1+bc}, \quad \frac{c-a}{1+ca}, \quad \frac{a-b}{1+ab}$$

is equal to their product.

III. Draw with the same axes the graphs of $y=2x+3$ and $y=x+x^2$ and write down the values of x at the points of intersection of the graphs to one place of decimals.

IV. (1) If $a=x^2-yz$, $b=y^2-zx$, and $c=z^2-xy$, show that $a^3+b^3+c^3-3abc$ is the square of $x^3+y^3+z^3-3xyz$.

(2) Solve
$$x+y+z=0$$
$$ax+by+cz=0$$
$$a^2x+b^2y+c^2z=(a-b)(b-c)(c-a).$$

V. The number of soldiers present at a review is such that they could be formed into a solid square and also could be formed into four hollow squares, each four deep and each containing *twenty-four* more men in the front rank than when formed with a solid square. Find the whole number of soldiers.

VI. A student goes daily from Mylapore to the Law College, a distance of five miles. He walks at the rate of four miles an hour until a tramcar, whose speed is 10 miles an hour, overtakes him; then he goes into it and completes the journey. Obtain an equation connecting (x) the number of minutes occupied by the whole journey, with (y) the number of minutes occupied in walking.

Draw the graph of this equation, specifying the scales neatly with axes. If on a certain day the whole journey takes 33 (thirty-three) minutes, find the distance he walks on that day.

1919.

I. (1) Determine a and c so that the terms containing x^3 and x may disappear in the product of

$$(1+x)(1+ax+cx^2).$$

(2) Resolve into factors;

$$(a-b)(a+b-2c)^2+(b-c)(b+c-2a)^2+(c-a)(c+a-2b)^2.$$

II. (1) What must be added to—

$(2a-3)(2a+1)(2a+3)(2a+5)$ to make the expression a perfect square.

Give the square root of the resulting expression.

(2) Find A, B, C D so that x^3-4x^2+6x-4 may be expressed in the form

$$A(x-2)^3+B(x-2)^2+C(x-2)+D.$$

III. A wire six feet long is bent so as to enclose a rectangle. If one of the sides is x feet and y the corresponding area in square feet, obtain y in terms of x. Draw the graph of this equation, and find from the graph the values of x to one place of decimals which make the area one square foot. Check the result by calculation.

IV. (1) If $x=a(y+z)$, $y=b(z+x)$, $z=c(x+y)$ show that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$$

(2) Solve :—

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

$$x+y+z=a+b+c$$

$$ax+by+cz=a^2+b^2+c^2$$

a, b, c being unequal.

V. Three motor cars leave Madras at 9, 9-30 and 10 A.M., travelling along the same road at uniform rates. The second travels three miles an hour faster than the first, and the third five miles an hour faster than the second. If all the three meet at a particular instant find the distance from Madras of the place of meeting and the rates at which they have travelled.

VI. In a school hall a certain number of boys are seated on benches of equal length. If there were ten more benches, the boys might be accommodated by seating them one less on each bench. If there were 15 fewer benches, two boys more must sit on each bench. In each of these arrangements the same number of boys are seated on each bench. Form the equations to find the number of benches and the number of boys on each bench, and solve the equations graphically.

1920.

I. (1) Substitute $y+3$ for x in the expression $x^4-x^3+2x^2-3$ and arrange the result in descending powers of x.

(2) Show that $\frac{1+x^3}{(1-x)^2} = 1+2x+3x^2+5x^3+\frac{7x^4-5x^5}{(1-x)^2}$

and use this result to evaluate the left hand side correct to four places of decimals when $x=0{\cdot}01$.

II. (1) Given that $(x+4)$ and $(2x-3)$ are factors of the expression $6x^4-11x^3-121x^2+106x+120$, find its other factors and hence find the values of x which will make the expression equal to zero.

(2) If $x=\frac{n+1}{mn+1}$ and $y=\frac{mn+n}{mn+1}$ find in its simplest form the value of $\frac{x-y+1}{x+y-1}$.

III. 1) If $x+y+z=0$ and $x+3y-4z=0$ show that

$$\frac{x}{-7}=\frac{y}{5}=\frac{z}{2};$$

given also that $px+qy+rz=0$, prove that $7p=5q+2r$. x, y, z not being zero.

(2) Solve the equation.

$$\frac{x-3{\cdot}5}{2x+5}=\frac{3x-1{\cdot}5}{2x-1}.$$

Verify your solution by actual substitution. Explain any peculiarity you may note in the process.

IV. Transform into algebraic language the following geometric problem: 'given the hypotenuse of a right angled triangle and the sum of the sides containing the right angle, find its sides.' Solve the problem algebraically; hence or otherwise construct the triangle.

V. Two motorists driving uniformly but at different rates start at 10 o'clock, one from P to Q and back and the other from Q to P and back. They meet at 2 o'clock for the second time at a point distant 20 miles from P. If the distance from P to Q is 60 miles, find when and where they meet for the first time. Give a graphical solution and check your result.

VI. A tenant rents a certain number of acres of land for Rs. 1,080. He cultivates 3½ acres himself, and letting the rest for Rs. 10 an acre more than he pays for it receives for this portion five-sixths of the whole rent. Obtain a quadratic equation to find the number of acres rented, and hence find that number.

1921.

I. (1) Find the remainder when the expression—
$x^5 - 7x^4 - 5x^3 - 11x^2 + 2x^2 - 8x + 3$ is divided by $x - 4$

(2) If $z = x - \frac{1}{x}$, express in terms of z the expressions

$$x^3 - \frac{1}{x^3} \text{ and } x^4 + \frac{1}{x^4}.$$

II. Extract the square roots of—

(1) $(2x^2 + 3x - 35)(6x^2 - 43x + 77)(3x^2 + 4x - 55)$;

(2) $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (bz - cy)^2 - (cx - az)^2 - (ay - bx)^2$.

III. (1) Solve the equation—

$$\frac{2x-1}{x+1} - \frac{1}{2} + \frac{3x+7}{2x-8} = \frac{4x+1}{3x+3} - \frac{x+4}{x-4} + \frac{1}{6}.$$

(2) If the straight line whose equation is $\frac{x}{h} + \frac{y}{k} = 1$ passes through the points $(2, -\frac{1}{2})$ and $(-3, 3\frac{1}{2})$, find the values of h and k and verify graphically.

IV. Prove that $\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2 = n$.

Obtain from this formula a geometrical method of extracting the square root of any number n. Use your method to find the square root of 7 to two decimal places.

V. In an action between two battleships A and B, A fired $2\frac{1}{2}$ times as many shells as B. The total number of hits was to the total number of misses as 3 : 25. The number of B's misses was 390, and B's hits exceeded A's hits by 65. Find the number of shells fired, and the number of hits made by each.

VI. The distances between two towns by two different routes are 84 miles and 103½ miles. A motor car 'taking' the

longer road travels 5 miles an hour faster than another taking the shorter road, and does the journey in ten minutes less. Find the rate of each car.

1922.

(1) (i) If $f(x)=2x^2-3x$. write down $f(x+1)$ and $f(x+h)$ and simplify $\dfrac{f(x+h)-f(x)}{h}$

(ii) $\dfrac{1}{a}+\dfrac{1}{c}=\dfrac{2}{b}$. show that $\dfrac{a}{c}=\dfrac{a-b}{b-c}$

(2) Given that (i) $v=u+ft$ (ii) $s=ut+\frac{1}{2}ft^2$ derive, by eliminating t from (i) and (ii) the result

$$v^2=u^2+2fs.$$

Also obtain an expression for s in terms of v, f, and t.

(3) (i) Show that $1+3x+x^2$ is the square root of $1+x(x+1)(x+2)(x+3)$. Assuming this result, write down the square root of $1+(x+2)(x+3)(x+4)(x+5)$.

(ii) Prove that

$$a^4+a^2b^2+b^4\equiv(a^2+ab+b^2)(a^2-ab+b^2)$$; use this identity to resolve x^4+9x^2+81 into two quadratic factors

4. (i) Solve the equation $\dfrac{x}{0\cdot4}+\dfrac{2}{0\cdot3}=\dfrac{x+2}{0\cdot6}$ and verify your solution.

(ii) An accountant in entering a sum of x as. y ps. in a column of figures writes it down by mistake as. Rs x y as. with the result that the column adds up Rs. 14 8 as. 4 ps. too much. Find x and y and verify your solution.

(5) $ABCD$ is a square of side 8" P is a point in the side AB, distant x" from A Squares are described on AP and BP within the given square. Express in terms of x the area of that portion of the square $ABCD$ which lies outside the squares on AP and BP. Tabulate the values of the area as x varies from 0 to 8, and represent the results on a graph. Also find the position of P for which the area is greatest.

(6) The length of a seat in a tram car is 18 ft. If 3 persons more than the prescribed number are seated, each person gets 3·6 ins. of space less than he ought to. What is the prescribed number of persons for the seat ?

1923

(1) (i) Find the co-efficient of x^4 in the product of $1-2x-x^3-7x^4$ and $3+5x+2x^2-6x^4$, without actually multiplying the two expressions.

(ii) Show that $x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$ is unaltered in value, if x, y and z are each increased by the same quantity.

(2) (i) If $x=t+1$, and $y=t^2+t$, express y as a function of x only. For what values of t are x and y equal ?

(ii) Find the values of p and q so that $2x^4-11x^3+px^2-qx+6$ may be divisible by x^2-4x+3 without a remainder.

(3) (i) Factorise :—

(a) $(x^2+x)^2-22(x^2+x)+40$.

(b) $a^{12}-b^{12}$

(ii) Find the L.C. M. of x^2+x-6, $2x^2+7x+3$ and $2x^2-3x-2$.

(4) (i) Define index, and prove the index law $(am)n=amn$, where m and n are positive integers.

(ii) Find the square root of—

$4x^4-12x^3+29x^2-30x+25$.

(5) (i) Given that the two variables R and S are connected by the relation $S=a+bR+\frac{c}{R^2}$, where a, b, c are constants, and that when R takes the values 1, 2, 3, S has the values, 4, 1·5, 0 respectively, find the value of S when $R=-1$.

(ii) A dinner was ordered for a certain number of persons, the cost being Rs. 67-8-0. As three failed to come, each of the remaining persons had to pay Rs. 1-2-0 more. How many were present ?

(6) Draw the graph of the function $(x-1)(x-4)$ for the range $x=-1$ to $x=6$, read off from your figure the least value of the function, and also the values of x for which the function is equal to 2.

1924

(1) (i) What integral value of x makes $3x-1$ greater than $x+5$, and also $4x-7$ less than $x+7$?

(11) If $\frac{x+5y}{3x+y}=\frac{4}{3}$, find the value of $\frac{x^2-xy+y^2}{x^2+xy+y^2}$

(2) (i) Simplify $\frac{4x^2}{x^4-1}+\frac{1-\frac{2}{x^2}}{x-\frac{1}{x}}-\frac{1+\frac{2}{x^2}}{x-\frac{1}{x}}$

(ii) Prove that the difference of the cubes of two consecutive even numbers is greater by 2 than six times the square of the intervening odd number.

(3) (i) Factorise (*a*) x^4+64y^4

(*b*) $a^3+b^3+c^3-3abc$.

(ii) Show that $x-1$ and $x-2$ are factors of $x^4+(a-3)x^3-(3a-5)x^2+(2a-9)x+6$ for all values of a. Find the value of a if $x+1$ is also a factor, and in this case find the remaining factor.

(4) (i) Find the square roots of

$x^6-4x^4-6x^3+4x^2+12x+9$.

(ii) Find the H. C. F of $6x^4-x^3-8x^2-4x-3$ and $2x^4+5x^3+4x^2-14x-15$.

(5) Two parallel chords of a circle on the same side of the centre are 8 ins. and 12 ins. in length, and the distance between them is 2 ins. Find the radius of the circle: correct to $\frac{1}{10}$ of an inch, and the distances of chords from the centre.

(6) (i) Solve, graphically, the equations $y=x^2-x-2$ and $2x-3y=3$, and check the result by an algebraical solution.

(ii) The diameter of the back wheel of a motor car is 7 ins. greater than that of the front wheel, and the front wheel makes 240 more revolutions than the back wheel in a mile. Find the diameter of each wheel

(take $\pi=\frac{22}{7}$)

1925.

(1) (i) If $x=\frac{2mf}{a^2+m^2}$ $y=\frac{2mg}{a^2-m^2}$ show that $\frac{f^2}{x^2}-\frac{g^2}{y^2}=a^2$

(ii) If $f(x)=\frac{x}{1+x}$, find the value of $f(x)+f(y)+f(z)$, given that $x+y+z=2$, $xy+xz+yz=1\frac{13}{16}$, $xyz=\frac{3}{16}$

(2) Resolve into factors :—

(i) $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$

(ii) $6x^2-xy-12y^2$

(3) (i) Find the L. C. M. of :—

$4x^3+16x^2-3x-45$ and $10x^3+63x^2+119x+60$.

(ii) Extract the square root of :—

$$x^6-16x^5y+88x^4y^2-194x^3y^3+160x^2y^4-24xy^5+y^6$$

(4) (i) Prove that the difference of the squares of the greatest and the least of four consecutive numbers is equal to three times the difference of the squares of the other two numbers

(ii) Solve

$$\frac{6x+13}{2x+4}-\frac{8x-1}{4x-1}=\frac{9x+13}{3x+4}-\frac{10x-1}{5x-1}$$

(5) The expression $x^4-x^3+ax^3+bx^2+cx+2$ is divisible by each of the factors $x-1$, $x-2$, and $x+1$. Determine the values of a, b and c, and find also the remaining factor of the expression.

(6) $ABCD$ is a square. Points E, F, G and H are taken in the sides BC, CD, DA and AB respectively, such that $BE=CF=DG=AH=6$ ins. Show that $EFGH$ is a square. If the sides of the squares $ABCD$ and $EFGH$ are in the ratio 7 : 5, find the length of a side of the square $EFGH$.

1926.

A

1. From $(u^2+v)(v^2+u)$ subtract $(u^2+u)(v^2+v)$ and find the value of the difference if $u+v=a$ and $uv=b$.

2. Find the continued product of $x+1$, $x+2$, $x+3$, $x-1$, $x-2$ and $x-3$ by employing any standard identities or formulae.

3. Divide 1 by $(1-2x)^2$ so as to obtain a quotient in ascending powers of x. What is the remainder after four steps of the division have been performed ?

4. If $a^3+b^3+c^3=3abc$, prove that either $a+b+c=0$ or $a=b=c$.

5. Express $(a^2+b^2)(c^2+d^2)$ as the sum of two perfect squares.

B

6. Simplify :—

$$\frac{(a-1)(a^2+a+1)}{(a-b)(a-c)}+\frac{(b-1)(b^2+b+1)}{(b-c)(b-a)}+\frac{(c-1)(c^2+c+1)}{(c-a)(c-b)}.$$

7. Resolve $2x^2+5xy+3y^2-14x-17y+24$ into two linear factors. Hence find graphically the co-ordinates of the point of intersection of the pair of straight lines represented by the equation $2x^2+5xy+3y^2-14x-17y+24=0$.

8. If $12x^2-52x+64$ is written in the form $A(x-1)^2+B(x-2)^2+C(x-3)^2$ find the values of A, B and C.

9. Two persons P and Q start at the same time from one extremity A of a diameter AB of a semicircular field ACB. P walks along the circumference, and Q along the diameter, so that they reach B the other extremity of the diameter in t hours after starting. Then P walks along the diameter from B to A, and Q along the circumference BCA both interchanging their paths. How many hours start must P give Q so that both of them may reach A at the same time? Assume that the radius of the semi-circle is r miles and that the ratio of the circumference to the diameter is Π. Frame an equation and solve the problem. What is your remark on the answer?

10. ABC is an equilateral triangle. Points D, E, and F are taken in BC, CA and AB respectively such that $BD=CE=AF=12$ inches. Show that the triangle DEF is equilateral. If the sides of the equilateral triangles ABC and DEF are in the ratio $5:\sqrt{7}$, find the lengths of their sides.

ANSWERS TO EXERCISES.

PART I

EXERCISE 1.

1. 14; 21; 128; 96; 18; 0. **2.** 4; 1; 8; 16; 625; 6.

3. 2; 7; 0; $\frac{1}{4}$; 3; $25\frac{3}{5}$. **4.** 0; 0; 0; 0.

5 13. **6.** 7. **7.** 61. **8.** 47. **9.** 9.

10 1. **11.** 33. **12.** 345. **13.** -444. **14.** 1452.

15. 1034. **16.** 517. **17.** 43. **18.** -402.

EXERCISE 2.

1 54,000; 2,520; 0. **2.** 0; 48; 0; 1440.

3. $22\frac{2}{9}$; $331\frac{2}{27}\frac{3}{7}$; $\frac{3}{8}\frac{n}{25}\frac{+}{}$. **4.** $\frac{1}{5}$; 6; $3\frac{1}{3}$; 0; 0.

5. $2\frac{1}{12}$. **6.** $5\frac{1}{2}$. **7.** $6\frac{1}{4}$. **8.** -9.

9. $-\frac{3}{35}$. **10.** 29. **11** 630. **12.** 192.

13. 58. **14.** $17\frac{3}{10}$. **15.** 10. **16.** 3.

17. $3\frac{61}{830}$. **18** $6\frac{19}{35}$.

EXERCISE 3.

1. 20; 6; 30; $6\sqrt{6}$. **2.** $\frac{1}{4}$; $\frac{71}{10}$; 3; 240.

3. $1\frac{1}{3}$; $\frac{5}{18}$; $\sqrt{2}$; $\frac{5}{6}$. **4.** 3. **5.** 55.

6. 4. **7.** $\frac{36}{741}$. **8.** -108. **9.** 5.

10 0. **11.** 3. **12.** 4. **13.** 59.

14. 3. [$\sqrt{x-2}$ should be $\sqrt{x-2}$.]

15. 76; 536. **16.** 60. **17.** 38 [$\sqrt{x-y}$ should be $\sqrt[3]{x-y}$]

18. 50. **19.** $17\frac{43}{485}$. **20.** 0.

21. 9. [$\sqrt[4]{b(x+y)^2}$ should be $\sqrt[3]{b(x+y)^2}$].

22. $\frac{1}{4}$, $1\frac{1}{4}$, $-\frac{3}{4}$, 0 and $-\frac{1}{2}$. **23.** $2\frac{1}{2}$, $4\frac{1}{2}$ $6\frac{1}{2}$, $8\frac{1}{2}$, and $10\frac{1}{2}$.

24. $\frac{1}{3}$, $6\frac{1}{3}$, $8\frac{1}{3}$, $2\frac{1}{3}$, $-1\frac{2}{3}$. **25.** 4, 3, 6, 9, 18.

26

x	-2	3	1	4
y	12	2	0	6

27.

x	0	1	2	3	-2	-3
y	-6	-4	-8	-18	-28	-48

28.

x	-1	-2	0	1	2
y	-12	-26	-6	-8	-18

29. 6. **30.** -9, 21.

30. -9, 21.

31. 0

32. Substitute 11 instead of 10 for y.

33. p^2 should be p^3.

EXERCISE 4.

1. $8a; -6b; 2a$.

2. $37ab; 13a^2b$.

3. $23a-29b$.

4. $12x-12y-12z$.

5. $6a^2-8ab+11b^2$.

6. $3\frac{1}{2}ab+5bc-4\frac{1}{3}$.

7. $2\frac{1}{4}x^2-3\frac{1}{4}xy+5\frac{1}{2}y^2$.

8. $11\frac{1}{2}-15\frac{1}{3}ax+8\frac{1}{2}by-7ab$.

9. $\frac{11}{4}m^3+\frac{7}{5}n^3-20m^2n+21mn^2$.

10. $6a^3+\frac{7}{4}b^3-\frac{11}{18}c^3$.

11. $40a-25$.

12. $18x^2+35x+27$.

13. $8x^2+12xy+16xz-22yz$.

EXERCISE 5.

1. 0.

2. $+2b$.

3. $3\frac{7}{12}b$.

4. $-\frac{61}{120}a^2$.

5. $3(a+b+c+d)$.

6. $2a+2b+2c+2d$.

7. $\frac{7}{12}a^2+\frac{7}{12}b^2+\frac{7}{12}c^2+\frac{7}{12}d^2$.

8. $-5a^4-a^3+6a^2+2$.

9. $2ax-4by+2cz$.

10. $2a^4+12a^2+2$.

11. $4a^3-\frac{1}{2}a^2x+\frac{7}{8}ax^2+\frac{1}{12}x^3$.

12. $3a^2+3b^2+3c^2-1ab+2ac-2bc$.

EXERCISE 6.

1. $3b-8a^2b+10ab^2$.

2. $6ac-7xy+10xy^2+9xy^3$.

3. $12x^2+4x-5$.

4. $25a^3+7b^3+23a^2b+ab^2$.

5. $2m^3+2n^3+2l^3-3mn+3ln-3lm+1$.
6. $a+b+c-x-y$. 7. $y^2(a+b+c)+cy+c$.
8. $2(x^3+y^3+z^3)$. 9. $6a^3-2a^2b-3ab^2-22ac^2-c^3-7b^3$.
10. $8\frac{1}{2}a-5\frac{7}{60}b+3\frac{1}{4}c-3\frac{11}{20}d$. 11. $\frac{19}{20}a-\frac{13}{12}b+\frac{5}{12}c+\frac{1}{4}d$.
12. $3+9x+4x^3-11x^4$. 13. $-x^2-\frac{4}{35}y^2-z^2+\frac{2}{5}y^3$.
14. $a^3+3a^2b+3ab^2+b^3+a^2+2ab+b^2+a+b$.
15. $x^4+4x^3y+6x^2y^2+4xy^3+y^4$. 16. $(a+b+c)$ pies.
17. $(x+3y+5z)$ rupees. 18. $(2p+3q+4r)$ pounds.

EXERCISE 7.

1. $8, -3, -17$. 2. $3c-2d$, $6y+4x$, $-9x+8z$.
3. $c+d-a-b$, $3x-6y$, $3x-14z-6c$.
4. $2a-2b$. 5. $4x+5y-6z$.
6. $-a^2+ab+2$. 7. $-x^3-7x^2+2x+2$.
8. $-6a^3b-2b^3$. 9. $8a^3b+8ab^3$; 624.
10. $\frac{1}{3}x-4a+\frac{1}{18}b+\frac{1}{4}y$. 11. $-a+10b-\frac{1}{4}c+\frac{7}{2}bc$.
12. $-2\frac{1}{2}y-\frac{59}{12}a-\frac{1}{72}x$. 13. $19am-16b^n-2cp$.
14. $a+2\frac{1}{8}b+1\frac{1}{3}$. 15. $a^3-a^2+2a-1\frac{1}{12}$.
16. $a+b+d$. 17. $2x^2$.
18. $-4a^2+5ab-7b^2-8bc$.
19. $6a^3-12a^2b^2+2x^2ya-7yab^2-9abxy$.
20. $3a^2-3ab-9b^2$.
21. $9x^2-8xy+4y^2-10x-12y-13$.
22. $5x^2-4xy-5yz+11y^2$.
23. $a^2+y^2+z^2-2xy-2yz-2zx$.
24. $2xy$. 25. -10.
26. $+5$ lakhs, -5 lakhs.

EXERCISE 8.

1. 0. 2. 0. 3. $6b$. 4. $-7c$. 5. x.
6. $-a$. 7. $6b$. 8. $2x$. 9. $\frac{1}{2}b-\frac{1}{12}x$.
10. $\frac{1}{6}a+\frac{7}{12}b$.

EXERCISE 9.

1. $4b$. 2. $5a-5b$.
3. 0. 4. $3c^2-3b^2+a^2$.
5. $2b$. 6. $7x$.
7. 6. 8. 10.

9. a.
10. $3a+b$.
11. 0.
12. $6x-6y-20z$.
13. $9y+3z$.
14. $-5x-6z$.
15. $-2a-2c$.
16. $-7a+7b+c$.
17. $7a-2b+4c+2d$.
18. $4+x$.
19. $8a-3b+5c+2d$.
20. $3ax-bx-3by+cy$.
21. $7a-1\frac{3}{8}b-1\frac{5}{8}c$.
22. $2x-y-z$.
23. $6x^3-3x^2-x$.
24. $2a+9b$.
25. $-14x+135=132\frac{7}{8}$.
26. $8-8a=-48$.
27. $5a+19b=1\frac{1}{8}\frac{3}{4}$.
28. $2x-8a+6b-12c=2x-8a-7\frac{1}{3}$.
29. $x-10y+23z=\frac{1}{1}\frac{0}{3}$.
30. $-8x+11y-6a+19b=52$.
31. $11a-2b-16c=70\frac{5}{6}$.
32. $-(8h-a-3c-7g)+(5e-2b-4d-6f)$.
33. $-(2b-a-3c)+(5e-4d)-(6f-7g+8h)$.
34. $-\{-a-(3c-2b)\}-\{6f-(7g-8h)\}$.
35. $-(a+b)-(d-c)-(e-f)$.

EXERCISE 10.

1. $30abc$.
2. $12b^2x^2$.
3. $\frac{1}{4}abc$.
4. $6p^2qr$.
5. $\frac{4}{3}a^3b^2c^2x^2$.
6. $\frac{1}{2}abcxy$.
7. $\frac{3}{4}a^2b^2xyz$.
8. $25b^2c^2yz$.
9. $\frac{1}{3}\frac{0}{}pqxy$.
10. $\frac{1}{2}\frac{0}{1}a^2bc^3d$.

EXERCISE 11.

1. $6a^9$.
2. $20b^5$.
3. $\frac{1}{8}x^3$.
4. $21p^8$.
5. $25y^{13}$.
6. $2d^6$.
7. $81z^{21}$.
8. $70q^{13}$.
9. $\frac{4}{2}\frac{5}{}r^{18}$.
10. $24e^{26}$.
11. $3a^{11}$.
12. $60p^{18}$.
13. $300x^{23}$.
14. $6y^{13}$.
15. $3b^{16}$.

EXERCISE 12.

1. $12a^5b^4$.
2. $40x^{11}y^7$.
3. $\frac{1}{6}x^3y^4z^2$.
4. $150a^7x^3$.
5. $\frac{5}{2}a^3b^8x^6y^3$.
6. $52p^{10}q^8r^2$.
7. $200c^7d^5a^4$.
8. $385x^3y^3z^2$.
9. $6a^5bc^3x^2$.
10. $14ap^7x^5$.
11. $6a^6x^3y^6$.
12. $a^4b^5x^4$.
13. $24p^6q^3r^2$.
14. $60x^6y^5z^3$.
15. $540a^4c^3d^5$.

EXERCISE 13.

1. $4a^2c+8abc-12ac^2$.
2. $12x^3a-15x^2ay-3a^2x^2$.
3. $a^2b^2+a^2b^2c-ab^2c$.
4. $a^5b^2+a^5bc$.
5. $2a^6b^2+2a^4b^4-2a^4b^2c^2$.
6. $21a^2x^2-6a^2x+12a^3$.
7. $8x^3y^2-4x^2y^3+8x^2y-6xy^2$.
8. $\frac{1}{3}x^3-\frac{2}{5}x^3+\frac{1}{8}x^2$.
9. $\frac{7}{2}x^6y^2-\frac{3}{2}x^5y^3-2x^4y^4$.
10. $\frac{1}{5}a^4b^2-\frac{2}{3}a^3b^3+2a^2b^4-\frac{1}{4}ab^5$.
11. $\frac{1}{8}p^4q^3-\frac{1}{12}p^5b^2$.
12. $20x^4y^3-\frac{8}{3}x^3y^4+\frac{3}{2}x^2y^5$.
13. $\frac{2}{3}x^4y-\frac{1}{15}x^2y^3$.
14. $6a^4bc+8a^2b^3c-10a^2bc^3+12a^2bcd^2$.
15. $8a^3b^3c-12ab^3c^3-2a^3bc^3$.
16. $\frac{1}{2}a^4x^2y^2z-\frac{1}{4}a^3y^4z-\frac{1}{8}a^3y^2z^3$.
17. $8m^3np-2mn^2p^2+12m^3p^3$.
18. $a^2bcfg+2af^2g^2h-a^2f^2g-abfg^3-acfgh^2$.
19. $\frac{1}{2}a^3xyz+\frac{1}{2}ab^2y^2z-\frac{1}{2}ac^2yz^2-\frac{1}{2}a^3b^2xy^2z$.
20. $2x^6y-8x^5y+12x^4y-8x^3y+2x^2y$.
21. $2a^2+ab+b^2$.
22. $a^2-2ab+b^2$.
23. 0.
24. 0.
25. $a^3+b^3+c^3-3abc$.

EXERCISE 14.

1. $-12ab^2c$.
2. $\frac{1}{6}a^3b^2c$.
3. $-40a^6b^4x^2$.
4. $3x^3y^2z^2a$.
5. $-\frac{1}{2}p^4q^2r^6$.
6. $-a^2b^2-a^2bc-ab^2c$.
7. $-2x^2y^2z+2xy^2z^2$.
8. $-a^3b^2c+a^3bc^2+ab^3c^2$.
9. $-a^4b^3c+a^3b^4c+a^3b^3c^2$.
10. $2a^3b^2c^2-12a^2b^2c^3$.
11. $-\frac{1}{4}a^2bc+\frac{1}{8}abc^2+\frac{1}{8}a^2c^2$.
12. $x^3y^2-\frac{3}{8}x^3y^2z-\frac{1}{3}x^3yz$.
13. $16a^3bcxy+8ab^2x^2y^2+12ac^2x^2yz$.
14. $-\frac{1}{2}a^3x^2yc+\frac{1}{4}ab^2xy^2c+\frac{1}{4}axyzc^3+\frac{1}{2}a^2bxyc^2$.
15. $-4a^4b^4c^2+3a^2b^4c^4+2a^4b^2c^4$.
16. $-x^3a^2yb+xy^3ab^3+x^3abz^2c^3+x^2y^2abz$.
17. $-\frac{1}{3}p^3q^3r^2+\frac{1}{4}p^3q^4r^3+\frac{1}{2}p^2q^3r^3$.
18. $-10a^6+15a^3+35a^{10}+20a^4$.
19. $2x^8-\frac{2}{5}x^9-x^6-6x^5$.
20. $-a^5b^6c+ab^6c^5+a^5b^2c^5+a^5b^3c+ab^6c^3$.
21. $3ab+4bc-3b^2-4c^2$.
22. $-\frac{1}{2}zx-\frac{1}{3}xy-yz$.
23. 0.

24. $a^2+b^2+c^2$.
25. $a^2+b^2+c^2$.
26. $a^2+\frac{3}{4}b^2-\frac{3}{2}c^2-\frac{5}{6}ab+\frac{3}{4}bc-\frac{1}{6}ca$.
27. $x^4-6x^3-3x+24$.
28. $2a^4-8a^3-16a-4$.
29. $-x^5-x^4-12x^3+18x^2-56x+100$.
30. $2ax^2y-3cxy^2+5bxy-2cy-3ax$.
31. $-x+6y-z$.
32. $3x^2(p^2+q^2+r^2)-x(pq+qr+rp)-4(pr+4pq+3qr)$.

EXERCISE 15.

1. $x^2+9x+20$
2. x^2+5x+4.
3. $a^2+20a+99$.
4. $y^2+18y+45$.
5. $y^2+16y+55$.
6. $y^2+8y-33$.
7. y^2-y-6.
8. y^2+y-30.
9. $x^2+2x-120$.
10. x^2-x-90.
11. $x^2-9x+20$.
12. $y^2-15y+50$.
13. $y^2-8y+16$.
14. y^2-16
15. x^2-y^2.
16. $x^2+5ax+6a^2$.
17. $x^2-2ax-3a^2$.
18. $x^2+bx-2b^2$.
19. $x^2+cx-2c^2$.
20. $x^2+2cx+c^2$.
21. $4a^2-b^2$ (2 should be 2 a).
22. $4x^2-16x-33$.
23. $4y^2-8y+3$.
24. $4a^2-1$.

EXERCISE 16.

1. $2m^2-5mn+3n^2$.
2. $12x^2-29x+15$.
3. $-6a^2+20a-16$.
4. a^2-y^2.
5. $-x^2+7xy-12y^2$.
6. $4x^2-9y^2$.
7. $-(4a^2+12ac+9c^2)$.
8. $-2x^2+33x-45$.
9. $-20+19c-3c^2$.
10. $2x^2+5ax+2x^2$
11. $-21a^2+37ac-12c^2$.
12. $12y^2-ay-6a^2$.
13. $4-49x^2$.
14. $20m^2+43mn-21n^2$.
15. $6b^2-abx-2a^2x^2$.
16. $3a^2c^2-10acxy+3x^2y^2$.
17. $6m^2n^2-13amn+6a^2$.
18. $b^2c^2-a^2d^2$.
19. $6x^2y^2-5abxy-6a^2b^2$.
20. $cxz(a+b)-axy(a+c)+yz(bc-a^2)+ab(z^2-y^2)$

21. $\frac{1}{40}xy - \frac{1}{30}xz + \frac{2}{120}yz - \frac{1}{12}(y^2 + z^2)$.
22. $ab + ac + bc - a^2 - b^2 - c^2$.
23. $-6(x^2 + y^2 + z^2) - (xy + xz + yz)$.
24. $a^2(b - c) + b^2(c - a) + c^2(a - b)$.
25. $(a^2 + ab - 2b^2)$ rupees.
26. $(2x^2 - xy - 6y^2)$ Sq. ft.
27. $(6a^2 - 13a + 6)$ annas.
28. $(2x^2 + x - 10)$ miles.
29. $(6a^2 + 7ab + 2b^2)$ rupees.

EXERCISE 17.

1. $a^3 + 2a^2b + 2ab^2 + b^3$.
2. $a^3 - 2a^2b + 2ab^2 - b^3$.
3. $x^2 - 4y^2 - z^2 + 4yz$.
4. $a^6 + b^6$.
5. $a^6 - b^6$.
6. $x^2y^2 - y^2z^2 - z^2x^2 - 2xyz^2$.
7. $x^3 - y^3 - z^3 + xy^2 + xz^2 - yx^2 - yz^2 - zx^2 - zy^2$.
8. $a^2 - 6b^2 - 12c^2 - ab + 17bc + ac$.
9. $3m^2 - 4n^2 - 5p^2 - mn - 9np - 2mp$.
10. $20a^4 - 31a^3 + 79a^2 - 52a + 56$.
11. $a^4 - 25a^2b^2 + 20ab^3 - 4a^4$.
12. $6x^4 - 2x^3y - 13x^2y^2 - 3xy^3 + 2y^4$.
13. $2x^4 - 8x^3 + 17x^2 - 18x + 10$.
14. $27a^3 - 108a^2b + 144ab^2 - 64b^3$.
15. $a^4 - 2a^3b^2 - a^3b + 2a^2b^3 - ab^3 + b^4$.
16. $2x^4 + 2x^3y - 3x^2y - 4xy^2 - y^3$.
17. $\frac{1}{8}a^2 + \frac{7}{12}ab - \frac{1}{6}b^2 - \frac{1}{4}ac + \frac{1}{16}bc$.
18. $\frac{3}{4}a^4 - \frac{1}{8}a^3b + \frac{17}{12}a^2b^2 - \frac{2}{3}ab^3 + \frac{1}{8}b^4$.
19. $x^2 - \frac{13}{12}xy + \frac{1}{2}y^2 - \frac{1}{2}xz + \frac{2}{3}yz$.
20. $v^3 + \frac{1}{4}xv^2 + v^2$.
21. $2a^4 - \frac{3}{4}\frac{1}{4}a^3 + \frac{1}{4}\frac{3}{2}a^2 - \frac{7}{4}a + \frac{1}{2}$.
22. $\frac{1}{4}x^4 - \frac{1}{4}v^3 + \frac{3}{16}v^2 - \frac{1}{4}v + \frac{1}{2}$.
23. $\frac{1}{4}a^4 + \frac{1}{2}a^3x^2 - ax^3 + 2x^4$.
24. $\frac{1}{3}m^4 - \frac{5}{4}m^3n + \frac{13}{6}m^2n^2 - \frac{1}{12}\frac{3}{3}mn^3 + \frac{1}{2}n^4$.
25. $\frac{1}{3}x^4 + \frac{1}{3}x^3b - \frac{1}{12}\frac{1}{2}x^2b^2 - \frac{2}{3}xb^3 + \frac{1}{4}b^4$.
26. $a^2 - 3c^2 + 2ab + 2ac + 6bc$.
27. 0.
28. $25ab$.
29. $22x^2 - 143y^2 - 26z^2 + 95xy + 191yz - 139xz$.
30. $\frac{1}{2}(lm + mn + nl - l^2 - m^2 - n^2)$.

EXERCISE 18.

1. a^4+a^2+1.
2. $x^4-2x^2y^2+y^4$.
3. $10x^5-21x^4+27x^3-16x^2+5x+2$.
4. $x^3+y^3+z^3-3xyz$.
5. $x^6+x^4-x^2-1$.
6. $x^6+x^4y^2-x^2y^4-y^6$.
7. $a^5+a^4-3a^3+1$.
8. $a^6-2a^5b+3a^4b^2-3a^3b^3+3a^2b^4-2ab+b^6$.
9. $c^5-5c^4+10c^3-10c^2+5c-1$.
10. a^4-b^4.
11. $-4y^6-5y^5+13y^4-3y^3-4y^2+5y-2$.
12. $2a^4x^4+11a^3x^3+6a^2x^2+ax+15$.
13. $x^5-x^3b^2+3x^2b^3-2xb^4+b^5$.
14. $-2y^6+11y^5-22y^4+26y^3-26y^2+7y+10$.
15. $pr x^4+(p^2+qr)x^3+(2pq+r^2)x^2+(q^2+pr)x+qr$.
16. $ap x^5+aq x^4-bp x^4+ar x^3-bq x^3-cp x^3-br x^2+as x^2$
$-cq x^2-cr x-bs x-cs$.
17. $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$. [for x^2-3a^2x read $3ax^2-3a^2x$].
18. x^6-2x^3+1.
19. $a^8+a^6b^2+a^4b^4+a^2b+b$.
20. $a^8+6a^5b^2-16a^4b^2+40a^3b^2+9a^2b^4+8a^2b-25a^2b^2$
$-10ab^5-b^8$.
21. $\frac{1}{4}x^5-1\frac{1}{3}x^4+3\frac{5}{24}x^3-3\frac{7}{12}x^2-2\frac{1}{4}x+9$. [For $\frac{x^3}{2}$ in $3+\frac{x^3}{2}$ $-x$ read $\frac{x^2}{2}$].
22. $\frac{x^5}{4}-\frac{11}{24}x^3-3\ \frac{5}{24}x-5\frac{3}{4}$.
23. $9x+43x^3y^2+y^4-18x^2y+6xy^3$.
24. x^4-x^3+6x-9.
25. a^4-b^4.
26. $4a^2+16a+16-9b^2$.
27. $3xz-3yz-6x^2y+21xy^2-2x^2y^2-3x^2z^2-2xyz$
$+3xyz^2+3xy^2z-5x^2yz+2x^2z-x^3+2x^3z+2x^3y$.
28. $b^2-2c^2+2a^2b+2ab^2-2bc^2-4b^2c+2ac-bc+2abc$.
29. $3(b^3c+bc^3-2b^2c^2)+5(c^3a+ca^3-2c^2a^2)+7(a^3b+ab^3-2a^2b^2)$.
30. $2(ab^3+ac^3)-3(bc^3+ba^3)+4(ca^3+cb^3)$.
31. $2\{x^3y^3+y^3z^3+z^3x^3-x^4-y^4-z^4\}$.
32. $2x^2y^2+2y^2z^2+2z^2x^2-2x^4-2y^4-2z^4$.
33. $3a^2-ab-15ac+b^2+3bc-7c^2$.

EXERCISE 19.

1. $a^4-2a^2b^2+b^4$. 2. $x^3+x^2(b-a-c)+x(ab-bc-ab)+abc$.
3. $a^2b+ab^2+b^2c+bc^2+ca^2+ac^2+2abc$.
4. $ab^2-a^2b+a^2c-ac^2+bc^2-b^2c$. 5. $x^3+5x^2-29x-105$.
6. $6x^3-41x^2-19x+84$. 7. $-8y^3-30y^2+99y-54$.
8. $24-34c+15c^2-2c^3$. 9. x^8-y^8.
10. x^4-16y^4. 11. $a^8+a^4b^4+b^8$. 12. x^8-47x^4+1.
13. $6b^2a+5abc-6ac^2-6b^3-5b^2c+6bc^2$.
14. $(x^2-y^2-z^2+2yz)(y+z-x)$.
15. $x^3-3x^2c-10xc^2+24c^3$.
16. $-8x^4+8x^3a-6x^2a^2+4xa^3-a^4$.
17. a^8-16.
18. x^8-16b^8. [For $x^2-2bx-2b^2$ read $x^2-2bx+2b^2$].
19. $x^3+x(ab+ac+bc-a^2-b^2-c^2)+(a-b)(b-c)(c-a)$
20. a^6-b^6. 21. $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$.
22. $1+x^5+x^{10}$.

EXERCISE 20.

1. $-x$. 2. $-ac$. 3. $3a^2b^2$.
4. $-\frac{1}{2}a^2bc$. 5. $\frac{5}{3}x^5$. 6. $-\frac{2}{3}y^5x^2$.
7. $\frac{5}{6}xy^4z$. 8. $-\frac{3}{4}a^{10}b^5c^6$. 9. $5a^5b$.
10. $-5a^7x^2$. 11. $\frac{1}{2}p^4n^2$. 12. $-\frac{7}{8}y^4z^{11}$.
13. $-\frac{2}{3}mnpq$. 14. $-3z^2a^7b^7$. 15. $-\frac{2}{5}mxz^8b^5$.
16. $\frac{a}{b-c}$.

EXERCISE 21.

1. $-x+3y$. 2. $-a^2+3ab-3b^2$. 3. x^5-x+2.
4. $-ax+a^3$. 5. $-c-a-b$. 6. $-\frac{1}{3}a-b+3ab$.
7. $-\frac{3}{2}a^4-\frac{11}{2}a^2+6ab^2+\frac{13}{2}b^3$.
8. $4x^3-2x^2y^4+6xy^5$. 9. $\frac{1}{4}a-\frac{1}{8}\frac{a^2}{b}+\frac{1}{8}a^2$.
10. $-6ab^2+3b-\frac{8}{3}a^2$. 11. $-\frac{3}{4}m+\frac{1}{3}n-\frac{3}{8}mn^2$.
12. $\frac{2}{5}b^5+\frac{1}{5}b^4-\frac{3}{5}b^3+\frac{3}{10}b$. 13. $-2bc+5b^2-\frac{8}{3}bc^2$
14. $-ax-\frac{1}{2}a^3+\frac{1}{2}x^3$.
15. $\frac{1}{5}m^3n^3p^3-\frac{1}{5}mn^3p^3+\frac{1}{5}m^3np^3$).

EXERCISE 22.

1. $a+2(a^2+4a$ should be $a^2+4a+4)$. 2. $a-5$.
3. $x-12$. 4. $x-7$. 5. $x-22$. 6. $3x-2$.
7. $2x-3$. 8. $a+11$. 9. $2x-3$. 10. $5x+4$.
11. $8x-3$. 12. $5a-2b$. 13. $3x-4y$. 14. $3b-5c$.
15. $7a+2x$. 16. $9x+2y$. 17. $11a+15b$.
18. $x+14y$. 19. $4a-b$. 20. x^2+2x+1.
21. $4x^2-4x+1$. 22. x^2-2x. 23. $2x^2-x$.
24. x^3-3x+2. 25. $27x^2+15x-10$. 26. $4x^2+x-2$.
27. $8a^2-2ab-b^2$. 28. $2x^2-5x+3$.
29. $8b^3-12b^2a+6ba^2-a^3$. 30. $x^2+2xy+y^2$.
31. $x^2-2xy+y^2$. 32. x^2+3x+3. 33. a^2-b^2+2ab.
34. a^2-b^2+2ab. 35. $8b^3-36b^2a+54ba^2-27a^3$.

EXERCISE 23.

1. $x-1$. 2. $3x+2$.
3. $x^3+2ax^2+2a^2x+a^3$. 4. x^2+4x+8.
5. $x-4$. [For $x-256$ read x^4-256].
6. $x^5+2x^4+x^3-4x^2-11x-10$.
7. $1+2x+3x^2+4x^3$. 8. $3x^3-4x^2y+4xy^2+2y^3$.
9. $3x^3-2x^2-5x-3$.
10. $1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9$.
11. $a^2+4b^2+9c^2-6bc+3ac+2ab$.
12. $x^2+y^2+z^2+xy-xz+yz$. 13. $m^2+n^2+1+m+n-mn$.
14. $x-1$. 15. $x+1$.
16. $2x^2-3x-1$. 17. $3x^2-9x+8$.
18. x^2-5x+6. 19. $x^3-2x^2+12x-18$.
20. $3x+6y+7z$. 21. $2a+5b-3c$.
22. $5a+6b+7c$. 23. $7x+6y+2$.
24. $3a-2b+4$. 25. $(x-y)(x^6+x^3y^3+y^6)$.
26. $(x+y)(x^6-x^3y^3+y^6)$. 27. $x^2+y^2+2xy-x-y+1$.
28. $x^5+xy^4-yx^4-y^5$.
29. $x^6-3x^5+9x^4-27x^3+81x^2-243x-729$.
30. $1+a^2+4b^2+a+2b-2ab$.
31. $x^8+x^4y^4+y^8$. 32. $a^4-2a^3+4a^2-8a+17$.
33. $a^3+\frac{1}{2}a+\frac{1}{4}$. 34. $\frac{1}{4}x^2-\frac{1}{2}xy+\frac{2}{3}y^2$.
35. $2a^3-4a^2+2a-8$. 36. $6a-\frac{1}{3}b-\frac{1}{4}$.

37. $2x - \frac{1}{3}y + \frac{1}{3}$. **38.** $\frac{3}{8}x^2 - \frac{1}{4}x - \frac{2}{3}$.

39. $\frac{1}{4}a^2 + b^2 + \frac{1}{3} - \frac{1}{2}ab + \frac{1}{3}b + \frac{1}{3}a$.

40. $\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{1}{3}b^2$.

EXAMINATION PAPERS.

First Series.

1

1. (1) 0 ; (2) 0. **2.** $6x$.

3. $-2a^3 + 3a^2 - ab + 7a^2b^2 + 2ab^3 + 4a^3b$.

4. (1) $14x^4 - 3x^3 + x^2 + x - 45$; (2) $\frac{1}{8}a^4 - \frac{2}{15}a^3x + \frac{2}{5}a^2x^2 - \frac{1}{3}ax^2 + x^3$.

5. $(2a^2 - 5ab + 2b^2)$ annas.

2

1. $a^2 + b^2 + c^2 - ab - ac - bc$; 3.

2. $-4a + 6b + 3c$. **4.** $a^6 - 4a^4 + 4a^2 - 16$.

5. $\frac{1}{2}, -2\frac{1}{2}, -7, -4$. **6.** $a^2 + b^2 + c^2 + ab - ac + bc$.

3

1. 0 ; $a + b + c$. **2.** -3.

3. $ab(a+b) + bc(b+c) + ca(c+a) - a^3 - b^3 - c^3 - 2abc$.

4. $\frac{(2x^2 + 7xy + 6y^2)}{9}$ sq. yds.

5. $(x-1)(x-3)$.

6. where $y = 2x^2 - x + 3$.

x	0	-1	2	-4
y	3	6	9	39

4

1. -132. **2.** $5x^2 - 3x + 3$.

3. $1 - 2x + 2x^2 - 2x^3 + 2x^4$.

4. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$; $-a^2 - b^2 - c^2 - 2ab - 2ac - 2bc$.

6. 0, 10, 36.

5

2. 0

3. $(x+2y)^2-(x+2y)(3x+z)+(3x+z)^2$.

5. $-x+5y-4z$. 6. $4ab$.

6

1. $x^3(a-1)+x^2(a-b)+x(b+c)+5$.

2. $a^5+2a^4x+9a^3x^2+18a^2x^3+81ax^4+162x^5$.

3. $a^2+4b^2+c^2-4ab+2ac-4bc$.

4. $-6, -21, -5, -15, -26$. 5. 0. 6. 0.

7

1. $3a^3-7a^2b+5ab^2-b^3$.

2. $3x^2-10x+9$. [$a-2$ should be $x-2$].

3. $1, 1\frac{2}{3}, \frac{17}{9}, 2\frac{1}{3}$. 4. $-34x^2-40x-25$.

5. $\frac{(x+1)}{3}$ yds. 6. $12(20x+y)$ pence.

8

1. $3\frac{1}{2}, 2, -13$. 2. $x+1$.

3. $4x^2-29x+50$. [$a-3$ should be $x-3$.]

4. a^8-b^8. 5. $12(x^2+1)$.

EXERCISE 24.

1. (*a*) 10. (*b*) 5. (*c*) 5. (*d*) -4.

2. (*e*) -9. (*f*) -2. (*g*) -5. (*h*) 4.

3. (*k*) $\frac{1}{10}$. (*l*) $\frac{7}{50}$. 2. 10. 3. 17.

4. 13. 5. 30. 6. 5. 7. 6

8. $4\frac{3}{4}$. 9. 12. 10. 7. 11. 10

12. 4. 13. 4. 14. 2. 15. -

16. 14. 17. 3. 18. 2. 19. 2

20. $3\frac{1}{2}$. 21. $1\frac{1}{5}$. 22. 7. 23. 5.

24. $1\frac{3}{4}$. 25. 6. 26. 50. 27. 21.

28. $\frac{9}{16}$. 29. $\frac{7}{3}$. 30. $\frac{9}{49}$. 31. $2\frac{3}{4}$.

32. 6. 33. $\frac{1}{4}$. 34. $2\frac{2}{3}$. 35. 0.

36. 1. 37. $3\frac{16}{55}$. 38. -4. 39

40. $2\frac{3}{10}$. 41. $4\frac{3}{4}$. 42. $2\frac{1}{3}$. 43. $7\frac{2}{17}$.

44. 12. 45. $1\frac{1}{2}$. 46. $-\frac{3}{10}$. 47. $\frac{1}{2}-$

48. 1. 49. $10\frac{7}{10}$. 50. $2\frac{1}{2}$. 51. 0.
52. $-1\frac{3}{5}$. 53. $\frac{1}{8}$. 54. $-7\frac{1}{2}$. 55. $2\frac{1}{4}$.
56. $-1\frac{1}{4}$. 57. -20. 58. $12\frac{1}{2}$. 59. 3.
60. $\frac{5}{9}$. 61. $\frac{1}{2}$. 62. -3. 63. $\frac{3}{4}$.
64. $-6\frac{1}{2}$. 65. -5.

EXERCISE 25.

1. $\frac{2}{3}$. 2. $\frac{3}{8}$. 3. 10. 4. $2\frac{4}{5}$.
5. $2\frac{1}{2}$. 6. $2\frac{1}{2}$. 7. 24. 8. $3\frac{2}{3}$.
9. 8. 10. 3. 11. 21. 12. $1\frac{7}{11}$.
13. 16. 14. $-218\frac{3}{5}$. 15. $1\frac{1}{2}$. 16. $-8\frac{1}{7}$.
17. $-9\frac{6}{11}$. 18. $11\frac{13}{17}$. 19. $-2\frac{1}{7}$. 20. -3.
21. -1. 22. $5\frac{5}{6}$. 23. $-\frac{3}{4}$. 24. $\frac{1}{2}$.
25. 36. 26. 13. 27. $41\frac{1}{7}$. 28. 7.
29. 13. 30. 10. 31. 15. 32. 8.
33. 9. 34. $-10\frac{60}{77}$. 35. $1\frac{1}{2}$. 36. $5\frac{1}{2}$.
37. -8. 38. 1. 39. 7. 40. 1.

41. 5

EXERCISE 26.

1. $p-q$. 2. $25-a$. 3. $x-2$.
4. $15+x$. 5. $m-n$. 6. $s-a-b$.
7. $c-(a+b)$. 8. $p+q-3$. 9. $4a+5c$.
10. $15+x$. 11. $\frac{y}{x}$. 12. $4x$.
13. $4x$. 14. $\frac{y}{x}$. 15. $\frac{a+b}{c}$.
16. (1) $10+x$. (2) $10-y$. 17. $x+15$.
18. $30+a-x$. 19. $2x+2$. 20. $25x$ yrs.
21. $x+1$; $x-1$. 22. $a+2$, $a+1$, a. 23. $b+1$, b, $b-1$
24. x, $x-1$, $x-2$, $x-3$.
25. $a+b$, $a+b+1$, $a+b+2$, $a+b+3$.
26. Eldest $-2x$, Middle $-\frac{1}{3}x+1$.
27. 10 yrs. hence A's age $=x+10$ yrs. and
B's age $=\frac{1}{4}x+10$ yrs ;
5 yrs. ago A's age $=x-5$.
B's age $=\frac{1}{4}x-5$.

28. mx miles. **29.** $\frac{x}{y}$ mi. an hr.

30. $6x$ mi. **31.** $x+3$ mi. ; $(x+3)\,y$ mi

32. $(p+q)\,a$ mi. **33.** $10-2\,(x+y)$ mi

34. (1) $x-y$ mi. ; (2) $3\,(x-y)$ mi. (3) $m\,(x-y)$ mi.

35. $\frac{c}{x-y}$ hrs. **36.** $\frac{4}{a+b}$ hrs.

37. $(192a+12b+c)$ pies.

38. $(a+\frac{b}{15}+\frac{c}{20})$ pounds. **39.** (1) $7m$ (2) $\frac{mx}{12}$ (3) my

40. $(192x-12y-z)$ pies.

41. $3a+4b$; $\frac{3a+4b}{a+b}$. **42.** (1) $\frac{144}{a}$. (2) $\frac{144c}{ad}$

EXERCISE 27.

1. $10x+4$. **2.** $100p+10q+r$. **3.** $120x+21y$.

4. $1111x+3211$. **5.** $100z+10y+x$. **6.** $\frac{tm}{p}$ men.

7. $\frac{ab}{a+b}$. **8.** $\frac{mx-ny}{m-n}$. **9.** $\frac{38b-ax}{38-x}$.

10. $\frac{ax+by+cz}{a+b+c}y$. **11.** $\frac{3}{4}\,(20x+y-10)$. **12.** $a-b=c$.

13. $y-x=\frac{1}{10}x$. **14.** $\frac{ax}{12}+\frac{by}{12}+p=\frac{1}{2}(a+b)$

15. $-5x-30=a$. **16.** $3\,(p+1)\,(p+2)=a$.

17. $5x-11=y$. **18.** $(10x+y)\,(10y+x)=253$.

19. $100z+10y+m=3\,(100m+10y+z)$ *i.e.*, $97x-209=229$

20. $9y-9x=d$. **21.** $(2x+10)-(x+10)=10$.
i.e., $x=10$.

22. $x-5=\frac{1}{5}\,(3x-5)$.
i.e., $2x-20=0$ **23.** $(x+y)\,t=a$

24. $t\,(x-y)=2$. **25.** $bn-am=3$.

26. $x-(a-b)=x+(c+d)$ **27.** $1200-(20x+12y+120)=c$

28. $\left(\frac{x}{2}-\frac{x}{3}\right)-\left(\frac{x}{5}+\frac{x}{4}\right)=y$ **29.** $\frac{x}{3}+\frac{x}{4}+a=3$.

EXERCISE 28

1. (i) 75 sq. in. (ii) 4 ft. (iii) 5 ft.

2. (i) $50\frac{2}{7}$ sq. in (ii) 7 ft. (iii) 6·2 cm.

3. (i) 264 in. (ii) 15 ft. (iii) 10·4 ft.
4. (i) 99·3896 sq. in. (ii) 5.
5. (i) 4928 c. in. (ii) 21·8 ft. (iii) 65·4 ft.
(iv) 6·57 ρ. (v) 21703·7 c. in.
6. (i) 400 yds. (ii) 7200 mi. (iii) 3600.
(iv) $14\frac{2}{3}$ ft. per sec. (v) 90 in.
7. (i) 438. (ii) 235·32.
8. (i) 210. (ii) 5050. (iii) 266. (iv) 7650.
9. (i) 210. (ii) 820. (iii) 4371.
(iv) 192. (v) 245. (vi) 693.
10. (i) 4. (ii). $d=2\frac{1}{2}$. **11.** 3600. **12.** 11.
13. (i) £54-15*s*. (ii) Rs. 57. (iii) £145.
(iv) Rs. 385. (v) $2\frac{1}{2}$.
14. (i) 792 sq. ft. (ii) 18 ft. (iii) 40 yds.

EXERCISE 29

1. 7, 28. **2.** 245. **3.** 12.
4. 210. **5.** 2,16. **6.** 10
7. 8, **8.** 60, 90. **9.** 38, 32.
10. A's share = Rs. 40. **11.** A's share = £ 19-13-0.
B's ,, = Rs. 50. B's ,, = ,, 4 -9-0
C's ,, = ,, 1-13-0.
12. 8, 9, 10. **13.** 5, 6, 7, 8, 9, **14.** 12, 13, 14, 15.
15. 24, 26, 28. **16.** 3, 5, 7, 9. **17.** 5, 7, 9.
18. 2, 7, 12, 17. **19.** 25, 35. **20.** 30, $7\frac{1}{2}$, 6.
21. 25. **22.** 34. **23.** 42.
24. 23. **25.** 5, 11. **26.** 100, 80.
27. 48. **28.** 9, 14. **29.** 54.
30. 10 yrs., 6 yrs.

EXERCISE 30.

1. 70. **2.** Rs. 600, Rs. 50. **3.** £ 1-15-0.
4. £ 10-2-0, £ 5-1-0, £ 4-16-0.
5. 26. **6.** 35. **7.** 24.
8. 20 yrs. **9.** 11 days. **10.** 99 miles.
11. When the father's age is 62 years.

12 A's share = Rs. 3,500.
B's " = " 4,500.
C's " = " 7,200.

13 3 miles an hour.

14 4 sovereigns, 8 half-crowns, 12 six pences.

15. 20 rupees, 60 two-anna coins and 30 one-anna coins.

16 5 sovereigns, 40 crowns.

17 2 sovereigns, 6 crowns and 8 florins.

18. A has £8 and B £ 6. **19.** Rs. 38.

20. A with Rs. 32, B with Rs. 16. **21.** 15 ft., 12 ft.

22. 400 sq. ft. **23.** 200 sq. ft.

24. 180 yds., 90 yds. **25.** 15 ft., 12ft.

26 $1\frac{5}{7}$ days. **27.** 10, 20. **28.** 4.

29. Carriage £ 45, horse, £ 60 ; harness £ 15.

30 60. **31.** 34, 16. **32.** 40, 10.

EXERCISE 31.

1 A (5, 5), B (10, 10), C (15, 5),
D (10, 0), E (−5, 2), F (−10, −2),
G (−5, −5) H (−12, −13), K (0, −13),
L (0, 3), M (10, −19), N (14, −15),
O (0, 0), P (10, −10).

2 D (1, 1), B (2 2), C (3. 1),
D (2, 0), G (−1, −1), H (−2·4, −2·6),
N (2·8, −3).

3. A (5·5), B (1, 1), C (1·5, ·5),
D (1, 0), L (0, ·3), E (−·5, ·2),
K (0, −1·3), H (−1·2, −1·3), N (1.4, −1·5),
P (1, −1).

5 (i) (2·4, 6); (ii) 28·45, 6·15. (iii) 3·25, 3·25

6. $(6, 7\frac{1}{7})$, (−9, 5), $(-7\frac{1}{2}, -7\frac{1}{2})$.

7 (i) $13\frac{1}{2}$, 15. (ii) $-2\frac{1}{2}$, $-2\frac{1}{2}$. (iii) 0, $7\frac{1}{2}$.

11 The points lie on a parallel to the axis of x at a distance of 4 units from it.

12 The points lie on a parallel to the axis of x at a distance of −5 units from it.

13 11·2, 15·5, 22·7 units.

15 (i) 30 sq. units. (ii) 40 sq. units (iii) 39 sq. units

16. 137 sq. units. **17.** (i) 90 sq. units (ii) 40 sq. units.
18. 264 sq. units **19.** 32·9 units. **20.** 9·5 miles nearly
21. 13 units.

EXERCISE 32.

4. (10, 25) lies on (ii)
(−5, −2) ,, (ii)
(−10, 25) ,, (i).

EXERCISE 33.

11. $x=-10$. **12.** $y=9$. **13.** $x=3y$.
14. $x+y=8$. **35.** 1·4
36. The value of the function is −1; when $x=6$; (i) $\frac{1}{2}$; (ii) $2\frac{1}{3}$
36. $x=2$, $y=2$. **37.** $x=1\frac{1}{2}$, $y=-2\frac{1}{2}$.
38. $x=4$, $y=5$. **39.** $x=2\frac{1}{2}$, $y=1\frac{1}{2}$.
40. $x=10$, $y=-12$. **41.** $x=2$, $y=3$.
42. $x=5$, $y=3$. **43.** $x=2$.
44. $x=12$. **45.** 10.
46. $3\frac{1}{2}$. **49.** (4, 4), $(1\frac{1}{2}, 1\frac{1}{2})$, (3, 2).
50. (1, −1), (6, 2), (2, −1).

EXERCISE 34.

1. $16x^2+24x+9$. **2.** $121x^2+176x+64$.
3. $9a^2+48ab+64b^2$. **4.** $16m^2+48mn+36n^2$.
5. $a^2x^2+4abxy+4b^2y^2$. **6.** $9a^2b^2c^2+6abc^3+c^4$.
7. $a^2x^2+2abcxy+b^2c^2y^2$. **8.** $m^2p^2+2mnqp+q^2n^2$.
9. $\frac{25}{49}x^2+\frac{6}{7}xy+\frac{25}{49}y^2$. **10.** $9a^4+42a^2b^2+49b^4$.
11. $9p^6+6p^3q^2+q^4$. **12.** 10404.
13. 10060·09. **14.** 360480·16.
15. $25x^2-30x+9$. **16.** $81x^2-126x+49$.
17. $a^2b^2-2a^2bc+a^2c^2$. **18.** $25a^2+40ab+16b^2$.
19. $16p^2-24pq+9q^2$. **20.** $a^2b^2+2abcd+c^2d^2$.
21. $\frac{1}{9}a^2-ab+\frac{9}{16}b^2$. **22.** $b^4c^2-2a^2b^2cd+a^4d^2$.
23. $p^6-4p^3qr+4q^2r^2$. **24.** $a^2b^2c^2+2abcde+d^2e^2$.
25. $16a^6-40a^3c^3+25c^6$. **26.** 996004.
27. 999600·04. **28.** 63·936016.
29. $a^2+9b^2+16c^2-6ab-8ac+24bc$.

30. $9x^2+16y^2+25z^2-24xy-30xz+40yz$.
31. $16p^2+9m^2+n^2-24pm-8pn+6mn$.
32. $\frac{1}{4}a^2+\frac{1}{9}b^2+\frac{9}{16}c^2-\frac{2}{3}ab-\frac{3}{4}ac+bc$.
33. $a^2b^2+b^2c^2+c^2a^2+2abc(a-b-c)$
34. $a^4-2a^3-a^2+2a+1$.
35. $a^4+4b^4+9c^4-4a^2b^2-6a^2c^2+12b^2c^2$.
36. $x^2+b^2+x^2+y^2-2ab+2ay+2ay-2bx-2by+2xy$.
37. $4a^2+9b^2+16x^2+25y^2-12ab-16ax+20ay+24bx-30by-40xy$.
38. $\frac{1}{4}a^2+\frac{1}{9}b^2+\frac{1}{16}c^2+\frac{1}{25}d^2-\frac{1}{3}ab-\frac{1}{4}ac-\frac{1}{5}ad+\frac{1}{6}bc+\frac{2}{15}bd+\frac{1}{10}cd$.
39. $a^4+4b^4+4c^4+d^4+4a^2b^2-4a^2c^2-2a^2d^2-8b^2c^2-4b^2d^2+4c^2d^2$.
40. $a^2b^2+a^2c^2+b^2c^2+2abc(a+b+c)$.
41. $a^2+9b^2+16c^2+6ab+8ac+24bc$.
42. $4m^2+9n^2+16p^2+12mn+16mp+24pn$.
43. $\frac{1}{4}a^2+\frac{1}{9}b^2+\frac{1}{16}c^2+\frac{1}{3}ab+\frac{1}{4}ac+\frac{1}{6}bc$.
44. $x^4+y^4+c^4+2x^2y^2+2x^2c^2+2y^2c^2$.
45. $a^4+b^6+c^8+2a^2b^3+2a^2c^4+2b^3c^4$.
46. $a^2+b^2+4x^2+9y^2+2ab+4ax+6ay+4bx+6by+12xy$.
47. $16a^2+9b^2+4c^2+d^2+24ab+16ac+8ad+12bc+6bd+4cd$.
48. $x^2+4y^2+9z^2+16v^2+4xy+6xz+8xv+12yz+16yv+24zv$.
49. $16m^2+9x^2+4p^2+9q^2+24mn+16mp+24mq+12pn+18nq+12pq$.
50. $3(a^2+b^2+c^2)-2(ab+ac+bc)$.
51. $14(a^2+b^2+c^2-ab-ac-bc)$.
52. $4(a^2+b^2+c^2+d^2)$.
53. $a^4-4a^2b+2b^2$.
54. $x^4+4x^2y+2y^2$.
55. $4x^2+4z^2+8xz$.
56. $4(a^2+b^2+c^2+d^2)-8(ac+bd)$.
57. 41^2.
58. 0.

EXERCISE 35.

1. $2(9a^2+49)$.
2. $\frac{1}{2}a^2+\frac{8}{9}b^2$.
3. $2(a^4+b^4)$.
4. $2(a^6+b^6)$. [b^2 should be b^3]
5. $96a$.
6. xy.
7. $4p^2q^2$.
8. $4x^2y^2$.
9. $2\{(a^2+b^2)^2+a^2b^2\}$.
10. $2(x^2+y^2)^2$.
11. $2\{a^2+(b-c)^2\}$.
12. $4xy(x^2+y^2)$.
13. $8ab(a^2-b^2)$.
14. $4x(z-y)$.

EXERCISE 36.

1. $a^2b^2-c^2d^2$.
2. $25a^2-49$.
3. $a^2x^2-y^4$.
4. $x^2y^2-y^2z^2$.
5. $16x^2-25y^2$.
6. $9x^4-49b^4$.
7. $16x^2y^2-25m^2n^2$.
8. $p^4-9q^2r^2$.
9. a^4-b^4.
10. $^4-1$.
11. $x^2-(y-z)^2$.
12. $(x+z)^2-y^2$.
13. $p^4+p^2q^2+q^4$.
14. a^8+a^4+1.
15. $(x^2-y^2)^2$.
16. $4a^2-9b^2-c^2+6bc$.
17. $a^2+b^2-c^2-d^2+2ab+2cd$.
18. $a^8+a^6+a^4+a^2+1$.
19. a^2+ab+b^2.
20. $x^2+2x\surd x+x-1$.
21. x^4+9x^2+81.
22. $(a^4-1)^2$.
23. x^8+81x^4+6561.
24. $a^8+388a^4+20736$.
25. 39991.
26. 2·8896.
27. 399·999984.

EXERCISE 37.

1. $64a^3+144a^2+108a+27$.
2. $27x^3+54x^2y+36xy^2+8y^3$.
3. $\frac{8}{27}a^3+2a^2b+4\frac{1}{2}ab^2+3\frac{3}{8}b^3$.
4. $64b^3-240b^2a+300ba^2-125a^3$.
5. $a^3b^3-3a^2b^2cd+3abc^2d^2-c^3d^3$.
6. $8p^3-60p^2q+150pq^2-125q^3$.
7. $\frac{1}{27}x^3-\frac{1}{8}x^2y+\frac{1}{4}xy^2-\frac{1}{8}y^3$.
8. $x^6+16x^4y+6x^2y^2+8y^3$.
9. $m^3a^3x+3m^2xa^2x+mx+3mx^2m+x+x^3a^3m$.
10. $8x^3a-36x^2ayb+54xay^2b-27y^3b$.
11. $x^3y^3+y^3z^3+z^3x^3+3xyz(x+y)(y+z)(z+x)$.
12. $a^3b^3-b^3c^3+a^3c^3+3abc(a-c)(a-b)(c+b)$.
13. $a^6-b^6-c^6-3(a^2-b^2)(a^2-c^2)(b^2+c^2)$.
14. $a^3+8b^3-c^3+3(a+2b(2b-c)(a-c)$.
15. $p^6+q^6-r^6+3(p^2+q^2)(q^2-r^2)(p^2-r^2)$.
16. $a^3+8b^3+27c^3+3(a+2b)(2b+3c)(3c+a)$.
17. $\frac{1}{8}a^3-\frac{1}{27}b^3-\frac{1}{64}c^3-3(\frac{1}{2}a-\frac{1}{3}b)(\frac{1}{2}a-\frac{1}{4}c)(\frac{1}{3}b+\frac{1}{4}c)$.
18. 10,61,208.
19. 98,80,47,936.
20. 127·26 4527.
21. 976191·488.
22. 217.
23. 504.
24. 18.

25. 140. **26.** 125. **27.** 8.

28. 27. **29.** 125.

30. $3(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+2abc)$.

31. $3(xy^2-x^2y+yz^2-y^2z+zx^2-z^2x)$. **32.** $125(a-b)^3$.

33. $(x-y)^3$. **34.** $8c^3$ [$(a+b+c)^2$ should be $(a+b+c)^3$].

35. $32y^3+ax^2y$.

EXERCISE 38.

1. x^3+8. **2.** x^3+27. **3.** $8a^3+1$.

4. $a^3b^3+c^3$. **5.** $125x^3+8y^3$. **6.** $64a^3+343b^3$.

7. $1-27x^3$. **8.** $a^3-8b^3c^3$. **9.** x^3-8.

10. $m^3-64p^3n^3$. **11.** $8a^3b^6-1$. **12.** $a^{12}-b^{12}$.

13. $a^3-\frac{1}{8}b^3$. **14.** $\frac{8}{27}a^3+\frac{27}{8}b^3$. **15.** $a^{\frac{3}{2}}-b^{\frac{3}{2}}$.

16. $27x^{\frac{3}{2}}-64y^{\frac{3}{2}}$. **17.** $\frac{m^6}{64}-\frac{n^6}{729}$ [$\frac{1}{3}x^2$ should be $\frac{1}{9}x^2$]

18. $a^6x^6-b^6y^6$. **19.** $729x^6-64y^6$.

EXERCISE 39.

1. $x^2+12x+35$. **2.** x^2-x-6. **3.** x^4-8x^2+15.

4. $x^2+xy-12y^2$. **5.** $4x^2+4x-15$. **6.** $a^2x^2-ax-156$.

7. $4x^2-\frac{1}{3}xy-\frac{1}{12}y^2$. **8.** $6x^2+17xy+12y^2$.

9. $x^3+9x^2+26x+24$. **10.** $x^3+4x^2-20x-48$.

11. $x^3+4x^2-47x-210$. **12.** $8x^3-26x+12$.

13. $x^3-9x^2+23x-15$.

14. $xyz-(xy+yz+zx)+(x+y+z)-1$.

15. $x^3+\frac{13}{12}x^2+\frac{3}{8}x+\frac{1}{24}$.

16. $x^3+x^2(ab+bc+ca)+abcx(a+b+c)+a^2b^2c^2$.

17. $2+\frac{l}{m}+\frac{m}{n}+\frac{n}{l}+\frac{l}{n}+\frac{n}{m}+\frac{m}{l}$. **18.** $\frac{q^2}{pr}+\frac{q}{p}+\frac{qr}{p^2}-\frac{q}{p}-\frac{q}{r}-\frac{r}{p}$

19. $x^3-x^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+x\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)-1$.

20. $x^3+8x^2+21x+18$. **21.** $x^3-5x^2-8x+48$.

22. $8x^3+24x^2+22x+6$. **23.** $27x^3-54ax^2+33a^2x-6a^3$.

24. $6a^3-25a^2+34a-15$. **25.** $24x^3+46x^2+29x+6$.

26. $a^6-6a^4+11a^2-6$. **27.** x^4-10x^2+9.

28. $a^6-14a^4+49a^2-36$.

29 $x^3+2x^2(a+b+c)+x(a^2+b^2+c^2+3(ab+bc+ca)$
$+(a+b)(b+c)(c+a)$

30 1061106. **31.** 941094 **32.** 84842505.

EXERCISE 40.

1 $x^6+y^6+z^6-3x^2y^2z^2$. **2.** $a^3-b^3-c^3-3abc$.

3. $8x^3-27y^3-z^3-18xyz$. **4.** $27p^3-125q^3-64-180pq$.

5. $m^3-n^3+1+3mn$. **6** $a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2$.

7. $x^2+\frac{1}{x^3}-2$. **8.** $4-\frac{l^3}{m^3}+\frac{m^3}{l^3}$.

9 $x^6-2+\frac{1}{x^6}$. **10.** $\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}-3$.

11 $\frac{x^3}{y^3z^3}-\frac{y^3}{z^3x^3}-\frac{z^3}{x^3y^3}-\frac{3}{xy}$

12 $2(a^3+b^3+c^3-3abc)$.

13 $(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a)$
or $2(a^3+b^3+c^3-3abc)$.

14. $(2x-y)^3+(2y-z)^3+(2z-x)^3-3(2x-y)(2y-z)(2z-x)$
or $7(x^3+y^3+z^3-3xyz)$.

EXERCISE 41.

1 $a^2+b^2+3ab+a^2b+ab^2+a+b$.

2 $\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)+1$.

3 $(x+2y+z)(2x+y+z)(x+y+2z)+(x+y)(y+z)(z+x)$.

4 $(x-z)(x+y-2z)(2x-y-z)+(x-y)(y-z)(x-z)$.

5 $-2(x+y+z)^3+9(x+y+z)(xy+yz+zx)-27xyz$.

6 0, [$c-a$ should be $c-b$] **7.** 0.

8 $x+y+x^2(y+1)+y^2(x+1)+2xy$ [a should be 1].

9 $2(b+c)^3+bc(b+c)$.

10 $-(2a-3b-4c)(12bc-6ab-8ac)+24abc$.

11 $(\frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z)(\frac{1}{6}xy+\frac{1}{8}xz+\frac{1}{12}yz)-\frac{1}{24}xyz$.

12 $-(2a+b+c)^3(bc-(b+c)(2a+2b+2c)^2+bc(2a+2b+2c)$

13 $x^3-8y^3-27c^3+4y^2(x-3c)+9c^2(x-2y)-x^2(2y+3c)$.

14 $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)-1$

15 $(a-2b+c)(b-2c+a)(2a-b-c)$.

EXERCISE 42.

1. $b(a+b+c)$.
2. $x(x+y+z)$.
3. $a(a-b-c)$.
4. $bc(a-d+b)$.
5. $abc(a+b-c)$.
6. $(a+b)(a+b-2)$.
7. $(x-y)\{1-4(x-y)\}$.
8. $(c-a)(c-b)$.
9. $x^2y(x^3-4xy-3y)$.
10. $an(a-1)$.
11. $4x(1+4x)$.
12. $2n(1+2n)$.
13. $3^{x+1}(1-3\cdot2^x+2)$.
14. $(b-c)\{a^2+bc(b-c)+(b-c)\}$.
15. $(x-y)\{(x-y)^2-2(x-y)-4\}$. [Read $y-x$ for $y+x$]

EXERCISE 43.

1. $(a+2b)^2$.
2. $(2x-3y)^2$.
3. $(3m+4n)^2$.
4. $(4p-5q)^2$.
5. $(5ac+bd)^2$.
6. $2(6x+7y)^2$.
7. $x^2(3x+7)^2$.
8. $\frac{1}{3}(9a-b)^2$.
9. $(\frac{2}{3}x-\frac{3}{4}y)^2$.
10. $\frac{1}{3}ab(2a^2-3b^2)^2$.
11. $(4l+n-m)^2$.
12. $(an+bn)^2$.
13. $x^m-x^n)^2$.
14. $(2x+2y)^2$.
15. $(3^2n-2n)^2$.
16. $4a^2$ or $4b^2$.
17. $4c^2$ or $4(a-b)^2$.
18. $49(a-b)^2$ or $(a+b)^2$.
19. $4a^2b^2$ or $4c^2(a-b)^2$.
20. $(a^2+b^2-2c^2)^2$ or $9(a^2-b^2)^2$.
21. 1. **22.** 4. **23.** 0. **24.** 1. **25.** 81.

EXERCISE 44.

1. $(a^4+9)(a^2+3)(a^2-3)$.
2. $(x^8+16)(x^2+2+2x)(x^2+2-2x)(x^2+2)(x^2-2)$.
3. $(5a-3)(5a+3)$.
4. $(a^2+9b^2)(a+3b)(a-3b)$.
5. $(7c-8d)(7c+8d)$.
6. $(2ax-3b)(2ax+3b)$.
7. $(a-b+c)(a+b-c)$.
8. $(a+b-c-d)(a+b+c+d)$.
9. $(3x+4y-7z)(3x+4y+7z)$.
10. $4(3a+b)(6b-a)$.
11. $(p-12q)(7p+2q)$.
12. $4a(b-c)$.
13. $8(a-b+c)(c-a)$.
14. $4(3x-z)(3y-3z-x)$.
15. $(2p-q-3r)(4p-7q+5r)$.
16. $(x^2+8+4x)(x^2+8-4x)$.
17. $(2a^2+9+6a)(2a^2+9-6a)$.
18. $(a^2+2b^2+2ab)(a^2+2b^2-2ab)$.
19. $(x^2+x+1)(x^2-x+1)$.
20. $(x^4-x^2+1)(x^4+x^2+1)$.
21. $(x^2+xy+y^2)(x^2-xy+y^2)$.
22. $(x^2+3+2x)(x^2+3-2x)$.
23. $(x^2-3+x)(x^2-3-x)$.

24 $(2a^2+3+3a)(2a^2+3-3a)$.
25. $(x^2+12+4x)(x^2+12-4x)$.
26 $(3x^2-4+3x)(3x^2-4-3x)$.
27 $(3x^2+4+5x)(3x^2+4-5x)$.
28 $(3a^2+5x^2+7ax)(3a^2+5x^2-7ax)$.
29 $(3x^2-4+x)(3x^2+4-x)$. **30.** $(x-2y+3z)(x-2y-3z)$.
31 $(a+2b+c)(a+2b-c)(c+a-2b)(c-a+2b)$.
32. $(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$.
33 $(a-c)(a+c-2b)$. **34.** $(a-d+b-c)(a-d-b+c)$.
35 $(a^2+c^2+b^2-2ac)(a+b+c)(a+c-b)$.
36 $(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$.
37 0. **38.** 0. **39.** 986820. **40.** 13732.

EXERCISE 45.

1. $(a+3)^3$, **2.** $(3x-1)^3$. **3.** $(ab-bc)^3$.
4 $(2p+\frac{1}{3}q)^3$. **5.** $(\frac{1}{3}b^2-c^3)^3$. **6.** $(\frac{1}{2}m-\frac{1}{3}n)^3$.
7. $(1+\frac{2}{3}x)^3$. **8.** $(l-m+n)^3$. **9.** $(2n+3n)^3$.
10. $(x+1)^3$. **11.** $8(a^2+b^2)^3$. **12.** $(a-c)^3$.
13. $-(a+b)^3$. **14.** $8c^3$. **15.** 100,00,00.

EXERCISE 46.

1. $(x-2)(x^2+2x+4)$. **2.** $(a+3)(a^2-3a+9)$.
3 $(6bc-a)(36b^2c^2+6abc+a^2)$.
4 $(4a-3bd)(16a^2+12abd+9b^2d^2)$.
5. $x^2y(a+3y)(a^2-3ay+9y^2)$.
6. $(x-y)(x^2+x^2y+xy^2-y^3)$.
7. $x(x+y)(x^2-3xy+3y^2)$.
8. $(a^2+bc)\{(a^2-bc)^2-2bc(a^2-bc)+4b^2c^2\}$.
9. $(x-1)(x^2+x+1)(x^3+2)$.
10 $(\frac{2}{3}a-\frac{3}{4}b)(\frac{1}{9}a^2+\frac{1}{2}ab+\frac{9}{16}b^2)$.
11. $(3x-4y)(3^2x+3\cdot4y+4^2y)$. [$27x-64y$ should be $27x-64y^3$].
12 $3(3a-5)(9a^2+15a+25)$.
13 $3n(2n+3n)(2^2n-6n+3^2n)$.
14. $(a+1)(a^3-4a+7)$.
15 $(a-c)\{(a+b)^2+(a+b)(b+c)+(b+c)^2\}$.
16 $(a+b)(37ab-19a^2-19b^2)$.
17. $(2m-5n+3p)(4m^2+7n^2+9p^2-2mn-12np-6pn)$.

18. $(a-2)(a^2+5a+13)$. 19. $(2a+1)(a^2+a+1)$.

20. $(y+z)\{(x+y+z)^2+x(x+y+z)+x^2\}$.

EXERCISE 47.

1. $(x+3)(x-2)$. 2. $(x+9)(x+16)$.
3. $(x-5)(x-1)$. 4. $(x-6y)(x-2y)$.
5. $(x+1)(14x+1)$. 6. $(x+a)(x+2b)$.
7. $(x-a)(x-a-b)$. 8. $(x-b+a)(x-b-a)$.
9. $\{x-(a+b)^2\}\{x-(a-b)^2\}$. 10. $(x-a)(x-b-c)$.
11. $\left(x+\frac{a}{b}\right)\left(x+\frac{b}{a}\right)$.
12. $(x+a+b-c)(x+a-b+c)$. 13. $(x-ab+bc)(x+ab+ac)$.
14. $(x-c)(x+a+b)$. 15. $(x+7)(x-2)(x+3)(x+2)$.
16. $(a^2-3a+1)(a+1)(a-4)$. 17. $(x^2+2x-2)(x+1)^2$.
18. $(m+6n)(m-5n)$. 19. $(x^2+7)(x+2)(x-2)$.
20. $(a^3-2)(a-2)(a^2+2a+4)$. 21. $(a^4+5)(a^2+4)(a-2)(a+2)$.
22. $(x+a^2+ab)(x+b^2+ab)$. 23. $(x+a^2-ab)(x+b^2-ab)$.
24. $(x-a+2b)(x-b+2a)$. 25. $(x-2z)(x+y-z)$.

EXERCISE 48.

1. $(7x+6y)(7x+y)$. 2. $3x(3x+5)$.
3. $(x+1)(2x-1)$. 4. $(4x+3)(2x-3)$.
5. $(2x-3y)(x+4y)$. 6. $(2x-5)(x+3)$.
7. $(3a-5)(2a+3)$. 8. $(3x-2)(2x-3)$.
9. $(2x+1)(4x-3)$. 10. $(3x-2a^2)(2x-5a^2)$.
11. $(4x+25)(5x+24)$. 12. $(4m+5n)(3m-4n)$.
13. $(5a-3b)(2a-7b)$. 14. $(6x-7)(3x-2)$.
15. $x^2(6a+x)(4a-3x)$. 16. $(7m-3n)(4m+5n)$.
17. $(11a+2b)(3a+5b)$. 18. $(7x+8y)(5x-6y)$.
19. $(7x-12y)(6x+11y)$. 20. $(7x-8y)(9x+13y)$.
21. $(12l-13m)(14l-15m)$. 22. $(15x+16)(17x+18)$.
23. $(8p^2+12p+3)(4p^2+6p-3)$. 24. $(2ab+2ca-c)(ab+ca-5c)$.
25. $(x^2-y^2-y-z)(x-y+1)$.

EXERCISE 49.

1. $(a+b)(c-d)$. 2. $(ax+y)(bx-y)$. 3. $2(x+2a)(x+3b)$.
4. $(2a-b)(a^2+2bx)$. 5. $(xy+a^2)(2x-3y)$.
6. $(x+1)(ax^2+bx+a)$. 7. $(1-x)(1-y)$. 8. $(x+1)(a+1)$.

9. $(a-b)(a-c)$. 10. $(p+q)(p+r)$. 11. $(x-y)(4+b)$.
12. $(a+b)(a+1)$.

EXERCISE 50.

1. $(x^2+ab)(3a-4b)$.
2. $(x+z)(x-z)(x^2+y^2+z^2)$.
3. $(x+y-z)(x-y+z+1)$.
4. $(x-y)(x-y-1)$.
5. $(a-b)(a+b+c)$.
6. $(x+a)^3(x-a)$.
7. $(v+y)(x-y)^3$.
8. $(b-c)(x+a)$.
9. $(a-b-c)(a+b+c+1)$.
10. $(a+b)(a^2+b^2)$.
11. $(a^2+b^2)(a-2b)$.
12. $(y+xz)(x+yz)$.
13. $(a+b)(c+a-b)(c-a+b)$.
14. $(x-y)(1+x)(1+y)$.
15. $(a+b)(a+b+1)$.
16. $(b+c)(a+b-c)(a-b+c)$.

EXERCISE 51.

1. $(b+c-a)(b^2+c^2+a^2-bc+ab+ac)$. [$a^2$ should be a^3]
2. $(c+1)(c^2-c+37)$.
3. $(c+4b)(c^2+7b^2-4bc)$
4. $(3a-b)(3a^2+b^2+3ab)$:
5. $(a-b-c)(a^2+b^2+c^2+ab+ac-bc)$.
6. $(x^2+y^2+z^2)(x^4+y^4+z^4-x^2y^2+x^2z^2+y^2z^2)$.
7. $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-\frac{a}{c}-\frac{c}{b}-\frac{b}{a}\right)$.
8. $(l-\frac{1}{2}m-\frac{1}{3}n)(l^2+\frac{1}{4}m^2+\frac{1}{9}n^2+\frac{1}{2}lm+\frac{1}{3}ln-\frac{1}{6}mn)$.
9. $(x+y+1)(x^2+y^2+1-x-y-xy)$.
10. $2a+2b+3c)(a^2+4b^2+9c^2-2ab-3ac-6bc)$.
11. $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.
12. $\left(x+\frac{1}{x}+1\right)\left(x^2+\frac{1}{x^2}-x-\frac{1}{x}\right)$.
13. $\left(p-\frac{1}{3p}-1\right)\left(p^2+\frac{1}{9p^2}+p-\frac{1}{3p}+\frac{1}{3}\right)$.
14. $(2a^2-3a+1)(4a^4+12a^3+5a^2+6a+1)$.
15. $(1+4a^2+3a^4)(1+16a^4+9a^8-4a^2-3a^4-12a^6)$.
16. $3(2+a+b)(2+b+c)(2+c+a)$.
17. $3(2-x-y)(2-y-z)(2-y-x)$.
18. $3(a^2+b^2-ac-bc)(b^2+c^2-ac-ab)(c^2+a^2-ab-bc)$.
19. $3(2x+y-z)(2x+z-y)(2z+x-y)$.
20. $3(a+2b+c)(b+2c+a)(c+2a+b)$.

EXERCISE 52.

1. $3(a^2-b^2)(b^2-c^2)(c^2-a^2)$.
2. $3(a+b-2c)(b+c-2a)(c+a-2b)$.
3. $3abc(a-b)(b-c)(c-a)$.
4. $3(x-a)(x-b)(x-c)(a-b)(b-c)(c-a)$.
5. $3(x+a)(x-b)(a+b)$.
6. $3abc\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{b}{c}-\frac{c}{b}\right)\left(\frac{c}{a}-\frac{a}{c}\right)$.
7. $3(x-2y)(2y-1)(1-x)$.
8. $3(ax-by)(by-cz)(cz-ax)$.
9. $3abc(bz-cy)(by-cz)(ay-bx)$.
10. $3(a-2b+c)(b-2c+a)(c-2a+b)$.
11. $3(a+2b-3c)(b+2c-3a)(c+2a-3b)$.
12. $-3c(s-a)(s-b)$.
13. $3(2a-b)(2b-c)(2c-a)$.
14. $3(3a-s)(3b-s)(3c-s)$.
15. $3a^3b^3c^3(a-b)(b-c)(c-a)$.

EXERCISE 53.

1. $x=5$; $y=3$.
2. $x=1$; $y=5$.
3. $x=3$; $y=2$.
4. $x=1$; $y=2$.
5. $x=3\frac{5}{14}$; $y=-\frac{4}{7}$.
6. $x=10$; $y=12$.
7. $x=\frac{1}{2}$; $y=\frac{1}{3}$.
8. $x=\frac{1}{4}$; $y=\frac{1}{5}$.
9. $x=-8$; $y=6$.
10. $x=3$; $y=-3$.
11. $x=-3$; $y=2$.
12. $x=3$, $y=-1$.
13. $x=10$; $y=4$.
14. $x=3$; $y=2$.
15. $x=5$; $y=-4$.
16. $x=5$; $y=6$.
17. $x=2$; $y=3$.
18. $x=1$; $y=2$.
19. $x=5$; $y=4$.
20. $x=4$; $y=-3$.
21. $x=8$; $y=3$.
22. $x=-1$; $y=2$.
23. $x=-3$; $y=4$.
24. $x=7$; $y=4$.
25. $x=6$; $y=-3$.
26. $x=4$; $y=8$.
27. $x=-\frac{1}{2}$; $y=2\frac{1}{2}$.
28. $x=5$; $y=-1\frac{5}{17}$.
29. $x=13$; $y=-7$.
30. $x=2$; $y=5$.
31. $x=1$; $y=2$.
32. $x=2$; $y=-2$; 16.
33. $x=3$; $y=4$; 2 and 59.
34. $x=\frac{1}{2}$; $y=\frac{1}{3}$; 12 and -1.
35. $x=\frac{1}{4}$; $y=\frac{1}{5}$; 0 and 3.
36. $a=3$; $b=-11$.

37. $m=\frac{1}{3}$; $c=1\frac{1}{3}$ (a and b should be m and c).

38. $a=-5\frac{1}{7}$; $b=\frac{5}{7}$.

39. $m=2$; $c=-1$.

40. $a=b=-1$.

41. $x=10$; $y=12$.

42. $x=16$; $y=12$.

43. $x=4$; $y=5$.

44. $x=\frac{440}{23}$; $y=\frac{425}{23}$.

45. $x=10$; $y=15$.

46. $x=15$; $y=16$.

47. $x=5$; $y=4$.

48. $x=-2$; $y=\frac{1}{2}$.

49. $x=2$; $y=3$.

50. $x=2$; $y=3$.

51. $x=3$; $y=5$. [1122 should be 40 and $-x+10$ should be $x+13$].

52. $x=\frac{1}{4}$; $y=\frac{3}{5}$.

53. $x=\frac{8}{1705}$; $y=\frac{203}{682}$. [$+\cdot06y$ should be $-\cdot06y$].

54. $x=7$; $y=1$.

55. $x=2$; $y=4$.

56. $x=3$; $y=4$.

57. $4y$ should be y.

58. $6x-1=a$; $x=2$; $y=5$.

59. $14-x=a$; $x=5$; $y=3$.

60. (i) $y+5z=0$ (ii) $x+4z=0$.

61. (i) $3y+z=2$ (ii) $3x-z=1$.

62. $x=5$; $y=10$; 20.

63. $x=10$; $y=8$; 24.

64.

EXERCISE 54.

1. 16 and 9.

2. 24 and 25.

3. 30 and 70.

4. 92 and 100.

5. 6 and 24.

6. 145° and 155°.

7. 20 and 25.

8. A, 10 Rs; B, 12 Rs.

9. A, 58 yrs.; B, 18 yrs.

10. A, 72 yrs.; B; 52 yrs.

11. Father, 30 yrs., son, 20. [*Twice the son's age by* 10 *years* should be *thrice the son's age by* 30 *years*].

12. $\frac{1}{18}$ [$\frac{11}{10}$ should be $\frac{1}{10}$].

13. $\frac{8}{30}$ [$\frac{1}{2}$ should be $\frac{1}{3}$].

14. $\frac{9}{10}$.

15. $\frac{5}{7}$.

16 54. [The second condition gives rise to the same equation; so it should be *the digit in the ten's place is greater than the digit in the unit's place by unity*.]

17. 47.

18. 64.

19. 96.

20. 12.

21. 6 yds., 4 yds. **22.** 16 ft., 12 ft.
[Omit *diminished* in the second line].
23. 20 ft., 12 ft. **24.** 180 sq. ft.
25. Tea, 2*s.* 4*d.* per lb; sugar, 3*d.* per lb.
26. Cow, Rs. 45; horse, 250 Rs. [5 horses should be 3 horses].
27. 15 beggars; 27*s.*
28. 9 men; £2,700.
29. A, 76 $\frac{16}{19}$ Rs.; B, 44$\frac{4}{19}$ Rs.
30. A, Rs. 20; B, Rs. 25.

EXAMINATION PAPERS.

Second Series

1.

1. $3a^3 - 29a^2 - 96a - 107$.
2. (i) $(2x+3)(3x-2)$; (ii) $5a(a-2b)$.
3. 0. **4.** $a = 10$ [*find n* should be *find a*]. **5.** 1.
6. $\left(\frac{b-a}{a}\right)$ 100 per cent.

2.

1. a^2+7. **2.** (i) $(x-10)(x+7)$; (ii) $(x+1)(x^2-x+2)$.
3. $2\left(\frac{x}{y}+\frac{y}{x}\right)$; $2(ab-mn)$. $\left[xy+\frac{1}{xy}\right.$ should be $\left.\frac{x}{y}+\frac{y}{x}\right]$.
4. $2(cx-ay-by)$ rupees.
5. $x = 5\frac{2}{3}$.

3.

1. $(a^4-a^2+1)(a^2+a+1)(a^2-a+1)$.
2. $2c-b$; b^2-2.
3. Father, 45; son, 15.
4. 6·5 ins. correct to $\frac{1}{10}$ of an inch.
5. $x=1$; $y=5$.
6. (2, 3).

4.

1. $x=9$. 2. $x=4$; $y=3$. 3 45.
4. 388114000000.
5. x^2+x-y^2-y; $(x-y)(x+y+1)$.
6. $(x+7)(x-2)(x+2)(x+3)$.

5.

2 (1) $(2a-3b+2c)(2a-3b-2c)$
(2) $(x^2+1)(x-1)(x^2+x+1)$
3 $x=8$; $y=10$.
4. $a^3-8a^2+23a-17$.
5. x^8+9x^4+81. [$3x$ should be $3x^2$].
6 Cow, 55 Rs.; horse, 170 Rs.

6.

2. (i) $a^3(a+1)^3$; (ii) $a(a+1)^2(a-1)(a^2+a+1)$.
3. (i) $x=1\frac{6}{67}$; (2) $x=0$; $y=3$.
4. $2a^2b^2+2a^2c^2+2c^2a^2-a^4-b^4-c^4$.
5. $\frac{4}{10}$.
6. $\frac{1}{8}(a+b+c)^3$. [$2a$ should be $2z$, $(x+2b)$ $(x+2c)$ should be $(y+2b)(z+2c)$ and *in terms of a* should be *in terms of a, b and c*].

7.

1. 48 and 20. [17 should be 27].
2. (1) $(ay-2bx)(by-2ax)$. (2) $(bx-a)(ax+b)$.
4. $h=12$. 5. A, 46 Rs; B, Rs. 30.
6. $a=7$; $b=-5$.

8

1. 3.
2. (i) $7x=4z$ (ii) $7y=3z$.
3. $a+x=14$; $x=5$; $y=3$.
4. 50 ft., 24 ft.
5. (i) $(x-y)^3(x+y)$ (ii) $(a+b)(a+b+1)$.

EXERCISE 55.

1. x^3-4x^2+7x-6.
2. $2x^4+7x^3-5x^2-14x-5$.

3. $3-8x-x^2(3+m)+(3m-5)x^3+15x^4$.
4. $36x^6-x^2+2x-1$.
5. x^8-2x^4+1.
6. $x^5-3x^4y+5x^3y^2-5x^2y^3+3xy^4-y^5$.
7. $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$.
8. $x^5+3x^4y+4x^3y^2+4x^2y^3+3xy^4+y^5$.
9. $1+18x+77x^2+40x^4+320x^5+300x^6$.
10. $\frac{1}{32}x^5+\frac{1}{8}x^3+\frac{1}{4}x^2+1$.
11. $6a^6-16a^4+15a^3-10a^2-10a+12$.
12. $a^7+a^5-2a^4-a^3(p+2)+a^2(p+1)+a(p-1)-1$.
13. $a^{16}+a^8+1$.
14. x^6-1.
15. $a^{16}+a^{12}+a^8+a^4+1$.
16. $x^6-a^2x^5+2abx^5-x^4(b^2+2ac)+2x^3(ad+bc)\quad x^2(c^2+2bd)$
$+2cdx-d^2$.

EXERCISE 56.

1. $ab-a^2(x+y)+b^2(x-y)-ab(x^2-y^2)$.
2. $x^2(a^2-b^2)+2bxy-y^2$.
3. $(m^2-n^2)x^2+2xy(an+bm)-y^2(a^2-b^2)$.
4. $x^3(a-b)+x^2(a^2-b^2-c)-2bcx-c^2$.
5. $(x-y)^3-(x-y)^2-2(x^2-y^2)+3x+y-1$.
6. $l^2(m^2-n^2)+l^2\{(m-n)^2+(m+n)\}+1$.
7. $a^3(x+y)+a^2y(x-2y)+a\{(x+y)^2-(x-y)(x^2-xy+y^2)\}$
$-(x^3+y^3)$.
8. $(a^3-b^3)x^3-bx^2(a+2b)-2bx-1$.
9. $(x^3-y^3)a^3+a^2(x^2-2xy+3y^2)-4ax-4$.
10. $(a^2-b^2)x^3+2x^2y(b^2-ac)+xy^2(a-b)(c+a)-(b^2-c^2)$
$+y^3(b-c)(c+a)$.
11. $\{(p+q)^2-r^2\}x^3+2rx^2(r-p+q)$
$+2x(pq+2pr-qr-q^2)+(p+q)^2-r^2$.
12. $m^4(x^2+2x-3)+2m^3(x^2+x-4)+m^2(3x^2-14)$
$+2m(x^2-x-4)+(x+1)(x-3)$.
13. $x^2(a^2-b^2)+y^2(b^2-c^2)+z^2(c^2-a^2)+2bxy(a-c)$
$+2cyz(b-a)+2azx(c-b)$.

14. $\{(a^3-1)(m^3-1)-(a^2-1)(m^4+m^2+1)\}p$
$+\{(m^3-1)(a^4+a^2+1)-(m^3+1)(a^3+1)\}q.$

15. $m^6(p^4+p^2q^2+q^4)-2q^3m^3-m^2(2q^3-q^2+4p^2q+p^2)$
$-4pqm-(p^2-q^2).$

EXERCISE 57.

1. $3x^2-x+2.$
2. $5x^2-10x+2.$
3. $7x^4-2x^3.$
4. $11x^2-2x+77.$
5. $p^6-3p^4+2p^2-2.$
6. $x^5+x^3-x-1.$
7. $a^6+a^5+a^4+a^3+a^2+a+1.$
8. $a^4+a^3+a^2+a+1.$
9. $a+b.$
10. $a^3-3a^2+9a-27.$
11. $\frac{1}{4}a^2-\frac{1}{3}ab+\frac{1}{9}b^2.$
12. $m^4-7m^3n+3m^2n^2-31mn^3+5n^4.$
13. $ax-b.$
14. $2a^3+3a^2+4a+5.$
15. $ax^2+bx+c.$

EXERCISE 58.

1. $(x^2-xy+y^2)a^2+(x-y)a+1.$
2. $(m^2+m+1)x^2+(m+1)x+1.$
3. $(a+1)^2+2(a+1)-3.$
4. $(p-q)^2-(q+r)(5q+3r+2p)+4(q+r)^2.$
5. $l^2(m-1)-l(m+1)+3.$
6. $a^2(x+3)-a(x-2)+(x+1).$
7. $(a-1)m^2-(a-1)m+3.$
8. $(a+4)x+(a-2)y.$
9. $(p+2q)m-(p-q).$
10. $(a-b)^2-2(a-b)-1.$
11. [Read $x(x-1)a^2$]. $(x-1)a+x^2.$
12. $(x^2+xy+y^2)a+(x+y).$
13. $c-b.$
14. $x^2(x-1)-xy(x+y-1)+y^2(y-1).$
15. $(b+c)x^2+(c+a)x+(a+b).$

EXERCISE 59.

1. $Q=8x^2-5x,\ R=3.$
$Q=8x^2+11x+6,\ R=9.$
$Q=8x^2-13x+21,\ R=-39.$
$Q=8x^2+19x+33,\ R=69.$

$Q = 8x^2 - 21x + 58$, $R = -171$.

$Q = 8x^2 + 27x + 76$, $R = 231$,

2. $Q = 7x^3 + 14x^2 + 22x + 51$, $R = 107$.

$Q = 7x^3 - 56x^2 + 442x - 3528$, $R = 28227$.

$Q = 7x^3 + 49x^2 + 337x + 2367$, $R = 16572$.

3. $Q = 2x^2 - 4x + \frac{11}{3}$, $R = -\frac{25}{3}$.

$Q = 2x^2 - \frac{4}{3}x + \frac{1}{9}$, $R = \frac{7}{9}$.

$Q = 3x^2 - \frac{17}{2}x + \frac{57}{4}$, $R = -\frac{171}{8}$.

$Q = 3x^2 + \frac{1}{2}x + \frac{9}{4}$, $R = \frac{23}{4}$.

$Q = 3x^2 - \frac{11}{2}x + \frac{17}{4}$, $R = -\frac{21}{4}$.

$Q = 3x^2 - \frac{5}{3}x + \frac{1}{4}$, $R = \frac{3}{4}$.

$Q = 2x^2 - 2x + \frac{1}{3}$, $R = -\frac{2}{3}$.

$Q = 2x^2 - \frac{10}{3}x + \frac{1}{4}$, $R = -\frac{10}{99}$.

4. Same answers as in (3).

EXERCISE 60.

1. 6, 17, 30, -10, -15, -18, $-4\frac{3}{16}$, $-1\frac{1}{64}$, $y^2 + 8y - 3$, $n^2 - 8n - 5$.

2. 0, 2, $y^2 - y + 1$, $y^2 - 5y + 6$, $y^2 - 2y + \frac{3}{4}$, $y^2 - \frac{3}{2}y + \frac{7}{16}$, $2y^2 - 6y + 12$.

3. 0, 54, $64\frac{8}{27}$, $123\frac{37}{64}$, $(2n^3 + 6nh^2 + 12nh + 6n)$, $2(-3n^2h - h^3 + 33n^3 + 3h^2 - 3h + 1)$.

4. 3024, 0, 0, 360, 24.

5. $n^4 + 6n^3 + 2n^2 - 24n + 5$,
$n^4 + 6n^3 + n^2 - 15n - 1$,
$n^4 + 6n^3 - 27n + 13$.

6. $(n+2)(n+1)n$, $n(n-1)(n-2)$, $(n+3)(n+2)(n+1)$, $(n-1)(n-2)(n-3)$.

7. 385, 91, $\frac{2n(2n+1)(2n+1)}{6}$, $\frac{(2n+1)(n+1)(4n+3)}{3}$.

8. $\frac{(n-1)n(4n-1)}{3}$.

9. $\left(\frac{1}{n^2} - 1\right)^3$.

10. $y^4 + 6y^3 + 16y^2 + 21y + 12$.

EXERCISE 61.

1. $a + b + c$.

2. $-ap^3 + bp^2 - cp + d$.

3. $= pa5 + pa1 = 1a + 4$.

4. $16a + 8b + 4c + 2d + e$.

5. $\mp mb^5+nb^4+pb^3+qb^2\pm rb+s$. 6. 5. 7. 4.
8. -4. 9. -3. 10. a; $3a$. 11. 12.
21 $x^6+x^4y^2+x^2y^4+y^6$.
22. $x^{12}-x^9y^3+x^6y^6-x^3y^9+y^{12}$.
23 $x^{12}-x^{10}y^2+\ldots-x^2y^{10}+y^{12}$.
24 $x^{18}-x^{16}y^2+\ldots\ldots+x^2y^{16}-y^{18}$.
35 $a=0$; $b=-1$. 36. $a=-9$; $b=18$.
37 $p=0$; $q=\pm\sqrt{-1}$.
$p=\pm\sqrt{2}$, $q=1$.
$p=\pm\sqrt{-2}$; $q=-1$.
38. $a=-9$; $b=2$.
39. $A=1\frac{1}{4}$; $B=2\frac{1}{2}$; $C=1\frac{1}{4}$. 40. $a=4$, $b=-6$; $c=3$.
41. $8(x-1)^3+33(x-1)^2+39(x-1)+18$; $(2x-1)^3+5\frac{1}{4}(2x-1)^2+6(2x-1)+5\frac{3}{4}$.
42 $(x-1)^3+3(x-1)^2+3(x-1)+1$.
43 $-10+10(x+1)-9(x+1)(x+2)+2(x+1)(x+2)(x+3)$.
44 $z^6-3z^5+9z^4-13z^3+18z^2-12z+8$.

EXERCISE 62.

1 $(2x+3y+4z)(3x+6y+7z)$.
2 $(3a-b+2c)(2a+5b-3c)$.
3. $(x+2y+3z)(2x+3y+z)$.
4 $(5a+6b+7c)(2a-4b-5c)$.
5. $(a+2b+c)(a-b+c)$.
6 $(a+b+c)(a+b+c)$.
7. $(2x+y-z)(x+2y+z)$.
8 $(2a+b+c)(a+2b+c)$.
9. $(2a-b+c)(a+b+2c)$.
10. $(3x-2z-3y)(x-2z-y)$.
11 $(2x+3a-4)(x-6a+7)$.
12 $(3x+4y+5)(x+y-3)$.
13. $(3a-2b+4)(2a+5b-7)$.
14. $(3x+3y-5)(x-2y+10)$.
15. $(2a+b+3)(a+2b-3)$.
16 $(7x+6y+2)(2x+3y+6)$.
17. $(3x+y+3z)(x-2y-2z)$.
18 $(2x+3y-z)(x-y+z)$.
19. $(2x+3y-z)(x-4y-4z)$.
20 $(a+b+c-d)(a+b+c-d)$.

EXERCISE 63.

1. $(a-1)(3a^2+3a+1)$.
2. $(x-1)(2x^2-3x+4)$.
3 $(x-1)(x+1)(2x+1)$.
4 $(x-1)(3x^3-4x^2+5x-7)$.
5. $(x-1)^3$.
6. $(a-1)(3a^2-2a+1)$.
7. $(x+1)(3x^3-5x^2+8x-8)$.
8. $(a-1)(4a^2-2a+3)$.

9. $(x+1)(11x^2-4x-1)$.
10. $(x+1)(7x^3-3x^2+x-2)$.
11. $(x-1)(x^2+1)$.
12. $(x-1)(ax^2+ax+a+bx+b+c)$.
13. $(x+1)^3(x^2-3x-5)$.
14. $(x+1)(x^2+2x+3)$.
15. $(x+4)(x+5)(x+6)$.
16. $(x-1)(x+4)(2x+1)$.
17. $(x-2)(2x-1)(3x+2)$.
18. $(x-1)(x+1)(3x+2)(6x-7)$.
19. $(x-1)(x+1)(2x-1)(3x+7)$.
20. $(x-1)(x+1)(x-2)(6x+3)$.
21. $(x+1)(x-3)(2x+3)(3x-2)$.
22. $(x+1)(x+1)(x-3)(4x+5)$.
23. $(x-2)(x+2)(x+2)(x+3)$.
24. $(x+2)(x+3)(x-3)(x-3)$.
25. $(x^2-x+2)(x^2-x-8)$.
26. $(x^2+10x+22)(x^2+10x+18)$. $(x+2)(x+4)(x+6)(x+8)$ should be $(x+2)(x+4)(x+6)(x+8)$.
27. $(6x^2-31x+37)(6x^2-31x+41)$.
28. $(x-1)(x-1)(x^2-2x-44)$
29. $x(x+4)(x^2+4x+7)$.

EXERCISE 64.

1. $(a+b)(b+c)(c+a)$.
2. $(a+b)(b+c)(c+a)$.
3. $(a-b)(b-c)(a-c)$.
4. $(a-b)(b-c)(c-a)$.
5. $(a+b+c)(a-b)(b-c)(a-c)$.
6. $(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+ac+bc)$.
7. $(a+b+c)(a-b)(b-c)(a-c)$.
8. $(ab+ac+bc)(a-b)(b-c)(a-c)$.
9. $(a-b)(b-c)(a-c)(a^2+b^2+c^2+ab+ac+bc)$.
10. $5(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-ac-bc)$.
11. $(a^2-b^2)(b^2-c^2)(a^2-c^2)$.
12. $(a-b)(b-c)(c-a)(a+b+c)$.
13. $(a^2-b^2)(b^2-c^2)(c^2-a^2)$.
14. $(a-b)(b-c)(c-a)$.
15. $(b^2-1)(b^2-c^2)(c^2+1)$.
16. $\left(\frac{a}{b}-\frac{b}{c}\right)\left(\frac{b}{c}-\frac{c}{a}\right)\left(\frac{a}{b}-\frac{c}{a}\right)$.
17. $(a-b)(b-c)(a-c)(a^3+b^3+c^3+a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+abc)$.
18. $(a-b)(b-c)(c-a)(a+b+c)$.
19. $(a^2-b^2)(b^2-c^2)(a^2-c^2)$.
20. $(a^2+b^2)(b^2+c^2)(c^2+a^2)$.
21. $(x^2+y^2)(y^2+z^2)(z^2+x^2)$
22. $\left(\frac{a}{b}+\frac{b}{c}\right)\left(\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{a}+\frac{a}{b}\right)$.
23. $\left(x^3+\frac{1}{x}\right)\left(x^3+\frac{1}{x^3}\right)\left(\frac{1}{x}+\frac{1}{x^2}\right)$.
24. $8abc$.
25. $3abc$.

26 $-(a+b+c)(a+b-c)(b+c-a)(c+a-b)$.

27. $2xyz[(y+z-x)^3+(z+x-y)^3$ should be $(x+z-y)^3+(y+z-x)^3]$

28 $-(a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2+ab+ac+bc)$.

29 $5(a+b)(b+c)(c+a)$.

30 $(\sum a^2+\sum ab)(\sum a^2-\sum ab)$.

EXERCISE 65.

1 b^2x^2. 2. $5x^2$. 3. $3ab^2$.

4. $3p^2q^3$. 5 x^2y^2. 6. $17a^4b^3c^3$.

7. $9a^3c^3$. 8 $7xp+qyq+r$. 9. $14a^mbp+q$.

10. $6xa+b+cypq+r$ ($\frac{1}{2}$q should be 29).

EXERCISE 66.

1. a^2-ab+b^2. 2. $a+1$. 3. $x+1$.

4. $x-2$. 5. $x-b$. 6. $x-y$.

7. x^2+2x+2. 8. $x-1$. 9. $2x+3$.

10 $3x-2$. 11. $a-2$. 12. $a+c$.

13 $x+y+z$ 14. $x-1$. 15. $2x-y$.

16. $3x+1$. 17. $x+5$. 18. $x-2$.

19. $(x+1)^2$. 20. $2(x^2+ax-2a^2)$.

EXERCISE 67.

1 x^2+2x+3. 2. $x-1$. 3. x^2+2x+3.

4. $3x-7$. 5. $2x-1$. 6. $(x+1)^3$.

7 x^3+4x+5. 8. $(x-1)^3$. 9. x^2-1.

10 $5x^2-1$. 11 $x-1$. 12. $x-1$.

13. $x-1$. 14 $1+a+a^2$. 15. x^2-3x-4.

16. $2(x^2-2ax+2a^2)$. 17. $x^2-(a-b)x+b^2$.

18. $x+a$. 19. $a+b$. 20 $x-2$.

21. $x+7$. 22 $2x+1$. 23. $x-5$.

24. $x-4$.

EXERCISE 68.

3. $\frac{s-q}{r-p}$ 7. $ab=a^2-c^2$. 13, $x-4$; $a=-4$. 14. 7. 15. 8.

EXERCISE 69.

1. $54x^4y^4z^4a$. 2. $120a^3b^3c^3$.

3. $b^3x^2(x^3-b^3)$. 4. $(x^2-2x-15)(x+1)$.

5. $(l-2m)^2(l+3m)$ 6. $x(x+3y)(2x+y)(3x-2y)$.
8. $-m(x+y)\{l^2(x-z)^2-m^2(y+z)^2\}$.
9. $(x^4-a^4)(x^2+ax+a^2)$. 10. $(x+1)(x-2)(x+3)(x+4)$.
11. $(x+a)(x+b)(x+c)$ 12. $(x^2-b^2)(x^2-c^2)(x^2-a^2)$.
13. $(4x^4-5x^2+1)$. 14. $(x+y)(2x-y)(3x-y)(4x-y)$.
15. $(x-1)(3x-2)(5x-4)(4x^2-1)$. 16. $6x^3+11x^2-3x-2$.
17. $(9x^2-1)(9x^4-1)(x^2-3)^2$.
18. $(a-b)^3(a+b)^3(a^2+ab+b^2)^2(a^2+b^2)$.
19. $3(a-b)(b-c)(c-a)$. 20. $(a^6-b^6)(b+c)(c+a)$.

EXERCISE 70.

1. $(3a^2-a+1)(2a-3)(3a-2)$.
2. $(x-4)(x-5)(x^2-5x+6)$.
3. $(x+c)\{2x^3+x^2(2a-3b)-x(2b^2+3ab)+3b^2\}$.
4. $(x+1)(2x+1)(3x^2-x+2)(2x^2-x-4)$.
5. $(a^2+2a+3)(3a^2+8a)(4a^2+9)$.
6. $(3x-1)(2x-3)(2x+1)(2x+9)$.
7. $(3a-2)(4a-1)(2a+3)(2a-5)$.
8. $(a^2-3a+20)(3a+4)(a+3)$.
9. $(x^5-2x^4-x-1)(x^3+x^2+2x+1)$
10. $(2a+3b+c)(a-b-c)(a-4b+4c)$.
12. $x^3+2ax^2-a^2x-2a^3$. 16. -2.
17. $(x-5)(x-6)$ and $(x-2)(x-5)$.
18. $(x+4)(x-3)$ and $(x+4)(x-5)$.
19. $(x+1)(x+2)$ and $(x+1)(x+3)$.

EXERCISE 71.

1. $\frac{4ac^2}{7b}$. 2. $\frac{3xyz}{5}$. 3. $\frac{3x}{4yz}$. 4. $\frac{a^4x^2z}{b^3x}$.
5. $\frac{x-a}{x}$ 6. $-\frac{1}{3+a}$. 7. $\frac{2a-3b}{2a}$ 8. $\frac{3a}{x+4a}$.
9. $a+\frac{b^2}{a+b}$. 10. $\frac{1}{x^2-y^2}$. 11. $\frac{x-1}{x-4}$. 12. $\frac{3x+4}{2x^2+5x-1}$.
13. $\frac{x-4}{x-5}$. 14. $x-3$. 15. $a+b$. 16. $\frac{4x+2}{x+1}$.
17. $\frac{3x-1}{2x-1}$. 18 $\frac{x-y}{x^2+y^2}$ 19. $\frac{x-2}{2x-1}$.

20. $\dfrac{x-10}{x^2-7x+10}$.

21. $\dfrac{x^2-12x+55}{3x-17}$.

22. $\dfrac{x^2+2xy+4y^2}{2(2x-3y)}$.

23. $(b+c-a)(c+a-b)$.

24. $\frac{1}{3}(a+b+c)$.

25. -3.

26. $\dfrac{1}{b+c}$.

27. $3(a-b)$.

28. 3.

29. $\dfrac{(a-1)(b-1)}{(a+1)(b+1)}$.

30. $\dfrac{a-b+c}{a-4b-4c}$.

EXERCISE 72.

1. $\dfrac{x^2y^2,\ y^2z^2,\ x^2z^2}{xyz}$.

2. $\dfrac{a^3,b^2,c^2}{abc}$.

3. $\dfrac{2(a-1),\ 3a-1,\ 10}{2(a^2-1)}$.

4. $\dfrac{(x-3),\ x-1),\ x-2)}{(x-1)(x-2)(x-3)}$.

5. $\dfrac{(y-z)\ (z-x),\ (z-x)\ (x-y),\ (x-y)\ (y-z)}{(x-y)\ (y-z)\ (z-x)}$.

6. $\dfrac{(a-b)\ (a^3+b^3),\ (a+b)\ (a^3-b^3),\ 1}{a^6-b^6}$.

7. $\dfrac{x(y+z),\ y(z+x),\ z(x+y)}{(x+y)(y+z)(z+x)}$.

8. 0.

9. $\dfrac{2}{(1+x)^2-x^2}$.

10. $\dfrac{2(4a^2+9b^2)}{4a^2-9b^2}$.

11. $\dfrac{8x+5}{(2x+1)(4x+3)(3x+2)}$.

12. $\dfrac{2ab}{a^2-b^2}$.

13. $\dfrac{4a+x}{a-x}$.

14. $\dfrac{2}{x(x+1)(x+2)}$.

15. $\dfrac{3x^2}{x^2-1}$.

16. $4a^3$.

17. $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

18. $\dfrac{2}{x(1-4x^2)}$.

19. $\dfrac{ab(3a-2b)}{(a-b)^2}$.

20. 0.

21. $\dfrac{bx^3+bc(a^2-ax+x^2)+a(a^3+x^3)}{b(a^3+x^3)}$.

22. 1.

23. $\dfrac{4x-3y}{4x^2-y^2}$.

24. $\dfrac{x}{(1-x)(1+x^2)}$.

25. $\dfrac{a^2}{(a-2)(a+1)^2}$.

26. $\dfrac{12}{(x-1)(x+4)}$.

27. $\dfrac{2(x+8)}{(1-x)(x+5)}$.

28. 0.

29. $\frac{(a-b)(b-c)(a-c)}{(a+b)(b+c)(c+a)}$. 30. $\frac{3x-11}{(x-1)(x-3)(x-5)}$.

31. 0. 32. 3. 33. $\frac{x^2}{(x-a)^2}$. 34. $\frac{2a^2-31a-2}{a^4-4}$.

35. $\frac{2}{(1-a^2)(1-a^4)}$.

EXERCISE 73.

1. $b(a^2+b^2)$. 2. $a+b$. 3. $\frac{(x-4)(x-7)}{x^2}$. 4. $\frac{4ab}{a^2-b^2}$.

5. $\frac{a^2(a-b)}{x}$ 6. $\frac{x^2}{a^2-x^2}$. 7. 1. 8. $\frac{b^2(a^2-1)}{a^2+1}$.

9. $\frac{x^3+y^2}{xy}$. 10. $\frac{x^4-y^4}{x^3y^3}$. 11. $\frac{(a+b+c)^2}{2bc}$.

12. $\frac{y(yz+1)}{(xy+1)(xyz+x+z)}$. 13. $\frac{a}{b}$. 14. $\frac{1}{a^2+b^2}$.

15. $\frac{(a+b-c)(c-a+b)}{4}$. 16. $\frac{(c+d)^2-(a-b)^2}{4(ab+cd)^2}$

17. $\frac{ax(a^2-ax+x^2)}{(a-x)(a^3-x^3)}$. 18. 1. 19. $\frac{a-c}{ac+1}$.

20. $\frac{b}{a}$. 21. $\frac{c}{a}$. 22. $a+b$. 23. $\frac{2a^4}{(a^2+b^2)^2}$.

24. $\frac{b^2-a^2+bc}{b^2-a^2-ac}$. 25. $\frac{5}{2(1+2x)}$. 26. $1\frac{1}{3}$.

27. 1; $(a^2-b^2)(a^2+ab+b^2)$.

28. $\frac{1}{(x-2)^2(x+1)^2(x-6)}$; $x^2(x^2-2x+4)(x^2-4)$.

29. $\frac{a-1}{(a^2+1)(a^3+1)(a+1)}$; $(a^3-1)(a^2-1)$.

30. $\frac{1}{abc(a+b)(b+c)(c+a)}$; $a^3b^3c^3(a-b)(b-c)(c-a)$.

EXERCISE 74.

1. 1. 2. 0. 3. 1. 4. $n+m$. 5. -1. 6. -1.
7. 1. 8. 1. 9. 0. 10. 1. 11. -3.
12. 4. 13. 1. 14. 1. 15. 0. 16. 1.
17. $\frac{(a+b+c)-(h+k)}{abc}$.

18. $a+b+c$.

19. $a^3+b^2+c^2+ab+ac+bc$.

20. $\dfrac{1}{(a-1)(b-1)(c-1)}$.

21. $\dfrac{1}{abc}$.

22. $\dfrac{1}{(a+x)(b+x)(c+x)}$.

23. $\dfrac{1}{(a-1)(b-1)(c-1)}$.

24. $\dfrac{1}{(a-1)(b-1)(c-1)}$.

25. $\dfrac{x}{(a-x)(b-x)(c-x)}$.

26. $\dfrac{x^2}{(a-x)(b-x)(c-x)}$.

27. $\dfrac{a+b+c-ab-ac-bc}{(a-1)(b-1)(c-1)}$.

28. $\dfrac{x^3-x+1}{(x-a)(x-b)(x-c)}$.

29. $\dfrac{(x+1)^3}{(a-x)(b-x)(c-x)}$.

30. $\dfrac{mx^3-nx+p}{(a+x)(b+x)(c+x)}$.

EXERCISE 75.

1. 0. 2. 0. 3. $\dfrac{2(3x^2-12x+11)}{(x-1)(x-2)(x-3)}$. 4. 0.

5. $\dfrac{2}{a-b}$. 6. 0. 7. x. 8. 2. 9. $\dfrac{1}{x}$. 10. -1.

11. 1. 12. $-n$. 13. 1. 14. 1. 15. $\dfrac{3(x+y)}{x-y}$.

16. $3(a+b+c)$. 17. $a^2+b^2+c^2$. 18. $2(a^2+b^2+c^2)$.

19. $-3(a+b)(b+c)(c+a)$. 20. $-\dfrac{3abc}{a+b+c}$.

21. 3. 22. $(a+b+c)$. 23. $ab+ac+bc$.

24. $\dfrac{4}{(x+1)(5x+1)}$; $\dfrac{n}{(x+1)\{(n+1)x+1\}}$.

25. $\dfrac{4}{(a-2b)(5a-6b)}$; $\dfrac{n}{(a-2b)\{(n+1)a-(n+2)b\}}$.

EXERCISE 76.

(53 $a^4+2b^2-4a^2b$, $a^6-6a^4b+9a^2b^2-2b^3$.

(54).

1. a^3-2b. 2. $a^3-3ab+3c$. 3. $(a^2-2b)^2-2(b^2-2ac)$.

4. $(a^2-2b)(a^3-3ab+3c)-a(b^2-2ac)+bc$.

EXERCISE 77.

1. $\dfrac{1}{x-a}-\dfrac{1}{x-b}$.

2. $\dfrac{2}{x-4}+\dfrac{1}{x+2}$.

3. $\dfrac{1}{2(x+2}-\dfrac{1}{2(x+6)}$.

4. $\dfrac{1}{x+2}-\dfrac{2}{x^2-2x+4}$.

5. $\frac{1}{2x-1}-\frac{1}{3x-1}$.

6. $\frac{1}{x+a+b}+\frac{1}{x-a-b}$.

7. $\frac{q-ap}{(b-a)(x+a)}+\frac{bp-q}{(b-a)(x-b)}$.

8. $\frac{4}{3(2x+1)}-\frac{1}{6(x+2)}$.

9. $\frac{3}{2(a+1)}-\frac{3}{a+2}+\frac{3}{2(a+3)}$.

10. $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$.

11. $\frac{1}{2(x-2)}-\frac{4}{x-3}+\frac{7}{2(x-4)}$.

12. $\frac{ap^2+bp+c}{(p-q)(q-r)(x-p)}+\frac{aq^2+bq+c}{(q+p)(q-r)(x-q)}$

$$+\frac{ar^2+br+c}{(r-p)(r-q)(x-r)}.$$

13. $\frac{3}{y-1}+\frac{2}{y-3}+\frac{5}{y-2}$.

14. $\frac{1}{x-2}-\frac{2}{x-3}+\frac{3}{x-4}$.

15. $\frac{4}{x+2}+\frac{5}{x+3}+\frac{3}{x+4}$.

EXERCISE 78.

1. 2. [Take $x=\frac{2ab}{a+b}$ for $\frac{a+b}{2ab}$].

2. $\frac{3a^2(b^2-a^2)}{b^2(a+2b)^2}$.

3. 0. 4. 1. 5. 0. 6. 1.

7. $\frac{1}{a}$. 8. 1. 9. $\frac{ab}{a-b}$. 10. 1. 11. $-\frac{b}{a}$.

12. $\frac{c^2+cd+d^2}{c+d}$. 13. $\frac{(a-b)^2}{(a+b)(a^2+b^2)}$. 14. 0.

15. $\frac{2}{b}$. 16. $\frac{ab}{a^2+b^2}$. 17. 1. 18. a. 19. 0.

20. $\frac{(n-1)^2}{4}$ 26. 0.

EXERCISE 79.

70. (*a*) $\frac{55}{29}$. (*b*) $\frac{39}{5}$. (*c*) $\frac{557}{11}$. (*d*) $\frac{6390}{471}$. (*e*) $\frac{6654}{29}$.

EXAMINATION PAPERS.

Third Series.

I

1. $2h(3x^2+12x-9+h^2)$ 2. $29\frac{1}{2}$.
3. 1. $(a-b)(b-c)(c-a)(a+b+c)$
   2. $3(a-b)(c-b)(a+c)$ [$+a^3$ should be $-a^3$].
4. $(4x^2-9y^2)(16x^4+36x^2y^2+81y^4)$.
5. $1\frac{1}{3}$. 6. $\dfrac{x(x^3+x^2-1)}{x^4-1}$.

II

1. $A=1$, $B=11$, $C=3$, $D=0$. 2. $2p=q\ (q+2)$.
3. $(a-1)^4+4(a-1)^3+6(a-1)^2+4(a-1)+1$.
4. [For $(x+y+z)^3+(y+z+u)^3$ read $(x+y-z)^3+(y+z+x)^2$].
5. $H.\ C.\ F.=x(x+5)$; $L.\ C.\ M.=6x(x-7)\ (x-9)(x+5)$.
6. $6x^3-44x^2+111x-98$. [Add x after "of"].
7. 1.

III

1. $a=2$; $b=3$.
2. $H.\ C.\ F.=4x^2+3$; $L.\ C.\ M.=(4x^2+3)(x-2)(3x+1)$.
3. $A=1$, $B=2$, $C=3$, $D=0$.
4. $\dfrac{5a+7}{a+1}$
5. $(a+b+c)\ (a+b-c)\ (b+c-a)\ (c+a-b)$.

IV

1. -9. 2. $A=1$, $B=1$, $C=1$.
3. 1. $(x+y)(z+x-y)(z-x+y)$
   2. $(2a-b+2)(3a+b-1)$.
4. $(4a^2-1)(3a+1)(2a-1)$. 6. 0.

V

2 y^3. 3. $a=5$. 5. 0. 6. $H.\ C.\ F.=a-1$
7. $(x-1)(x-2)(x-3)(x-4)$. [42 should be 24].

VI

2. $(x-3)(x-2)$.
4. $3(a-b)(b-c)(c-a)$. [For $b-c$ read $(b-c)^3$]
8. $A=\frac{1}{4}$, $B=-1$, $C=\frac{1}{2}$.

VII

1. $x-1$.
3. $(a-4)(a-1)$; $(a-4)(a-2)(a-3)$.
4. $A=2$, $B=3$, $C=-5$.
6. 1. 7. -1.
8. $(ax+by)^2+(bx-ay)^2$, $(ay+bx)^2+(by-ax)^2$.

VIII

1. $16-5a^2$.
2. $12xyz$.
4. $\frac{3x-8}{x^2-7}$. [For $3x^2$ read $3x^3$] 5. x^3+7x+1.
6. $5(x^2+y^2)-7xy$.
8. O. [For x^4+y^4 read x^4-y^4].

IX

1. $p=\frac{1}{c(c+1)(c+2)}$, $q=-\frac{c+3}{c(c+1)(c+2)}$, $r=\frac{1}{c}$.
2. 0.
3. $\frac{(x^2-6x+21)(x+1)}{(x-1)^2(x-3)}$.
4. $a-1$.
5. $-2(a-b)(b-c)(c-a)(a+b+c)$.
6. $(2x-1)^3+3(2x-1)^2-2(2x-1)+1$.
7. 36.

X

1. (a) $4(x-y)(y-z)(x-z)$.
   (b) $(x-2y+3)(4x+y-2)$.
2. $3(x^2+y^2+z^2)-5(xy+xz+yz)$.
4. (1) $kl-3m$ (2) $l^2-3m(kl-m)$.
5. $\frac{(x+y+z)^2}{2yz}$.
6. $\frac{x^3-2x^2+4x-8}{x^3-2x^2+12x-18}$.
7. $\frac{a^4-10a^3-6a-1}{a^4+10a^3+6a-1}$.
8. $\frac{a^2-ap-aq-pq}{a-p}$; $\frac{b^2+bp+bq-pq}{b+q}$.

EXERCISE 80.

1 $125a^9b^{12}$; $-32a^{10}b^{15}$ 2. $\frac{l^6m^3n^3}{a^2b^6c^9}$; $\frac{4x^4y^2z^6}{9p^6q^4r^2}$

3 $\frac{a^n}{b^6n}$. 4. $\frac{16z^6}{72xy^6}$.

5 a^{3n}. a^{n-m}. $+12n(n-1)$

21 $2(a^6+20a^3b^3+6ab^6)$. 22. $2(x^7+21x^5y^2+35x^3y^4+y^7)$.

23 $3ab(a+b)(a^2+ab+b^2)$. 24 $7ab(a+b)(a^2+ab+b^2)^2$.

25 $80a$. 26. $-12y$. 27. $-5103a^8$.

28 16. 29. 6. 30. 0.

EXERCISE 81.

1 $a^2b^3c^4$; $4mn^6$. 2. $3a^7b^2$; $-2x^4y$.

3 $2p^2q^4$; $-3lm^2n^5$. 4. $9ab+c^2$.

5 ax^2+bx+c. 6. $3-x+2x^2$.

7 b^2-2ba^2-a. 8. $(a-b)(ab+1)$.

9 x^3+4x-1.

10 $\frac{x^2}{2}-2x+\frac{a}{3}$. 11. $x^2-x+\frac{1}{4}$.

12 $\frac{x^2}{2y^2}+1+\frac{2y^2}{x^2}$. 13. $\frac{x^2}{2}+\frac{2x}{3}+\frac{2}{x^2}$

14 $\frac{x^2}{y^2}-\frac{x}{y}+1-\frac{y}{x}+\frac{y^2}{x^2}$. 15. $x+\frac{1}{x}-1$.

16 $x^2+\frac{1}{x^2}-2$. 17. $\frac{1}{a^2}+\frac{1}{a}-1+a$.

18 $a+b$ 19. $a^2-b^2+c^2-d^2$.

20 $a^2+(2b-c)a+c^2$.

21 $a+\frac{3}{2a-1}$. 22. $\frac{2x^2+x-2}{2x-1}$.

23 $a+\frac{1}{a+2}$. 24. 0. 25. 0. 26. $2a$

27 $3x^2+x+1$; 311.

EXERCISE 82

1. $3x^2-2x+1$. 2. $x^2-x+\frac{1}{4}$. 3. $x+\frac{a}{3}-\frac{b}{2}$

4. $x^2-3xy+2y^2$. 5. $(x+y)^2$.

6. [Read—$192x+64$]. $x^3-6x^2+12x-8$.

7. $x^4-a^2x^2-a^4$. 8. $27-54x^2+36x^4-8x^6$.

9. $4a^2+2a-1-\frac{3}{a}$. **10.** $\frac{3a^2}{b^2}+\frac{4a}{b}+3-\frac{2b}{a}$.

11. $2(x-1)^2+5(x-1)-6$. **12.** $(a-1)^2+3(a-1)-2$.

13. $a+b-c$. **14.** $a^2+c^2-2b^2$. **15.** $a-\frac{1}{a}-2$.

16. $a^2+\frac{1}{a^2}+1$. **17.** $x+\frac{1}{x}+2$. **18.** $\frac{x}{y}+\frac{y}{x}+1$.

19 $x=10$. **20** $x=2$. **21.** $x=\frac{a^4+b^4}{b^3}$. **22.** $x=3$.

23. $x=6a$. **24.** $x=9\frac{7}{8}$. **25.** $2x^2-x+1$.

26. x^2-3x+3. **27.** $2x^2+2x+3$. **28.** a^3+a^2+3a+3.

29. $c-3ab+2a^3=0$; $3b-2a^2=\sqrt{d}$. **30.** $p^2=4q$.

31. $r^2=p^2s$; $p^3+8r=4pq$. **32.** $p-q=3p^2q^2$; $p^3=6q^3$.

33. $a=1$; $b=-2$. **34.** 25. **35.** -24. **36.** -6; 9. **37**

38. 13. **39.** $4x^2(a+b)^2$.

40. $1-\frac{a^2}{2}-\frac{a^4}{8}-\frac{a^6}{16}-\frac{5a^8}{128}$; $1+\frac{a^2}{2}-\frac{a^4}{8}+\frac{a^6}{16}-\frac{5a^8}{128}$.

41. $\left(1-\frac{x^2}{2a^2}-\frac{x^4}{8a^4}-\right)$; $a\left(1+\frac{x^2}{2a^2}-\frac{x^4}{8a^4}+\right)$.

42 $1+\frac{x}{2}+\frac{3x^2}{8}-\frac{3x^3}{16}+\frac{3}{128}x^4$; $1-\frac{x}{2}+\frac{3}{8}x^2+\frac{3}{16}x^3+\frac{3}{128}x^4$.

43. $a^{\frac{1}{2}}\left(1+\frac{b}{2a}-\frac{b^2}{8a^2}+\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}\right)$;

$a^{\frac{1}{2}}\left(1-\frac{b}{2a}-\frac{b^2}{8a^2}-\frac{b^3}{16a^3}-\frac{5b^4}{128a^4}\right)$.

EXERCISE 83.

1. x^2-3x+2. **2.** $2x^2+4cx-3c^2$. **3.** $(x-1)^2$.

4 x^2+x+1. **5.** $x^2+2+\frac{1}{x^2}$. **6.** $3x^2-2x+1$.

7. (a) $x-\frac{1}{x}$ (b) $x-2$, **8.** $b=a^2$; $c=a^3$.

9. $p^2=3q$; $\sqrt{r}=\frac{p}{3}$. **10.** $4c^2=9bd$; $bd=16ae$.

EXERCISE 84.

1. $a^{-3}+a^3$. **2.** $a-1$ **3.** $x^{\frac{1}{4}}(x^{\frac{3}{4}}-y^{\frac{2}{3}})$.

4 $a^{\frac{1}{3}}+b+c^{\frac{1}{2}}+a^{\frac{1}{3}}b^{\frac{1}{2}}-a^{\frac{1}{3}}c^{\frac{1}{4}}+b^{\frac{1}{2}}c^{\frac{1}{4}}$. 5. $x^{\frac{p}{2}}+x^{\frac{p}{2}}y^{\frac{p}{2}}+y^{\frac{p}{2}}$

6 1 7. 0.

8 $(x^{-1}-4)(x^{-1}-3)$; $(x^{\frac{1}{4}}-y^{\frac{1}{4}})(x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}})$;
$(a^2+b^2+ab\sqrt{2})(a^2+b^2-ab\sqrt{2})$;
$(x+x^{-1}+1)(x^2+x^{-2}-x-x^{-1})$;
$(a^{-1}+a^{-2})(a^{-2}+a^{-3})(a^{-3}+a^{-1})$.

9 $x+y+\sqrt{xy}$; $(x-y)\times$ 2nd Expn.

10 $\left(\frac{a}{b}\right)^{\frac{p}{q}}$; $p=q$. 11. $y^{\frac{1}{2}}x^{-\frac{1}{4}}+y^{\frac{1}{4}}x^{\frac{1}{4}}-\frac{1}{2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$.

12 $2x^{\frac{1}{2}}-3y^{\frac{1}{3}}+4z^{\frac{1}{4}}$. 13. $x^{\frac{2}{3}}+1x^{\frac{1}{2}}+x-4x^{\frac{7}{12}}-4x^{\frac{3}{4}}+2x^{\frac{5}{6}}$

14 $x^2+b^2c^{a^2+b^2}$. 15. $(a^{\frac{1}{2}}-b^{\frac{1}{2}})m$.

16 $1\div x^{a^2+b^2}$. 17. y^2 [Read + as ×].

18. 1. 19. $1\div x^{p^2+q^2}$.

20. 1. 21. $a^{x^{\frac{1}{2}}-1}$.

22 $a^{\frac{1}{7}\frac{2}{2}\frac{5}{8}}b^{\frac{2}{8}\frac{5}{8}}$. 23 $a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{5}{2}}$.

EXERCISE 85.

1 2. 2. a.

EXERCISE 86.

1 $a\div(1-b+a)$. 2. $ab\div(a+b-c)$.

3 $(10-2a)\div(3a-3)$ 4 $-2ab\div(a+b)$.

5. $(3a^2-7ab)\div(3b-a)$. 6. $5a\div 4$.

7. $-5ab\div 2(a+b)$. 8. $ab\div(a+b)$.

9. $\frac{10a^2+b^2}{8(a+2b)}$. 10. 1.

11 $\frac{1}{3}(a+b+c)$. 12. $-\frac{1}{3}(a+b+c)$.

13 $(a+b+c+d)\div(m+n)$. 14. $\{cd(a+b)-ab(c+d)\}\div(ab-cd)$.

15. $(c^2-ab)\div(a+b-2c)$. 16. $\frac{1}{3}(a+b+c)$.

17. $\frac{1}{3}(a+b+c)$. 18. $\frac{1}{3}(x+b+c)$.

19. -8. 20. 7. 21. 5.

22. 3. 23. 5. 24. -2

25. $\frac{80}{17}$. 26. 8. 27. 1.

28. 4. 29. -31. 30. 5.

31. $4\frac{1}{2}$. 32. b. 33. $4\frac{1}{8}$.
34. $4\frac{1}{6}$. 35. 3. 36. $2\frac{3}{2}\frac{1}{0}$.
37. $2\frac{3}{4}$. 38. $(ab+bc-2ac)\div(a+c-2b)$.
39. $\frac{2}{7}$. 40. $2(c-b-a)$. 41. $a^2+b^2+c^2$.
42. $-(a+b+c+d)$. 43. $ab+a+b$.
44. $ab+ac+bc$. 45. $6ab+3ac+2bc$.
46. $-3a$. 47. $3a$. 48. 2.
49. 10. 50. $\{c^2(a-b)+b^2(a-c)\}\div(ab+ac-2bc)$.
51. $2ab\div(a+b)$. 52. $\{b^2(c-a)+a^2(c-b)\}\div(ac+bc-2ab)$
53. 5. 54. 13. 55. $\frac{1}{2}(a+b)$. 56. $a^2\div(b-a)$.
57. $(bn-am)\div(m-n)$. 58. $-\frac{2ab}{c}$.

EXERCISE 87.

1. 4. 2. 0. 3. $1\frac{3}{4}$. 4. 5. 5. 10. 6. 0.
7. 5. 8. -9. 9. 0. 10. 3. 11. 3. 12. 0.
13. $1\frac{1}{1}\frac{3}{7}$. 14. -2. 15. 5. 16. 1 17. -10.
18. $\frac{1}{3}$. 19. 4. 20. $-b$. 21. $2ab\div(a-b)$. 22. 5.
23. 2. 24. 4. 25. 6. 26. 1. 27. $-9\frac{1}{2}$. 28. 0.
29. $2b$. 30. $\{ab(c+d)-cd(a+b)\}\div(ab-cd)$.
31. $(c^2-ab)\div(a+b-2c)$. 32. $\frac{1}{2}(a+b-3)$. 33. 4.
34. $1\frac{1}{2}$. 35. $\frac{1}{2}(a+b)$. 36. $-2\frac{1}{2}$. 37. $2\frac{1}{3}$.
38. 7. 39. $2(a+b)$. 40. 8. 41. $3\frac{1}{3}$.
42. $\frac{5}{4}$. 43. $\frac{6}{3}\frac{5}{6}$. 44. $2\frac{1}{2}$. 45. $\frac{5a}{2}$.
46. $2\frac{3}{1}$. 47. $8\frac{1}{2}$. 48. 1. 49. $-3a$.
50. $(3ab-a^2-b^2)\div(a+b)$.

EXERCISE 88.

1. $\frac{1}{a}$. 2. $\frac{(a+b)(n-m)}{2n-2m+a-b}$. 3. $\frac{b+c+q+r}{a+p}$.
4. $\frac{cn}{m(b-a)}$. 5. $\frac{cq-br}{ar-cp}$. 6. $\frac{a(ab-cd)}{ad-bc}$.
7 $\frac{3}{4}$. 8. $2(a+b)$. 9. $\pm ac$.
10 $\frac{1}{a}$. 11. $-\frac{7}{8}$. 12. $ab\div(a+b)$.
13. $1\frac{1}{3}$. 14. $-6\frac{1}{2}$. 15. -10.
16. $-(a^2+b^2+3ab)\div(a+b)$. 17. 6.
18. $2\frac{1}{2}$. 19. $-5\frac{1}{2}$. 20. $-4\frac{1}{2}$

EXERCISE 89.

1. 4. 2. 50. 3. $\frac{13}{6}\frac{3}{1}$. 4. . 5. $6a$.

6. $\frac{17}{18}a$. 7. 20. 8. $(a-b)^2 \div (2a-b)$. 9. $\frac{a}{4}$.

10. $\frac{4a}{5}$. 11. $(a-b)^2 \div 2b$. 12. 25.

13. $\frac{1}{a}\left(b - \frac{cn}{n-1}\right)^2$. 14. 6. 15. ± 1.

16. $2r - p^2 - q^2$. 17. $2c - a - b$. 18. $10\frac{1}{2}$.

19. 4. 20. 1. 21. 0 22. $\frac{1}{2}(b-c)$. 23. $\frac{3}{16}$.

24. $\sqrt{\frac{2a-b}{a^2b}}$. 25. $\frac{a(a^2+1)}{(a+1)^2}$. 26. $\frac{4}{81}$. 27. $\frac{4a}{3}$.

28. 1. 29. $a(m^2+n^2) \div (m^2-n^2)$. 30. $a(b-1)^2 \div 4b$.

31. $\frac{a}{2b}(b-1)^2$. 32. $\frac{\sqrt[4]{a+1} + \sqrt[4]{a-1}}{\sqrt[4]{a+1} - \sqrt[4]{a-1}}$

33. $\frac{b(p^2+1)^2 + 4p^2}{a(p^2-1)^2}$. [Read $\sqrt{ax+b}$ for $\sqrt{a+b}$]

34. $\frac{3a-1}{\sqrt{(1-a)(9a-1)}}$ 35. 5. 36. $2a^2$.

37. 0. 38. $(b^{\frac{1}{3}} + b^{\frac{2}{3}})^3$. . $\frac{4}{b-2}$. 40. $-\frac{6}{7}$.

41. $\frac{27abc - (a+b+c)^3}{3(a^2+b^2+c^2-ab-bc-ac)}$. 42. $-a$. 43. ab.

44. 5. 45. $\frac{1}{p+1}$. 46. 1. 47. 78.

EXERCISE 89. Problems.

1. 16 ; 2.

2. $\frac{AB}{(B+1)^2} - n$; $\frac{AB}{(B+1)^2} + B$; $\frac{A}{(B+1)^2}$; $\frac{AB^2}{(B+1)^2}$; 18, 22, 10 & 40.

3. Rs. 25. 4. $\frac{2}{3}$. 5. 654. 6. 240.

7. 48 of each. 8. 740 ; 1,053 votes. 9. 120 bushels.

10. 5 : 1. 11. 40. 12. $6p$ and $4p$ 13. £2 15s.

14. 23. 15. 1,000 men. 16. 25 lbs.

17. 16 ; 20 coins. 18. (1) $38\frac{7}{11}m$; (2) $21\frac{9}{11}m$, (3) $5\frac{5}{11}m$. past 7.

19. 156 days. 20. $\frac{a+nq}{p+q}$; $\frac{pn-a}{p+q}$.

21. The hound makes 960 leaps and the hare 1 200 leaps.

22. 20 days. 23. $1\frac{5}{13}$ hrs. 25. 7*s.* ; 11*s.* 8*d.*

26. 193. 27. 180,000 men. 28. 760 men.

29. 600 men. 30. 600. 31. 2 miles an hour.

32. 4hrs. and 6hrs. 33. 6 miles. 34. 6,400.

36. 14 miles from *B*. After 6 hrs. 37. 2 hours after *B*.

38. 121 yds. 39. 99 yds ; 77 yds.

40. 10 hrs. 95 m. from *L*. 41. 36 miles.

42. 60 miles ; Passenger train, 30 miles an hour ; Goods' train, 20 miles an hour.

43. 150 miles ; 40 miles an hour.

44. 12 o'clock ; 125 miles from *A*.

EXERCISE 90.

1. $x=\frac{1}{2}$; $y=\frac{1}{2}$. 2. $x=3$; $y=4$. 3. $x=4$; $y=2$.

4. $x=8$; $y=4$. 5. $x=\frac{1}{2}$; $y=\frac{1}{3}$. 6. $x=a$; $y=b$.

7. $x=\frac{c(a^2+b^2)}{a^2-b^2}$; $y=\frac{c(a+b)}{2a}$. 8. $x=a+b$; $y=a-b$.

9. $x=\frac{cq-br}{aq-bp}$; $y=\frac{ac-cp}{aq-bp}$.

10. $x=\frac{ac(dn+bm)}{ad+bc}$; $y=\frac{bd(cn-am)}{ad+bc}$.

11. $x=\frac{abc(ab+ac-bc)}{a^2b^2+a^2c^2-b^2c^2}$; $y=\frac{abc(ac-ab-bc)}{a^2b^2+a^2c^2-b^2c^2}$.

12. $x=\frac{c-d-a+b}{2(bc-ad)}$; $y=\frac{a+b-c-d}{2(bc-ad)}$.

13. $x=\frac{b^2-c^2}{a(a-c)}$; $y=\frac{b^2-ac}{b(c-a)}$. 14. $x=\frac{n}{a^2-b^2}$; $y=\frac{n}{b^2-a^2}$.

15. $x=\frac{ab}{a+b}$; $y=\frac{ab}{b-a}$. 16. $x=\frac{b-a}{b+a}$; $y=\frac{a(a+b)^2-4b^2}{a(a^2-b^2)}$.

17. $x=y=1$. 18. $x=\frac{b^2+c^2-a^2}{2a}$; $y=\frac{c^2+a^2-b^2}{2b}$.

19. $x=b-c$; $y=c-a$ 20. $x=\frac{b-c}{a-b}$; $y=\frac{c-a}{a-b}$.

21. $x=1$; $y=-1$. 22. $x=y=1$.

23. $x=a^2$; $y=b^2$. 24. $x=3$; $y=2$.

25. $x=\frac{1}{5}a$; $y=\frac{1}{5}a$.

26. $x=-\frac{16}{15}a$; $y=\frac{7}{15}a$.

27. $x=(a+b)^2$; $y=(a-b)^2$

28. $x=y=343$.

29. $x=a+b$; $y=a-b$.

30. $x=a$; $y=b$.

31. $x=\dfrac{b^2(a^2+1)(a^{\frac{1}{2}}-1)}{(a^2b^2-1)(a^{\frac{1}{2}}-b^{\frac{1}{2}})}$; $y=\dfrac{a^2(b^2+1)(b^{\frac{1}{2}}-1)}{(a^2b^2-1)(a^{\frac{1}{2}}-b^{\frac{1}{2}})}$.

32. $x=\dfrac{a(a^2+ab+b^2)}{b(a+b)}$, $y=-\dfrac{a^2}{2+b}$.

33. $x=-ab$; $y=a+b$.

34. $\dfrac{15x}{14}-\dfrac{9}{14}$.

35. 6 ; 12.

36. $a=-1$; $b=-1$.

37. $2y+3x+1=0$.

38. $225x+132y+42=0$.

39. $2x^2+3xy+2y^2$.

40. 1.

41. $y=3x$.

42. $x=1$.

44. $k=1$.

45. $p=3$. [Read +13 for −13 in the second equation.]

EXERCISE 91.

1. 1 ; 2 ; 3.

2. 1 ; 2 ; 3.

3. 35 ; 30 ; 25.

4. 2 ; 3 ; 4.

5. 7 ; 10 ; 9.

6. 24 ; 60 ; 120.

7. 1 ; 2 ; 3.

8. 8 ; 10 ; 12.

9. 5 ; 7 ; −3.

10. 24 ; 60 ; 120.

11. $x=y=z=1$.

12. $x=\dfrac{2mnpa}{n(p+m)-pm}$, &c.

13. $\frac{1}{7}$; $\frac{1}{56}$; 1.

14. $\frac{1}{3}$; $\frac{1}{3}$; $\frac{1}{3}$.

15. $1\frac{1}{3}$; $3\frac{1}{2}$; $62\frac{1}{10}$.

16. 10 ; 6 ; 2.

17. -2 ; $-\frac{7}{3}$, $\frac{1}{3}$.

18. $\dfrac{1}{(c-a)(c-b)}$; $\dfrac{1}{(b-a)(b-c)}$; $\dfrac{1}{(a-b)(a-c)}$.

19. $\dfrac{1}{x}=\dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}\right)$: $\dfrac{1}{y}=\dfrac{1}{2}\left(\dfrac{1}{c}+\dfrac{1}{a}-\dfrac{1}{b}\right)$; $\dfrac{1}{z}=\dfrac{1}{2}\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)$

20. abc ; $ab+ac+bc$; $a+b+c$.

21. $x=\dfrac{b^2+c^2-a^2}{2bc}$, &c.

22. $x=y=z=\dfrac{a+b}{c}$.

23. $\dfrac{6-b}{2b-3}$; $\dfrac{3}{2b-3}$; $\dfrac{6-2b}{2b-3}$.

24. $\dfrac{1}{x}=-\dfrac{1}{2}\left(\dfrac{1}{b}+\dfrac{1}{c}\right)$, &c.

25. $x=y=z=1$.

26. $x=\dfrac{b^2+c^2-a^2}{2bc}$, &c.

27. 1 ; 2 ; 3 ; 4.

28. $\frac{1}{6}$; $-\frac{1}{6}$; $\frac{1}{18}$, $-\frac{1}{18}$.

29. 1 ; 2 ; 3 ; 4.

30. 1 ; $\frac{1}{2}$; $\frac{1}{3}$; $\frac{1}{4}$.

31. Consider $\dfrac{1}{xy}$, $\dfrac{1}{yz}$, $\dfrac{1}{xz}$ as the unknown quantities.

32. $x=y=z=a$. 33. $\frac{2}{3}$, 1, $\frac{1}{3}$. 34. $3x^2-5x-1$. 35. 0

36. 4, 1, −2.

37. $x+4y$: for 9 read 10 and for (1,2) read (2, 2)

38. $6x^2-\frac{37}{12}x+\frac{11}{6}$.

39. $x^3+y^2+z^2+2(xy+yz+zx)+9$.

40. $y=2x+3$.

EXERCISE 92.

1. $x=3$; $y=4$.
2. $x=12$; $y=5$.
3. $x=12$; $y=13$.
4. $x=14$, $y=20$.
5. $x=3$; $y=2$.
6. $x=\frac{q+ap}{a^2+1}$; $y=\frac{p-aq}{a^2+1}$.
7. $x=2$; $y=-3$; $z=1$.
8. $x=1$; $y=3$; $z=4$.
9. $x=c-b$; $y=a-c$; $z=b-a$.
10. $x=b-c$; $y=c-a$; $z=a-b$.
11. $z=\frac{a(b+c)}{(a-b)(c-a)}$, &c.
12. $x=b+c-a$; $y=c+a-b$, $z=a+b-c$.
13. $x=\frac{1}{a}$; $y=\frac{1}{b}$; $z=\frac{1}{c}$.
14. $x=a$; $y=b$; $z=c$.
15. $x=\frac{1}{(a-b)(a-c)}$, &c.
16. $x=b+c-a$; $y=c+a-b$; $z=a+b-c$.
17. $x=\frac{1}{a}$; $y=\frac{1}{b}$; $z=\frac{1}{c}$.
18. $x=a$; $y=b$; $z=c$.
19. $x=a$; $y=b$; $z=c$.
20. $x=a^2-b^2$; $y=c^2-a^2$; $z=b^2-c^2$.

EXERCISE 93.

1. 82.
2. 376.
3. 76 lbs; 40 lbs.
4. 9 gallons and 6 gallons.
5. $106\frac{2}{3}$ yds.
6. 457.
7. 36 and 27 miles per hour; 756 miles.
8. £56; £33.
9. $4\frac{7}{8}m$.
10. $\frac{5}{24}$; $\frac{3}{16}$.
11. A in $\frac{2abc}{bc+ac-ab}$ days; B in $\frac{2abc}{ab+bc-ac}$ days; C in $\frac{2abc}{ca+ab-bc}$ days.
12. 32 days; 40 days; 53$\frac{1}{3}$ days.

13. (i) 20, 30 and 60 days; (ii) 10 days.

14. 4 *yds.*; 5 yds. **15.** 72 lbs.

16. $40\frac{7}{22}m.$; $34\frac{7}{22}m.$ **17.** 3 miles and 6 miles an hour.

18. 2 miles an hour. **19.** 3 miles an hour.

20. $17\frac{5}{36}$, $12\frac{5}{36}$; $14\frac{17}{36}$ gals. **21.** $6m.$; $5\frac{5}{11}m.$

22. 150 *m.* **23.** $3m.$

24. 33 *m.*; $6m.$; $5m.$ **25.** *A* $1\frac{7}{8}m.$; *B* $1\frac{1}{8}$ *m.*

26. 27 *m.*, $3\frac{1}{2}m.$; **27.** 12 ft.; 9 ft.

28 10 yards, 7 yards. **29** 18 feet, 12 feet; 15 feet.

30. 255. **31.** £6; 20 persons.

32. 8 and 12 gals. **33.** 90 miles.

34. *A* in 5 minutes; *B* in 6 minutes.

35. $2\frac{1}{2}$ and 2 miles; distance 5 miles.

36. 100 *m.* and 25 *m.* **37.** $b(n-1)\div(a-c)$ miles.

38. 50 lbs.: £70. **39.** 6 and 10 qrs.

40. 45 *m.* and $22\frac{1}{2}$ *m.* **41.** 10 and 12 miles an hour.

42. 12 *m.* **43.** $33\frac{1}{3}m.$; $48\frac{1}{3}m.$

44. 45 and 30 miles. **45.** 30 and 50 miles.

46. 4550 *I*; 1750 *C*; 3853 *A*. **47.** 432.

48. Rs. 1,250; 1,500; 1,250. **49.** 63, (20; 21; 22).

50. 36 *m.*; 12 *m.*

52. Rs 510; Rs. 360. **53.** 200 yds; 100 yds.

54 A's money = Rs. 70, B's money 130.

55 $4\frac{1}{2}$ mi; 4 mi; $3\frac{3}{4}$ mi. **56.** 13 as 4*p*.

57. 220 yds. 176 yds. per mi. **58.** £4,680. £4,720.

59. Forewheel $1\frac{1}{3}$ yds; Hind wheel $1\frac{1}{7}$ yds.

60. *A* – 25 miles, *B* – 30 miles *C* – 35 miles per hour; distance = 100 miles.

61. 1,000 ft. per second.

62. 10 minutes fast.

63. 6 as. up to 15 seers and 6 ps. per every additional seer.

EXERCISE 95.

1. ± 1. **2.** $\pm\sqrt{5}$. **3.** $\pm\frac{1}{\sqrt{5}}$. **4.** $\pm\frac{a}{\sqrt{2}}$.

5. $\pm\frac{1}{2}$. **6.** $\pm 2\sqrt{\frac{3}{5}}$. **7.** ± 1. **8.** $\frac{4}{3}$; 0.

9. ± 3. **10.** ± 2. **11.** ± 6. **12.** ± 1.

13. $\pm\dfrac{a(n-1)}{\sqrt{2n-1}}$. **14.** 4; 0. **15.** $-2a$.

16. $\pm\frac{5}{6}$. **17.** $\pm\sqrt{2ab-b^2}$. **18.** $\pm\frac{5}{8}$.

19. $\pm\dfrac{a(b-c)}{2\sqrt{bc}}$. **20.** $\pm\sqrt{\dfrac{a^2-pb^2}{p+1}}$ where $p=\left(\dfrac{c+d}{c-d}\right)$

21. $\pm\dfrac{a\sqrt{3}}{2}$. **22.** $\pm\sqrt{3}$. **23.** $\pm\dfrac{3}{5\sqrt{19}}$.

24. $\pm a$. **25.** $\pm\dfrac{4}{\sqrt{5}}$.

EXERCISE 96.

1. 4; −2. **2.** 20; −6. **3.** 8; −40. **4.** 1; −8.

5. 1; −20. **6.** 25; −136. **7.** 10; −2. **8.** 3; −1.

9. 7; $-\frac{3}{2}$. **10.** $\frac{5}{3}$; $-\frac{8}{2}$. **11.** 4, 3. **12.** 13, −12.

13. $-a$, $-b$. **14.** ab, ac. **15.** 2, $2\frac{2}{5}$. [Read $-14x$ for -14].

16. $\frac{8}{5}$, $-\frac{7}{6}$ [Read $-13x$ for $-7x$]

17. $\dfrac{1}{a}$, $\dfrac{1}{b}$. **18.** 2; $-4\frac{1}{2}$. **19.** 3; $-\frac{4}{5}$.

20. $\frac{1}{8}(27\pm\sqrt{57})$. **21.** 1; $10\frac{2}{3}$. **22.** $\dfrac{-9\pm\sqrt{27}}{2}$.

23. 7; $-1\frac{3}{7}$. **24.** $b;\dfrac{a^2}{b}$. **25.** $a;\dfrac{1}{a}$.

26. $-(a+b)$; $b-a$. **27.** $-a$; $-3ab$.

28. $0;\dfrac{2ab}{a+b}$. **29.** $c;-\dfrac{a^2+ac+b^2+bc}{a+b+2c}$.

30. $\dfrac{a}{2}(-3\pm\sqrt{3})$. **31.** $\dfrac{a+b}{2}\pm\dfrac{a-b}{2}\sqrt{\dfrac{p+2q}{p-2q}}$.

32. $2a-b$; $2b-a$. [Read $(x-a)^2$ for $(x-a)^3$].

33. $\pm\sqrt{(a^2+10ab+b^2)}$. **34.** $\pm\sqrt{(a^2+b^2)}$.

35. $\dfrac{a^2+b^2}{2b}$; $\dfrac{2ab+b^2-a^2}{2a}$. **36.** $0;\dfrac{2ab-bc-ac}{a+b-2c}$.

37. $a+b$; $\dfrac{2ab}{a+b}$. **38.** 0; $a+b$. **39.** $\dfrac{a}{b}$; $\dfrac{b}{a}$.

40. $\dfrac{1\pm\sqrt{(1+4ac)}}{2(a+b)}$

41. $\dfrac{1}{a+b+c}\left\{ab+ac+bc\pm\sqrt{(a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)}\right\}$.

42. 49 ; 9. **43.** 4 ; $-4\frac{4}{5}$.

44. $-(2a-3b)\pm\sqrt{(5b^2-12ab-12b^2)}$.

45. 8 ; 20. **46.** 5; −8.

47. 7 ; −1. [Read +11 for −11.] **48.** 11 ; −5.

49. $\frac{1}{3}(2a+b+b\sqrt{1+4a^2})$. **50.** 3 ; 4.

51. 0 ; $1\frac{5}{8}$. **52.** 6 ; $-2\frac{1}{2}$. **53.** 1 ; 3.

54. $\frac{1}{2}(-1\pm\sqrt{33})$.

55. $\frac{1}{2}(-1\pm\sqrt{-3})$; $\frac{-(1-b)\pm\sqrt{(1-b)(1+7b)}}{2(1-b)}$.

56. 5 ; 3.

57. $\frac{1}{2}\left(\frac{m^2+n^2}{m^2-n^2}\pm\sqrt{\left(\frac{m^2+n^2}{m^2-n^2}\right)^2-\frac{4mn}{m^2-n^2}}\right)$. [Read m^2+n^2 for m^2-n^2 in the denominator.]

58. $a;\frac{a}{8}(-3\pm\sqrt{-7})$. **59.** $0;\pm\frac{\sqrt{2a-1}}{a}$ **60.** $\pm2;\pm\sqrt{-2}$.

EXERCISE 97.

1. 5 ; 1. **2.** $-\frac{7}{3}$; 1. **3.** 0 ; $\frac{3}{4}$. **4.** $-\frac{7}{8}$; 0.

5. $1-a$; a^2-a+1. **6.** $\frac{b^2-c^2}{b^2-a^2}$; $\frac{c^2-a^2}{a^2-b^2}$.

7. 3, 12. **8.** 7, 14. **9.** −5, −40. **10.** 9, −36

11. 9, −99. **12.** −2, 3. **13.** 1, −3. **14.** $\frac{3}{2}$, 6.

15. $\frac{2}{3}$, 3. **16.** $\frac{1}{3}$, −13. **17.** $x^2-x-12=0$.

18. $x^2+(a-b)x-ab=0$. **19.** $x^2-2x+1-a^2=0$.

20. $x^2+(c-a)x+(a-b)(b-c)=0$.

21. $x^2-4x+1=0$.

22. $-\frac{b}{c}$. **23.** $\frac{b^2}{ac}-2$. **24.** $\frac{b^2}{c^2}-\frac{2a}{c}$. **25.** $\frac{3b}{a}-\frac{b^2}{ac}$.

26. $\frac{b^2-3ac}{a^2}$. **27.** $\frac{b^2-ac}{a^2}$. **28.** $\frac{(b^2-ac)(b^2-3ac)}{a^3}$.

29. $-\frac{4}{5}$. **30.** $-3\frac{1}{2}$. **31.** $4\frac{3}{4}$. **32.** −91.

33. $2x^2-3x+1=0$. **34.** $12x^2-7x+1=0$.

35. $2x^2+3x-9=0$. **36.** $4x^2-12x+5=0$.

37. $2b^2=9ac$. **38.** $3b^2=16c$.

39. $4p^2=-25q$. **40.** $\frac{1}{4}$.

41. $(x-1)(a^2x-b^2x+a^2-c^2)$. **42.** $(x+a-b)(x+b-c)$.

43. $(x-1)\{(b+c-2a)x-a-b+2c\}$.

44. $(x-1)\{ax(b-c)-c(a-b)\}$. **45.** $(x+a^2+ab)(x+b^2+ab)$.

46. $(x+a^2-ab)(x+b^2-ab)$.

EXERCISE 98.

1. 20 **2.** -55. **3.** 184.

4. b^2-24a. **5.** b^2-4a^2. **6** 5.

7. 29. **8.** -4. **9** $4(2b^2-a^2)$.

10 $(4c-a-b)^2+4(ab-2c^2)$; the R. H. S. is $2(x-c)^2$.

11. $(a^2-b^2)^2$. **12.** Real and unequal.

13. Imaginary. **14.** Real and unequal.

15. Real and equal. **16.** $\frac{1}{16}$. **17.** $\frac{7}{9}$.

18. $-1, 3$. **19.** $-5, 3$.

20. $(a+c)(a+c-4b)+4b^2=0$.

EXERCISE 99.

1. $2; \sqrt{-6}$. **2.** $(x^2-x)^2-(x^2-x)-132=0$

3. $\pm 3; \pm\frac{1}{2}\sqrt{-11\frac{1}{2}}$. **4.** $1; 4$.

5. $-1; \frac{1\pm\sqrt{-3}}{2}$. **6.** $x+\frac{1}{x}=\frac{-2a\pm\sqrt{2a+2}}{a-1}$.

7. 2. **8.** 1. [Read $2x$ for $3x$]. **9.** $\frac{1}{2}$.

10. $y^2+3y-4=0$; $(y=\sqrt{x^2+5x+7})$.

11. $y^2+y-20=0$; $(y=\sqrt{x^2+9x+3})$.

12. $y^2-2y-35=0$; $(y=\sqrt{x^2-8x+40})$. [Read x^2 for y^2].

13. $y^2-\frac{1}{2}y-1\frac{1}{2}=0$; $(y=\sqrt{x^2-2x-3})$.

14. $9; -2; \frac{1}{2}(7\pm\sqrt{173})$. **15.** 1. **16.** $4; -9$.

17 $2; -5; \frac{1}{2}(-3\pm\sqrt{241})$. **18.** $2; -\frac{1}{2}$.

19. $y^2-2y+1=0$ where $y=\sqrt{3x^2-2ax+4}$.

20. $\left(\frac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$; $\left(\frac{a-b}{a+b}\right)^{\frac{2pq}{q-p}}$

21. $\frac{w}{1-w^2}$; $\frac{w^2}{1-w}$; w = imaginary cube root of 1.

22. $-2; 6; 3\pm\sqrt{21}$. [Read $+144$ for -144].

23. Divide by x^2; and put $x-\frac{1}{x}=y$.

24 $\frac{3}{5}(1\pm\sqrt{5})$; $3(\pm\sqrt{2})$. **25.** 1; $\frac{1}{2}(-3\pm\sqrt{5})$

26 $x+\frac{1}{x}=1\frac{2}{3}$ or $-\frac{1}{3}$. **27** Divide by x^2; and put $x+\frac{1}{x}=y$.

28 6; -1; $\frac{1}{2}(5\pm\sqrt{-39})$.

29 $8a$; $-a$; $\frac{a}{2}(7\pm\sqrt{\ -71.})$ **30.** $(\frac{3}{2}; \frac{1}{2})$; $(3\pm\sqrt{10})$

31 $x^2+5ax=-5a^2\pm\sqrt{a^4+c^2}$.

32. $(x^2-9x)^2+38(x^2-9x)+336=0$.

33 $(x^2+21x)^2+158(x^2+21x)+5565=0$.

34 $y^2-y+7\frac{1}{2}=0$; $(y=\sqrt{x^2+4x-5})$.

35 $x^2+3x=\frac{1}{4}$ or $\frac{9}{4}$. **36.** $x=5$ (obvious).

37 ±5. **38.** 1; $\dfrac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}$.

39. $(x^2+5x)^2+6(x^2+5x)-40=0$.

40 $x^2+3x=4$ or -2. **41.** $4x^2-3x=1$ or $2\frac{1}{2}$.

42. $x^2-5x+7=0$ or 1. **43.** $x^2+x=6$ or -7.

44 $\frac{2}{3}(-4\pm\sqrt{34})$, **45.** 2. **46.** $x^2+x=\frac{1}{2}(-1\pm\sqrt{3})$.

47 1; $\dfrac{-47\pm\sqrt{-1665}}{46}$. **48** $\frac{3}{2}\pm\sqrt{\left(\dfrac{25\pm\sqrt{1345}}{72}\right)}$.

49. $\frac{1}{2}$; $-4\frac{1}{2}$; $-2\pm\sqrt{-6}$. **50.** 1; -3; $-\frac{1}{2}$.

51. 0; -3; 12; -15. **52.** m.

53. a; b; c; $(x+a)(x+b)(x+c)=x^3$.

54. $-(a+b)$; $\dfrac{1}{a+b}\times\left\{-(a^2+b^2)\pm\sqrt{(a+b)^4+4a^2b^2}\right\}$.

55. 2; $2\pm\sqrt{3}$. **56.** 0; $\frac{4}{3}a$. **57.** 1; -3.

58. 0; $\frac{3}{8}$; $\frac{1}{8}(1\pm\frac{1}{2}\sqrt{-3})$. **59.** 1; $2+\sqrt{2}$; $2+\sqrt{-1}$.

60 0; $\pm\sqrt{5}$; $\pm\sqrt{\frac{1}{2}1}$.

EXERCISE 100.

1. $x=1-b$; $y=b$. **2.** $x=\dfrac{1}{\sqrt{2}}$; $y=\sqrt{2}$.

3. $x=4$; $y=3$. **4.** $x=5b$; $y=4b$.

5. $x=y=2$. **6.** $x=2$; $y=0$.

7. $x=7$; $y=5$. **8.** $x=10$; $y=1$.

9 $x=\dfrac{2}{a}$, $y=\dfrac{2}{b}$; $z=\dfrac{2}{c}$. **10.** $x=1$; $y=2$; $z=3$.

11. $x=1$; $y=2$; $z=3$. **12.** $x=p$; $y=q$; $z=r$.

13. $z=\sqrt{\dfrac{(c+a-b)(a+b-c)}{2(b+c-a)}}$, &c.

14. $x=2$; $y=3$; $z=4$.

15. $x=\frac{1}{2}\left(\dfrac{ac}{b}+\dfrac{ab}{c}-\dfrac{bc}{a}\right)$. 16. $\dfrac{1}{x^2}=\frac{1}{2}(b^2+c^2-a^2)$.

EXERCISE 101.

1. $x=3\frac{5}{7}$; $y=10\frac{14}{169}$. 2. $x=\pm 3$; $y=\pm d$. 3. $x=3$; $y=1$.
4. $3x^2y^2-196xy+2380=0$. 5. $x=\pm\sqrt{7}$; $y=\pm 2$.
6. $x=1$ or $1-(1\pm\sqrt{2})^3$; $y=0$ or $-1\pm\sqrt{2}$.
7. $x=5$ or $-2\frac{1}{3}$; $y=8$ or $\frac{1}{2}$.
8. $xy=4$; $x-y=-9$. 9. $x=\pm 6$; $y=\pm 3$.
10. $x=-\frac{5}{17}\frac{2}{7}a$; $y=-\frac{1}{17}\frac{3}{7}a$. 11. $x=\pm 7$; $y=\pm 3$.
12. $x=4$; $y=2$. 13. $x=3$ or $-\frac{1}{3}$; $y=3$ or $-\frac{1}{3}$.
14. $x=1$ or $\frac{5}{3}$; $y=2$ or $\frac{2}{3}$. 15. $x=-1$ or 3; $y=1$ or -3.
16. $x=\pm\dfrac{a(m+n)}{\sqrt{m^2+n^2}}$; $y=\pm\dfrac{a(m-n)}{\sqrt{m^2+n^2}}$.
17. $xy=\dfrac{b^2-a^2}{2}$; $x+y=b$. 18. $x=3$; $y=2$.
19. $x=0$; 1; $\pm\dfrac{1}{\sqrt{2}}$; $y=0$; 2; $\pm\sqrt{2}$.
20. $x=\pm 2$; $y=\pm 1$.
21. $xy=\frac{1}{16}\left(5a^2\pm\sqrt{5a^4\pm\dfrac{20b}{a}}\right)$ and $x+y=a$.
22. $x=\dfrac{m+n}{mn+1}$ or $\dfrac{mn+1}{m+n}$; $y=\dfrac{n-m}{mn-1}$ or $\dfrac{mn-1}{n-m}$.
23. $x=3a$ or $-2a$; $y=2b$ or $-3b$. 24. $x=2$; $y=1$.
25. $x=3$; -4; 2 or -5; $y=2$ or -5; 3 or -4. 26. $x=2$; $y=1$.
27. $x=y=1$. 28. $x=7$; $y=11$. 29. $x=4$; $y=1$.
30. $x=8$; $y=2$. 31. $x=3$; $y=-3$. 32. $x=125$; $y=1$.
33. $x=2$; $y=1$. 34. $x=3$; $y=2$.
35. $x^3=\dfrac{a^9}{a^6+b^6}$; $y^3=\dfrac{b^9}{a^6+b^6}$.
36. $x=a$; $y=b$. 37. $x=4$; $y=2$. 38. $x=4$; $y=5$.
39. $x=\dfrac{a}{2}(1\pm\sqrt{3})$; $\dfrac{a}{2}\left(1\pm\dfrac{1}{\sqrt{3}}\right)$; $y=\dfrac{a}{2}(1\pm\sqrt{3})$; $\dfrac{a}{2}\left(1\pm\dfrac{1}{\sqrt{3}}\right)$.
40. $x=a$; b; $y=b$; a.

EXERCISE 102

1. $x=1 \quad y=1 \; ; \; z=2.$

2. $x=\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}, \; y=\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}; \; z=\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}.$

3. $x=4 \; ; \; y=1 \; ; \; z=2.$

4. $x^2=\frac{a(b+c)}{(a-b)(c-a)}$, &c.

5. $x=b-c \; ; \; y=c-a \; ; \; z=a-b.$

6. $x=1 \; ; \; y=2 \; ; \; z=3.$

7. $x=2 \; , \; y=3 \; ; \; z=4.$

8. $x=1 \; ; \; y=2 \; ; \; z=3.$

9. $x=\frac{1}{2}b+c-a+\sqrt{\frac{1}{4}(2bc+2ac+2ab-a^2-b^2-c^2)}\div 2p$

where $2a^2=a+b+c+3\sqrt{\frac{1}{4}(2bc+2ac+2ab-a^2-b^2-c^2)}$.

10. $x=b+c$; or $b+c+\frac{2abc}{a-b-c}$, &c.

11. $x=b-c \; ; \; y=c-a \; ; \; z=a-b.$

12. $x=\pm\frac{a(b+c-a)}{\sqrt{(b+c-a)(c+a-b)(a+b-c)}}$, &c.

13. $x=a^{\frac{1}{9}}b^{-\frac{5}{9}}c^{\frac{1}{3}} \; ; \; y=a^{-\frac{7}{9}}c^{-\frac{1}{3}} \; ; \; z=a^{\frac{1}{9}}b^{-\frac{2}{9}}c^{\frac{1}{3}}.$

14. $x=\pm\sqrt{\frac{1}{2}(a+\sqrt{a^2-(b-c)^2}}$; &c.

15. $x=y=z=\pm\sqrt{\frac{c}{z}}.$

16. $x=a \; , \; y=b \; ; \; z=c.$

17. $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}=\frac{2(a^2+b^2+c^2)}{3(a^2+b^2+c^2)-2(bc+ca+ab)}$ or 0.

18. $x=6 \; ; \; y=7 \; ; \; z=8.$

19. $x=3 \; ; \; y=1 \; ; \; z=2.$

20. $x=3 \; ; \; y=-2 \; , \; z=0.$

EXERCISE 103.

1. 7 ; 3.
2. 20 ; 20.
3. 9 ; 8.
4. $\frac{1\pm\sqrt{5}}{2}$; $\frac{3\pm\sqrt{5}}{2}$.
5. $\frac{-1\pm\sqrt{-3}}{2}$.
6. 1; 6.
7. 44; 110.
8. 24; 40.
9. 220; 165.
10. 50.
11. 16.
12. 10; 15.
13. 20; 25.
14. 2; 5; 8.
15. 3.
16. 400; 1600
17. 3; 4.
18. 252.
19. 73 or 37.
20. $\frac{3}{7}$.
21. $5m$.
22. 248.
23. 54; 10.
24. $2d.$; $3d.$
25. $\frac{\sqrt{5}}{2}$, $\frac{\sqrt{5}(1+\sqrt{5})}{2}$.
26. 180.
27. £40.
28. 6; 9.
29. 15.
30. 25; 5.
31. 117; 130.

32. $13\frac{3}{4}$ ft.; $16\frac{1}{2}$ ft. 33. 10; 15. 34. $30m$.
35. A 4·30; B 5 A. M. 36. 10. 37. Bisect it.
38. $-1; -3; 3$. 39. 66; 22. 40. $9d$. 41. 28.
42. A 40 hrs.; B 60 hrs. 43. 5 hrs.; 3 hrs.
44. 160 qrs.; £2. 45. 24 days; A 4s. B 3s.
46. 10; 12. 47. 30 yds.; 40 yds.;
48. 100. 49. 1: 9; 1: 4. 50. £60; £40.

EXERCISE 104

3. (i) 2·25. (ii) 5·76. (iii) 3·24.
(i) 1·7. (ii) 3·2. (iii) 3·5, (iv) 3·7.
4. (i) 2·65. (vi) 3·46. (iii) 4·24.
5. (4, 16,) (−1, 1). 6. 1; $-\frac{1}{4}$.
7. (1, 2), $(\frac{1}{2}, \frac{1}{2})$. 8. −2, 1.
10. 3, $2\frac{1}{4}$; 2, 1. 13. (3, 9); (−1, −1).
14. 1; $-\frac{1}{4}$. 16. $(3, -4\frac{1}{2})$; (2, −2).

EXERCISE 105.

1. (i) $-\frac{2}{3}, -\frac{2}{3}$. (ii) −2, −4. (iii) 1, −1.
(iv) −1, 0. (v) $\frac{2}{3}, \frac{1}{6}$. (vi) $-\frac{1}{6}, 2\frac{1}{12}$.
(vii) −1, 4. (viii) $-\frac{3}{8}, -\frac{9}{16}$. (ix) $-\frac{1}{6}, \frac{1}{18}$.
2. (i) 5. (ii) −4. (iii) 6. (iv) 1.
5. $\frac{2}{3}$.
6. (i) 2, −1. (ii) 3, −2. (iii) 3, $-\frac{1}{2}$.
(iv) 0, $3\frac{1}{2}$. (v) −3, 1. (vi) −2, $\frac{1}{3}$.
7. The min. value = −1; −1, −3.
9 2, −1. 10. 1, 3.
11. (i) 1, 2. (ii) −2, 1. (iii) 2, $2\frac{1}{2}$. (iv) −1, $\frac{1}{3}$.
12 −2, $1\frac{1}{3}$. 13. −2, $1\frac{1}{2}$. 14. −1, −1.

15. Maximum when $x = 1\frac{1}{2}$ and zero when $x = 3, -\frac{1}{2}$. The function is $+ve$ for values of x which lie between $-\frac{1}{2}$ and 3 and negative for values of x which lie outside $-\frac{1}{2}$ and 3. The max. value = $6\frac{1}{4}$.

16. 2, $-\frac{1}{2}$. 17. 2, $-\frac{1}{2}$.
18. The max. value = $6\frac{1}{4}$; 1, −2.
19. The min. value = $-10\frac{1}{8}$.
20. x should be between −2 and 1. The max. value = $2\frac{1}{4}$.
21. Imaginary. 22. Real and equal.

24. The min. value $=2$.
25. The max. value $=-1$.
26. $x=1,\ 1;\ y=3,\ 3$.
27. $x=-3,\ 1;\ y=10,\ -2$.
28. $x=4,\ -3;\ y=3,\ -4$.
29. $x=6,\ 8;\ y=8,\ 6$.
30. $x=3,\ 1;\ y=1,\ 3$.
31. $x=1,\ 2;\ y=-2,\ -1$.
32. Read the equations as $x^2+y^2=25$; $xy=12$; $x=3,\ 4$; $y=4,\ 3$.
33. 20 days.

EXERCISE 106.

1. $a^3+ab+bc=0$.
2. $\sqrt{\frac{q}{p}}=\sqrt[3]{\frac{s}{r}}$.
3. $(br-cq)(aq-pb)=(pc-ar)^2$.
4. $p^2-q^2=1$.
5. $\sqrt[3]{\frac{b}{a}}=\sqrt[4]{\frac{q}{p}}$.
6. $la^2-am+n=0$.
7. $p^2-q^2=4$.
8. $a^2-b^2=4p$.
9. $(bc_1-b_1c)(ab_1-a_1b)=(a_1c-ac_1)^2$.
10. $(qn-mr)^2(pm-lq)=(lr-pn)^2$.
11. $\frac{A-B}{A-C}=\frac{a-b}{a-c}$.
12. $a(b^2-c^2)+2b(c^2-a^2)+4c(a^2-b^2)=0$.
13. $5(a^3-b^3)(2a^3+b^3)=9a(a^5-c^5)$.
14. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^3=c^3$.
15. $(a-b)^3(a^2+b^2)=a^2b^2$.
16. $(a+b)^{\frac{3}{3}}-(a-b)^{\frac{2}{3}}=(8c)^{\frac{2}{3}}$.
17. $b^2=ac$.
18. $a^4-4ac^3+3b^4=0$.
19. $a^4-2a^2b^2-b^4+2c^4=0$.
20. $q^2+4rp=k^2(p+r)^2$.
21. $ab+bc+ca+2abc=1$.
22. $\frac{2}{c}=\frac{1}{a}+\frac{1}{b}$.
23. $2p^3-3pq^2-2r^3+2s^3=0$.
24. $d^2(a+b+c)+abc=0$.
25. $a^2+b^2+c^2+2abc=1$.
26. $a^2+b^2+c^2+2abc=1$.
27. $a^3+b^3+c^3-3abc=0$.
28. $a^2b^2c^2(a^3+b^3+c^3+2abc)=a^3b^2c^3$.
29. $a^2+b^2+c^2=-2(ab+ac+bc)$.
30. $abc=(1-a-b-c)^2$.

REVISION PAPERS.

I

1. 3.
2. $a=1,\ b=2,\ c=15,\ d=4$.
3. $x^3-y^2=2$.
4. $(a-b)(b-c)(a-c)$.
5. (1) $x=2,\ y=3,\ z=4$; or $x=-4,\ y=-5,\ z=-6$ (2) $x=0$.
8. $x^2-4\frac{1}{2}x+21$.

II.

1. (i) -11; (ii) $-6\frac{7}{9}$.
4. $p=\frac{4}{3}$, $q=-\frac{3}{2}$.
5. $mn-2=l$
6. $(x+y)(y+z)(z+x)(x+y+z)$.

III.

2. $(4x+3y)^2-(x+7y)^2$.
3. $(x-2)^3-3(x-2)+2$.
4. $(x+y^2)(y+xz)(z+xy)$. [For xy^2 read xyz.]
5. $x=y=z=12$.
6. -152.
7. $a=22$.
8. 2 miles per hour.

IV.

1. $l=\dfrac{1}{k(k+1)(k+2)}$, $m=-\dfrac{k+3}{k(k+1)(k+2)}$, $n=\dfrac{1}{k}$.
3. $(x^2+2)(x^2+2x+2)$.
4. $\dfrac{x}{(x-a)(x-b)(x-c)}$.
5. (i) $(3x+y-1)(x-5y+2)$. (ii) $y^2(3x+y)(2x-3y)$.
6. $a(3b^2-a^2)=2c^3$.
7. $(2x+1)^3-3(2x+1)^2+\frac{31}{9}(2x+1)+2\frac{1}{2}$.
8. $x=5$ or -2.

V.

2. $(y^2-y-1)(3y^2-3y+2)$.
4. $(x^2+8x+11)^2$.
5. $y=x+7$.
6. (i) $x=7$. (ii) $x=\frac{1}{7}^2$, $y=\frac{1}{8}^3$, $z=-12$.
8. $9\frac{5}{8}$ miles.

VI.

2. $20b+19$ and $12b+15$; $b=-\frac{1}{2}$.
3. $(3y+2x)(3y-2x)(x+2)(x-2)(x^2+2x+4)(x^2-2x+4)$. [Read $4x^6$ for $4x^3$.]
4. $(2x-1)^3+2(2x-1)^2-2(2x-1)+2$.
5. $x(x-1)^2$.
6. (i) $x=3$, $y=2$; (ii) $x=-1\frac{5}{7}$, $y=2\frac{2}{5}$, $z=1\frac{13}{11}$.
7. $3\frac{7}{8}$.
8. Brandy, 45 gallons; wine 75 gallons, water, 150 gallons.

VII.

1. $(y-3x+2)(y+2x-1)$; $(\frac{3}{5},-\frac{1}{5})$.
3. $16a^4$.
5. (i) $x=\pm\dfrac{a^3}{\sqrt{a^3+b^3+c^3}}$, etc. (ii) $x=\dfrac{10\pm\sqrt{15}}{3}$.
6. $x^2+y^2+z^2\pm2xyz=1$.
7. $x+1$.
8. A in 40 hours; B in 60 hours.

VIII.

1. $1+5x+15x^2+45x^3+145x^4$. 2. $3(x+2)^2-17(x+2)+23$.
4. $(x-4)(x-2)(x-\frac{1}{2})(x-\frac{1}{3})$. 5. $(x-1)^2(x+1)(x^2+5)(2x+5)$.
6. $a=c$; $4b=a^2+8$. 7. 0. 8. 1 or $-\frac{1}{3}$.

IX.

1. 1. [For $p^2-(qr)^2$ read $p^2-(q-r)^2$.]
3. (i) $x=\frac{c(b^2+a^2)}{b^2-a^2}$, $y=\frac{c(a^2+b^2)}{2ab}$ (ii) $x=a$ or $\frac{1}{a}$.
4. $a^2+\frac{ab}{2}-\frac{3}{4}$. 5. $\frac{a+b}{a-b}$. 7. $7x^2+2xy+7y^2$.
8. $\frac{1}{4}$.

X.

2. 15. 3. $m^{\frac{2}{3}}-n^{\frac{2}{3}}=4$.
4. (i) $x=\pm\sqrt{\frac{a^2+b^2}{2}}$; $y=\pm\frac{(a^2-b^2)}{\sqrt{2(a^2+b^2)}}$.
(ii) $x+y=6$, $xy=3$; or $x+y=3$, $xy=6$.
5. $(2x^2-1)(x+3)(3x^2+2)$. 6. $x^3-3x+\frac{3}{x}-\frac{1}{x^3}$.
7. $p=-3$, $q=4$. 8. 200 Rs.

XI.

2. (i) $(3a+4)(2a-1)(3a+2)(6a+1)$.
(ii) $2(a-b-c)(b-c-a)(c+a+b)$.
3. $(xp-yq)^2+(xq+yp)^2$.
4. $(a-1)(a+2)$.
(ii) $x=1$ or $-\frac{26}{15}$.
5. (i) $x=q$, $y=p$.
6. $t=x$.
7. $a=1$, $b=0$, $c=1$.
8. $x=\frac{1}{3}$ or -1.

XII.

2. (1) $(2x+y-3)(x-11y+1)$.
(2) $(x^2+y^2)(x+y)(x-y)(x+2y)(2x-y)$.
3. $(x-2)^3+6(x-2)^2+6(x-2)+1$.
4. $3(x^2+y^2+z^2)-5(xy+xz+yz)$.
5. $4(ax+by+cz)$.
6. $x=a$, $y=b$.
7. $x(p+q)-y(p-q)$.
8. 480 oranges at 16 for a shilling.

XIII.

3. $\frac{A(a-b)(a-c)}{A+B+C}$
4. $x=1, y=1, z=1$.
5. x^2.
6. $2y=3x$.
7. $xyz(x+y+z)$.
8. $x=1\pm\sqrt{-2}$.

XIV.

1. $\frac{x}{y}=2$ [$y-3$ should be in the denominator].
2. $p^2+q^2+r^2-pqr=4$.
3. $x=2\frac{5}{9}$, $y=-3\frac{2}{3}$, $z=6\frac{1}{9}$.
4. $a^2-ab-ac-bc$.
6. $a=0$, $b=3$, $c=6$.
7. $x-5$.
8. 288.

XV.

1. 3.
2. $\frac{2}{x+1}+\frac{3}{x+5}+\frac{5}{x+7}$
6. (i) $x=a+b$ (ii) $x=\frac{11}{2}$.
8. Each child gets £1920-12s; each brother gets £960-6s.

Ex. Pp. 463—464

1. For 'books' Read 'articles'; Rs. 6, 6 articles.
2. Rs. 2-8-0.
3. Read S. P. in both cases. Re. 1-4-0; Re. 1-14-0.
4. £1-12-0.
5. (*a*) 62½ lbs.: (*b*) 3½ maunds.
6. 5 cms; 1·2 ins.
7. 22 lbs.
8. 1 a. 6 ps., 2 as. 8 ps., 4 as , 6 ps.,
9. 24 min.; 11$\frac{5}{6}$ min.
10. 8 miles per hour, 10$\frac{2}{3}$ miles.
11. (1) 80; (2) 55.
12. (i) 80; (ii) 57 nearly.

Ex. Pp. 469—470.

4. 59.5 ins. nearly.
5. 15.9 sq. ft. nearly.

ANSWERS.

TO

S S L C EXAMINATION QUESTION PAPERS.

1911.

1. $f(-a), -p-120; -120.$
2. $p=-\frac{11}{8}, q=\frac{3}{8}.$
   $r=\frac{1}{4}$
3. $A=\frac{15}{8}, \quad B=\frac{17}{8}$
   $\frac{15}{8(x-5)}+\frac{17}{8(x+3)}.$
4. 12 acres of wet land and 40 acres of dry land.
5. 10 miles an hour and 4 miles an hour. Or, $8\frac{2}{11}$ miles an hour and $4\frac{4}{33}$ mi. an hour.
6. $22x-x^2$ The greatest height = 121 ft.; $2\frac{1}{2}$ seconds.

1912.

1. $q^2x+2qy+1. \qquad x=\frac{1}{2}, y=\frac{1}{4}.$
2. $p^3-2p^2+3p-4; A=8, B=-12, C=4.$
3. $-x^2+4x+3.$ The maximum value is 4 when $x=\cdot5.$
   The function vanishes when $x=1\cdot5$ and $-\cdot5.$
4. Rs. 406 nearly.
5. Length 220 yds. and breadth = 44 yds.
6. One condition is wanting.

1913.

I. $\frac{2+x^5}{1-3x^2-x^3}=2+6x^2+2x^3+18x^4+\frac{-13x^5+15x^6+18x^7}{1-3x^2-x^3}$

2·062.

IV. Rs. 360, Rs. 510.

V. 166 yds., 125 yds.

VI. Rs. 1—4 as., Rs. 2.

1914.

I. (1) $(x+1)(x-2)(x+3)(3x-2).$
   (2) $-30.$

III. A fills the cistern in 10 min. and B in 12 min.

IV. (1) $(a^2+5a-5)^2.$ (2) $A=\frac{1}{8}, B=-\frac{1}{8}, C=\frac{1}{2}.$

V. 12 in. and 10 in.

VI. At 7-20 P.M. at a distance of $58\frac{2}{3}$ miles.

1915.

I. (2) $(a-b)(b-c)(a-c).$

II. (1) $4x-9y+24=0.$

III. (1) $ax^2+by^2+cz^2+dxy+eyz+fzx$.

(2) $a=b=1$, $c=2$. IV. (1) $A=7$, $B=4$;

$$\frac{7}{x+7}+\frac{4}{x-3}.$$

V. They meet at 6-30 P.M. at a distance of 27 miles from A. They will be 28 miles apart at 4-30 P.M.

VI. Forewheel = 4 yds., hindwheel = 5 yds.

1916.

I. $a^0=1$. II. (1) $x-\frac{x^2}{2}+\frac{x^3}{3}$.

(2) $a=1$, $b=-1$, $c=1$, and $d=-1$. (3) $3abc$.

III. The maximum value = 4; $x=-1\cdot5$ and $0\cdot5$.

IV. (2) $\frac{1}{3}$, 2, 3.

V. 40 miles and 30 miles an hour; distance = 420 miles.

VI. 264.

1917.

I. (1) $-(a-b)(b-c)(c-a)(a+b+c)$.

(2) $(4a^2-8ax-x^2)^2$ II. (1) 23.

(2) $\frac{2a}{1-a^2}$. III. (1). 11.

IV. (1) 5 and -3 (2). $A=3$, $B=4$.

$$\frac{3}{x+5}+\frac{4}{x-3}.$$

V. 25 miles an hour; 100 miles. VI. Rs. 558; Rs. 5

1918.

I. (1) $x(30\frac{1}{2}-a)+(39-c)$;

$a=30\frac{1}{2}$, $c=39$.

(2) $a=x$. III. $2\cdot3$, $-1\cdot5$.

IV. (2) $x=c-b$, $y=a-c$, $z=b-a$. V. 6400.

VI. $\frac{4y}{60}+\frac{10(x-y)}{60}=5$; the distance is 1,750 ft.

1919.

I. (1) $a=-2$, $c=1$. (2) $-9(a-b)(b-c)(c-a)$.

II. (1) 256; $(4a^2+12a-11)$.

(2) $A=1$, $B=2$, $C=3$, $D=0$.

III. $y=x(3-x)$; $2\cdot6$ and $\cdot4$. IV. (2) $x=a$, $y=b$, $z=c$.

V. 30 miles; 12, 15, 20 miles an hour.

VI. The number of benches = 90 and the number of boys on each bench = 10.

1920.

I. (1) $y^4+11y^3+47y^2+93y+69$.

(2) 1·0203.

II. (1) $(x+4)(2x-3)(3x+2)(x-5)$.

The expression will be zero when $x=-4, \frac{3}{2}, -\frac{2}{3}$ and 5.

(2). $\frac{1}{n}$. III. (1) $x=\frac{1}{2}$ or $-5\frac{1}{2}$.

IV. Solve $x^2+y^2=a$.

$x+y=b$.

Where x and y are the sides of the triangle

$$x=\frac{b+\sqrt{2a^2-b^2}}{2}, \quad y=\frac{b-\sqrt{2a^2-b^2}}{2}.$$

V. They meet at 20 min. after 11 A.M. at a distance of $33\frac{1}{3}$ miles from P. VI. 12.

1921.

I. (1) 31043. (2) z^3+3z; z^4+4z^3+2.

II. (1) $(x+5)(2x-7)(3x-11.)$ (2) $ax+by+cz$.

III. (1) $x=-3$. (2) $h=\frac{5}{3}$, $k=\frac{5}{4}$.

IV. Take a straight line $=\frac{n+1}{2}$ units and on it describe a semi-circle. With one end of this line as centre and a radius equal to $\frac{n-1}{2}$ describe an arc cutting the circumference at a point. Measure the distance of this point from the other end of the diameter. This length is the square root of n. $\sqrt{7}=2{\cdot}65$ nearly.

V. The total number of shells = 1820.

The numbers of hits by A = 65.

" " " B = 130.

VI. The rate of the faster = 23 miles an hour and the rate of the slower 18 miles an hour.

1922.

1. i) $2x^2+x-1$; $2x^2+4xh+2h^2-3x-3h$; $4x+2h-3$.

2. $S=vt-\frac{1}{2}ft^2$. 3. (i) $x^3+7x+11$.

(ii) $(x^2+3x+9)(x^2-3x+9)$.

4. (i) $x=-4$. (ii) $x=15$, $y=8$.
5. $(16x-2x^2)$ square inches. The area is greatest when AP = PB.
6. 12.

1923.

1. (i) -32. 2. (i) $y=x^2-x$; $l=-1$. (ii) $p=20$, $q=17$.
3. (i) (a) $(x-4)(x-1)(x+2)(x+5)$ (b) $(a^2+b^2)(a^4-a^2b^2+b^4)$, $(a+b)(a-b)(a^2+ab+b^2)(a^2-ab+b^2)$.
   (ii) $(x+3)(x-2)(2x+1)$. 4. (ii) $2x^2-3x+5$.
5. (i) $\frac{1}{11}$. (ii) 12 persons.

1924.

1. (i) $x\nearrow 3$, $x\swarrow 4\frac{2}{3}$; $\therefore x=4$. (ii) $\frac{103}{301}$.
2. (i) $\dfrac{2(2x^2+1)}{x(x^2+1)}$.
3. (i) (a) $(x^2+4xy+8y^2)(x^2-4xy+8y^2)$
   (b) $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.
   (ii) $a=4$; $x+3$. 4. (i) x^2-2x-3.
   (ii) $(x+1)(2x-3)$.
5. 7·2″; 4″ & 6″. 6. (ii) 21″ and 28″.

1925.

1. (ii) $1\frac{9}{76}$. 2. (i) $(a-b)(b-c)(c-a)$.
   (ii) $(3x+4y)(2x-3y)$.
3. (i) $(x+3)(2x+5)(2x-3)(5x+4)$. (ii) $x^3-8x^2y+12xy^2-y^3$.
4. (ii) $x=0$, $2\frac{1}{2}$ or $-\frac{3}{7}$.
5. $a=-2$, $b=-1$, $c=1$; x^2+x+1. 6. 10″ or $7\frac{1}{2}$″.

1926.

1. $(u+v)(u-v)^2$; $a(a^2-4b)$. 2. $x^6-14x^4+48x^2-36$.
3. $1+4x+12x^2+32x^3$ is the quotient; $R=80x^4-128x^5$.
5. $(ac+bd)^2+(bc-ad)^2$ or $(ad+bc)^2+(ac-bd)^2$.
6. $a+b+c$. 7. $(2x+3y-8)(x+y-3)$; (4, 2).
8. $A=3$, $B=4$, $C=5$.
9. $\left(\frac{\pi}{2}-\frac{2}{\pi}\right)t$ hours' start; the result is independent of v.
10. 30″, 20″, $4\sqrt{7}$″, $6\sqrt{7}$″.

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